# 彈性連接桿之穩態振動
# Steady State Vibration of a Flexible Connecting Rod

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**摘要**

**本計畫探討一具阻尼之彈性連接桿之全域動態行為。文中強調剛性曲柄長度對連接桿振動的影響。首先我們導出連接桿軸向及側向振動的非線性運動方程式。數值分析的結果顯示當曲柄長度增加時，可能存在多重解。當多重解存在時，我們利用細胞對應法計算對應於各解的吸引區域。此外我們也發現彈性連桿振動有混沌的現象。混沌振動開始發生的轉速會隨著曲柄長度的增加而降低。此研究結果顯示前人利用線性模式預測彈性連接桿之振動行為只在轉速及曲柄長度均低時才有意義。**

**關鍵詞：彈性連接桿，全域動態行為，混沌振動**

**Abstract**

Global dynamics behavior of a damped flexible connecting rod is considered in this paper with emphasis on the effects of the rigid crank length. Nonlinear equations of motion in terms of axial and transverse deflections are derived based on Lagrangian strain formulation. When the crank length is small compared to the connecting rod, it is found that only one periodic solution exists in the speed range up to 1.5 times the first bending natural frequency of the connecting rod. As the crank length increases, however, multiple solutions may exist and the associated domains of attraction can be identified by cell-to-cell mapping technique. Moreover, the steady state response may become chaotic, which renders precise prediction of the dynamics response meaningless. The onset rotation speed of chaotic vibration decreases when the crank length increases. This result shows that previous researches utilizing simplified or linearized model can predict the dynamics response of the flexible connecting rod only when both the crank length and the rotation speed are small.

Keywords: Flexible connecting rod, global dynamics, chaotic vibration

## Introduction

The flexible connecting rod of a slider-crank mechanism can be considered as a beam undergoing large rigid body motion and may deform in both the axial and transverse directions. Previous researches of interest include transient vibration, steady state response, and dynamic stability analysis. A complete formulation should take into account the coupling effect of the axial and transverse vibrations. However, due to mathematical complexity, it was very common in the past to adopt various assumptions in the formulation to facilitate analytical study. The simplest approach is to assume that the axial load in the connecting rod is a function of time only and can be obtained by assuming that the connecting rod is rigid. The resulted equation of motion in terms of the transverse deflection is an inhomogeneous Mathieu equation [1].

The second approach is to relate the axial force to the transverse displacement by integrating the axial equilibrium equation. This formulation is an improvement of the previous one, and has been adopted by Viscomi and Ayre [2].

The third approach is to consider the effect of axial vibration, but assume that the axial force is proportional to the linear axial strain [3]. The two equations of motion involve both the axial and transverse deflections. To simplify the solution procedure the coupled equations of motion are then linearized by deleting all the terms containing products of deflections.

Recently Chen and Chen [4] reformulated the equations of motion of the flexible connecting rod by assuming that the axial force is proportional to the Lagrangian axial strain, which involves both the axial and transverse deflections. They found that previous simplified or linearized formulations considerably overestimate the transient response, especially when the crank speed is comparable to the first bending natural frequency of the connecting rod because terms of significant order of magnitude are ignored inadequately. In most of the publications cited above, the length of the rigid crank is assumed to be small to facilitate analytical study. To our best knowledge, the effects of crank length on the dynamic response of the connecting rod have not been discussed in detail yet.

In this paper we extend Chen and Chen's work [4] to consider the global dynamics behavior of a damped flexible connecting rod with emphasis on the effects of the rigid crank length. Complete nonlinear equations of motion in terms of axial and transverse deflections are derived based on Lagrangian strain formulation. Bifurcation diagrams based on Poincare mapping are presented. Multiple steady state solutions and the associated domains of attraction are studied by cell-to-cell mapping technique. Possible chaotic responses of a flexible connecting rod with moderate and long crank lengths are demonstrated. It is believed that this interesting phenomenon has not been reported in the literature.

## Equations of Motion

The dimensionless equations of motion of a flexible connecting rod can be written as [4]

$$\varepsilon^2(u_{,tt}-\dot{\theta}^2u)-\varepsilon\left(2\dot{\theta}v_{,t}+\ddot{\theta}v\right)-\frac{1}{\pi^4}\left(u_{,xx}+v_{,x}v_{,xx}\right)$$
$$-\frac{\zeta}{\pi^4}\left(u_{,xxt}+v_{,xt}v_{,xx}+v_{,x}v_{,xxt}\right)-x\dot{\theta}^2-a\Omega^2\cos(\Omega t-\varphi)=0 \quad (1)$$

$$\varepsilon^2(2\dot{\theta}u_{,t}+\ddot{\theta}u)+\varepsilon\left[v_{,tt}-\dot{\theta}^2v+\frac{1}{\pi^4}\left(v_{,xxxx}-u_{,xx}v_{,x}-u_{,x}v_{,xx}-\frac{3}{2}v_{,x}^2v_{,xx}\right)\right]$$
$$+\varepsilon\frac{\zeta}{\pi^4}v_{,xxxxt}$$
$$-\varepsilon\frac{\zeta}{\pi^4}\left(u_{,xxt}v_{,x}+2v_{,xx}v_{,xt}v_{,x}+u_{,xt}v_{,xx}+v_{,x}^2v_{,xxt}\right)$$
$$+x\ddot{\theta}-a\Omega^2\sin(\Omega t-\varphi)=0 \quad (2)$$

It is noted that by this scaling method the small parameter $\varepsilon$ arises naturally, which can then be used to compare the order of magnitude of each term in the dimensionless equations.

We assume a one-mode approximation for $u$ and $v$ as following,

$$u(x,t)=f(t)\sin\frac{\pi x}{2} \quad (3)$$
$$v(x,t)=g(t)\sin\pi x \quad (4)$$

After substituting Eqs.(3) and (4) into Eqs.(1) and (2), multiplying Eqs.(1) and (2) by $\sin(\pi x/2)$ and $\sin\pi x$, respectively, and then integrating by parts from $x$=0 to 1 we obtain

$$\varepsilon^2(\ddot{f}-\dot{\theta}^2f)-\varepsilon\frac{8}{3\pi}(2\dot{\theta}\dot{g}+\ddot{\theta}g)+\frac{1}{\pi^2}\left(\frac{f}{4}+\frac{7}{15}g^2\right)+\frac{\zeta}{\pi^2}\left(\frac{\dot{f}}{4}+\frac{14}{15}g\dot{g}\right)=$$
$$\frac{8}{\pi^2}\dot{\theta}^2+\frac{4}{\pi}a\Omega^2\cos(\Omega t-\varphi)$$
$$-\frac{2}{\cos\varphi}\left(\Theta-m_sa\Omega^2\cos\Omega t\right) \quad (5)$$

$$\varepsilon^2\frac{8}{3\pi}(2\dot{\theta}\dot{f}+\ddot{\theta}f)+\varepsilon\left(\ddot{g}-\dot{\theta}^2g+g+\frac{14}{15\pi^2}fg+\frac{3}{8}g^3\right)$$
$$+\varepsilon\zeta\left(\dot{g}+\frac{14}{15\pi^2}\dot{f}g+\frac{3}{4}g^2\dot{g}\right)=-\frac{2}{\pi}\ddot{\theta}+\frac{4}{\pi}a\Omega^2\sin(\Omega t-\varphi)$$
$$(6)$$

## Short Crank Length, $a$=0.1

Equations (5) and (6) are two nonlinear coupled equations with periodic coefficients and forcing terms. In the following sections we use Runge-Kutta method to study the steady state solutions of the flexible connecting rod. The time interval of the integration is chosen to be one-hundredth of the crank rotation period. Satisfactory convergence is always checked and ensured. We first choose the following parameters for a slider-crank mechanism with relatively short crank: $\varepsilon=0.05$, $\zeta=0.024$, $m_s=0.1$, $a=0.1$. After specifying initial conditions for $f,\dot{f},g,\dot{g}$ at certain crank speed $\Omega$, we examine the Poincare map of the response in the $g-\dot{g}$ space. The deflection is recorded when the crank is in the direction of the slider motion. If there exists a periodic steady state solution with frequency $n\Omega$, the points in the Poincare map converge to $n$ separated points. We call this periodic steady state vibration a P-$n$ solution. With damping ratio $\zeta=0.024$ the steady state solution can be reached after 300 cycles. We record these steady state solutions in Fig.1 for another 100 cycles. In other words, each point for a periodic solution in Fig.1 actually represents 100 points at the same location in the Poincare map. In the case when $a$=0.1, Fig. 1 shows that the P-1 solutions bifurcate to P-2 at $\Omega$=1.05, and then change back to P-1 again at $\Omega$ =1.37. The $\Omega$ increment in the bifurcation diagram is 0.01. To examine whether different initial conditions result in different steady state solutions we choose four points in $f-\dot{f}$ space, $(f,\dot{f})=(\pm3,\pm3)$. For each of these points in $f-\dot{f}$ space, we choose 25 evenly-separated points in $g-\dot{g}$ space within the ranges $|g|\le 4.5$ and $|\dot{g}|\le 4.5$. For these 100 different initial conditions the solutions settle to the same steady states in this case. In other words, no multiple solution can be found in the specified range of initial conditions.

**Moderate Crank Length, $a$=0.4**

We next consider a slider-crank mechanism with moderate crank length $a$=0.4 while the other parameters remain the same. Figure 2 shows the bifurcation diagram from Poincare sampling similar to Fig.1. The same 100 initial conditions as in Fig.1 are used in the calculation. It is noted that multiple periodic steady state vibrations exist. For instance, in the speed range $0.42<\Omega<0.46$ there exist two P-1 solutions. In Fig.3 we choose the initial point ( $f,\dot{f}$ )=(0.5,0.5) and use cell-to-cell mapping technique to identify the domains of attraction of these two steady state solutions in the $g-\dot{g}$ space when $\Omega=0.43$. The $g-\dot{g}$ space is divided into $150\times150$ cells. The attractors for the two P-1 solutions are denoted by black dots. It is noted that the basin boundary of the two attractors changes as the rotation speed of the rigid crank changes. Therefore, by fixing an initial point and changing rotation speed, the response may settle to different attractor for different speed. Although it is impractical trying to locate all the attractors in the entire state space, the 100 initial conditions should cover most of the responses for those initial points in the state space of interest. More initial points can always be examined whenever faster computing facility becomes available.

In the higher speed range $\Omega$>0.86, it is noted that the response becomes chaotic. After identifying chaotic response we record the deflections for 500 cycles in Fig.2. In other words, each black strip in Fig.2 represents 500 points on the Poincare map. In the speed range 0.86<$\Omega$<0.96 one P-1 attractor and a chaotic attractor coexist. The onset of chaotic response at $\Omega$=0.86 is so abrupt that typical period doubling phenomenon was not observed even by an $\Omega$ increment of 0.0005. After ignoring the solutions of the first 300 cycles, the Poincare map at $\Omega$=0.9 with 60000 points is shown in Fig.4. The typical fractal distribution can be observed.

In the speed range 0.96<$\Omega$<0.99 the response becomes periodic again. We magnify this speed range with $\Omega$ increment of 0.0005 in the left inset of Fig.2. These solutions belong to a single attractor undergoing period bifurcation from chaos to P-12, P-6, P-3, and back to P-6, P-12, and chaos again.

**Conclusions**

We examine the steady state vibrations of a damped flexible connecting rod. Lagrangian strain formulation considering both the axial and transverse vibrations are performed to derive the coupled nonlinear equations of motion. The results from linear strain formulation are also presented for comparison. Several conclusions can be summarized in the following.

(1) When the crank length is small compared to the connecting rod ($a$=0.1), only periodic solutions exist in the speed range up to $\Omega$ =1.5. The periodic solutions can be P-1 or P-2. Linear strain formulation gives satisfactory results when $\Omega$<0.6.
(2) For a moderate crank length ($a$=0.4), there exist multiple steady state solutions. Moreover, the response can become chaotic when $\Omega$ >0.86. Linear strain formulation gives satisfactory results when $\Omega$<0.3.

**References**

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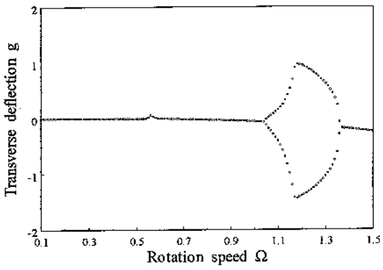


Fig.1

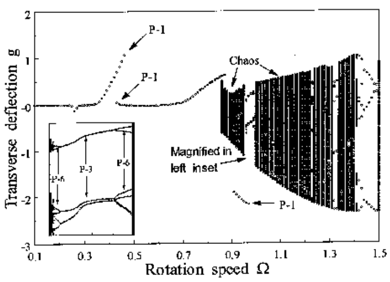


Fig.2

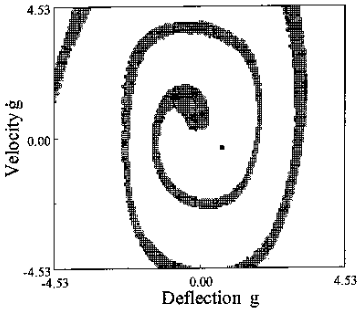


Fig.3

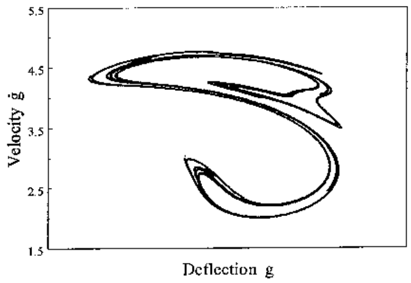


Fig.4