拱形元件於邊界受指定運動時之動態反應
Dynamics of Shallow Arches under Prescribed End Motion

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摘要

本篇論文主要在研究半正弦的拱形元件（中點高度h）的穩定性。首先探討拱形元件一端點以等速c移動距離e後的穩定性。在指定e的情況下，拱形元件最多有兩個穩定的平衡位置：P_0、P_1^-，兩者皆保持半正弦的形狀，其中P_0的凹口方向與初始形狀相同，而P_1^-則相反。當端點移動的速率可忽略時，拱形元件保持在P_0位置。當端點移動速率不可忽略時，拱形元件有可能由P_0跳至P_1^-，這個現象稱為動態折斷式挫曲。藉由能量屏障的觀念，可以在h-e平面上決定一範圍，使得端點不論以多少速率移動，拱形元件都不會跳至P_1^-。

關鍵詞: 拱形元件，動態折斷式挫曲，指定邊界運動

Abstract

In this project we consider a shallow arch with rise parameter h, free of lateral loading, but subject to prescribed end motion e with constant speed c. Attention is focused on finding out whether dynamic snap-through will occur. Quasi-static analysis is first performed to identify all equilibrium configurations and their stability properties when e and h are specified. It is found that there are at most two stable equilibrium configurations. One of them is P_0, which is always stable. The other is P_1^-, which is stable only in certain range of e and h. If the arch is stretched quasi-statically, it will be straightened up and no snap-through will occur. However, when the speed c is not negligible it is possible for the arch to snap from P_0 to P_1^- dynamically. After determining the energy barrier preventing the arch from moving from P_0 to P_1^- and the upper bound of total energy gained by the arch during prescribed end motion, one can specify the sufficient condition against dynamic snap-through.

Keywords: shallow arch, dynamic snap-through, prescribed end motion

Introduction

In general it is very difficult to determine the necessary and sufficient condition for dynamic snap-through to occur. However, it is possible to propose a sufficient condition against dynamic snap-through. To do this we first determine the energy barrier between two stable configurations. For dynamic snap-through to occur, the total energy gained by the arch during the prescribed end motion must exceed this energy barrier. This is only a necessary condition because even the arch gains enough energy to surpass the energy barrier, it is still possible for the arch to snap back to the original stable configuration. On the other hand, if the total energy gained by the arch during the prescribed end motion is smaller than the associated energy barrier, then it is obvious that no snap-through will occur. This can be used as a sufficient condition against dynamic snap-through.

Equations of Motion

The dimensionless equation of motion of an arch can be written as following,

$$u_{,²²} = -\left(u-u_0\right)_{,¶¶¶¶} + pu_{,¶¶} \qquad (1)$$

$$p = e + \frac{1}{2®}\int_0^® \left(u_{,¶}^2 - u_{0,¶}^2\right) d¶ \qquad (2)$$

The initial shape of the arch is $u_0(x)$. At time $²$=0, one of the ends starts to move a distance e with constant speed c. The shape of the stretched arch is $u(¶,²)$.

The initial shape of the arch is assumed to be in the form

$$u_0 = h \sin \P \quad (3)$$

h is the rise parameter of the arch. It is assumed that the shape of the arch after stretching can be expanded as

$$u(^2) = u_0 + \sum_{n=1}^{\infty} \sim_n (^2) \sin n\P \quad (4)$$

After substituting Eqs.(3) and (4) into (1) and (2) we obtain the equations governing $\sim_n$,

$$\ddot{\sim}_1 = -\sim_1 - (G+e)(h+\sim_1) \quad (5)$$

$$\ddot{\sim}_n = -n^4 \sim_n - n^2 (G+e) \sim_n \quad n = 2,3, \quad (6)$$

where

$$G = \frac{1}{4} \sum_{k=1}^{\infty} k^2 \sim_k^2 + \frac{h}{2} \sim_1 \quad (7)$$

$$e(^2) = c^2 \quad (8)$$

Equilibrium Configurations

One-Mode Solutions:

It is easy to show that if $\sim_j \neq 0$ and $\sim_n = 0$ for all $n \neq j$, then j=1. In other words, the one-mode solution must be in the form $u = \sim_1 \sin \P$, where $\sim_1$ satisfies a cubic equation.

(1) If h<4, or $h \geq 4$ but $e > e_1$, where

$$e_1 = \frac{h^2}{4} - \frac{3}{4} (2h)^{\frac{2}{3}} - 1 \quad (9)$$

then there is only one equilibrium configuration, denoted by P_0.

(2) If h>4 and $e < e_1$, then there are three equilibrium configurations are denoted by P_0, P_1^+, and P_1^-, respectively.

Two-Mode Solutions:

For this case the solutions can be written explicitly,

$$\sim_1 = \frac{-j^2 h}{j^2 - 1} \quad (10)$$

$$\sim_j = \pm \frac{2}{j} \sqrt{e_j - e} \quad (11)$$

where

$$e_j = \frac{(j^2 - 2) j^2 h^2}{4(j^2 - 1)^2} - j^2 \quad (12)$$

These configurations are denoted by P_{1j}^+ and P_{1j}^-. The special h which renders $e_1 = e_j$ is denoted by $\bar{h}_j$, where

$$\bar{h}_j^2 = 2(j^2 - 1)^3 \quad j=2,3,4,\ldots \quad (13)$$

Stability Properties of Equilibrium Configurations

First of all, the dimensionless total energy H of any configuration can be calculated as,

$$H = 2(G+e)^2 + \sum_{n=1}^{\infty} \left[\dot{\sim}_n^2 + n^4 \sim_n^2 \right] \quad (14)$$

Several theorems regarding the stability of the equilibrium positions can be proved.

Theorem 1: Equilibrium configuration P_0 is stable. Equilibrium configuration P_1^+ is unstable. Equilibrium configuration P_1^- is stable if and only if $e < e_1$ and $\bar{p} + 4 > 0$. If $4 < h \leq \bar{h}_2$, then P_1^- is stable if and only if $e < e_1$. On the other hand, if $h > \bar{h}_2$, then P_1^- is stable if and only if $e < e_2$. Equilibrium configurations P_{1j}^+ and P_{1j}^- are unstable.

Theorem 2: In the case when $4 < h \le \bar{h}_2$ and $e_2 \le e < e_1$, $U(P_1^+)$ is the global minimum in the sub-space $(0, a_2, a_3, \ldots)$. In the case when $e < e_2$, $U(P_{12}^{\pm})$ is the global minimum in the sub-space $(0, a_2, a_3, \ldots)$.

Theorem 3: The total energy gained by the arch during prescribed end motion with finite speed is less than $H_\infty \to 2e^2$.

Snap-Through Criterion

After establishing the upper bound of the total energy, we can restate the sufficient conditions against dynamic snap-through in a more conservative way as follows. *Case (1)* $4 < h \le \bar{h}_2$: If $e_2 \le e < e_1$, then the sufficient condition against snap-through is $\sim_1(2_s) > \sim_1(P_1^+)$ and $H_\infty < U(P_1^+)$. If $e < e_2$, then the sufficient condition against snap-through is $\sim_1(2_s) > \sim_1(P_1^+)$ and $H_\infty < U(P_{12}^{\pm})$. *Case (2)* $h > \bar{h}_2$: If $e < e_2$ then the sufficient condition against snap-through is $\sim_1(2_s) > \sim_1(P_{12}^{\pm})$ and $H_\infty < U(P_{12}^{\pm})$.

Theorem 4: No snap-through will occur if $4 < h \le \bar{h}_2$. No snap-through will occur if $h > \bar{h}_2$ and $e < e_{cr}$.

Figure 1 shows the total energy as a function of e for various speeds. The H_∞ curve represents the upper bound of total energy for any finite speed. In Fig.1(a) for $h = 5 < \bar{h}_2$, the energy barriers are $U(P_{12}^{\pm})$ and $U(P_1^+)$ when e is in the ranges $0 < e < e_2$ =1.56 and $e_2 < e < e_1$ =1.77, respectively. The total energy upper bound H_∞ can never surpass the energy barrier in time when e changes from 0 to 2. In Fig.1(b) for $h = 10 > \bar{h}_2$, H_∞ curve meets $U(P_{12}^{\pm})$ curve at e_{cr} =12.33, which is smaller than e_2 =18.22. $U(P_{12}^{\pm})$ is the energy barrier in this case. The energy histories for four speeds c=15, 31.9, 50, and 100 are shown to approach H_∞ as c increases. Figure 1(c) shows the special case when $h = \bar{h}_2$, at which H_∞ curve meets $U(P_1^+)$ and $U(P_{12}^{\pm})$ curves at $e = e_{cr} = e_1 = e_2$ =8.

Figures 2(a), 2(b), and 2(c) show the deformation history for an arch with h=10, and e=15. Th speed c is set to be 50. For these cases $2_s = 0.3$, as signified by the black dots on the response curves. The damping « used in Figs. 2(a), 2(b), and 2(c) are 0.005, 0.01, and 0.05, respectively. For the small damping case in Fig. 2(a) the energy gained by the arch is large enough to surpass $U(P_{12}^{\pm})$ and $\sim_1$ can reach $\sim_1(P_1^-)$. However, due to small damping, the arch is snapped back and finally settles to the configuration P_0. For the medium damping case in Fig. 2(b), the arch not only gains enough energy to surpass $U(P_{12}^{\pm})$ and $\sim_1$ can reach $\sim_1(P_1^-)$, the damping also prevents it from snapping back to P_0. The arch settles to P_1^- eventually. This phenomenon is called dynamic snap-through under prescribed end motion. In Fig. 2(c) the damping is so large that it prevents the arch from surpassing the energy barrier $U(P_{12}^{\pm})$, and the arch has no choice but to settle to P_0.

Conclusions

In this project we consider a shallow arch with rise parameter *h*, free of lateral loading, but subject to prescribed end motion *e* with constant speed *c*. Some conclusions can be summarized in the following.

(1) There are at most two stable equilibrium configurations for any give *e* and *h*. One of them is P_0, which is always stable. The other is P_1^-, which is stable only in certain range of *e* and *h*.

(2) When $4 < h \le \bar{h}_2$ and $e_2 \le e < e_1$, the energy barrier preventing the arch from snapping from P_0 to P_1^- is the strain energy of P_1^+.

(3) In the case when $e < e_2$ the energy barrier is the strain energy of $P_{12}^{\pm}$.

(4) The total energy gained by the arch has an upper bound $2e^2$, which corresponds to the case when *c* approaches infinity.

(5) The only possible situation when dynamic snap-through may occur is $h > \bar{h}_2$ and $e_{cr} < e < e_2$.

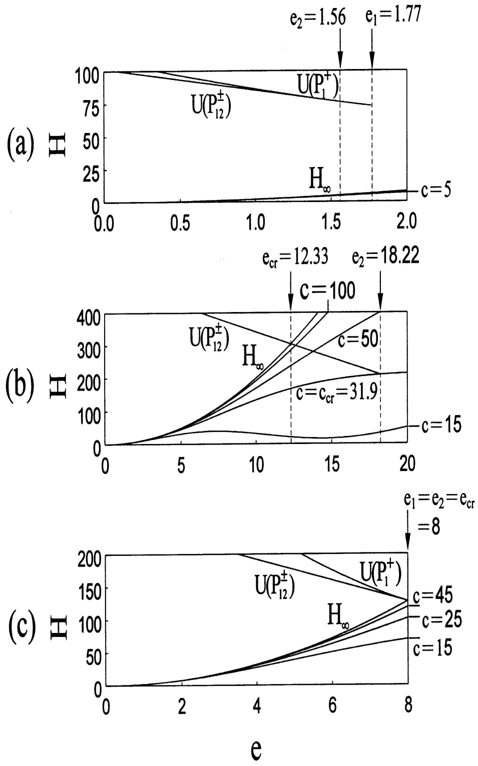

Figure 1

Figure 2