Transient Analysis of Dynamic Crack Propagation With Boundary Effect

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In this study, a dynamic antiplane crack propagation with constant velocity in a configuration with boundary is investigated in detail. The reflected cylindrical waves which are generated from the free boundary will interact with the propagating crack and make the problem extremely difficult to analyze. A useful fundamental solution is proposed in this study and the solution is determined by superposition of the fundamental solution in the Laplace transform domain. The proposed fundamental problem is the problem of applying exponentially distributed traction (in the Laplace transform domain) on the propagating crack faces. The Cagniard's method for Laplace inversion is used to obtain the transient solution in time domain. Numerical results of dynamic stress intensity factors for the propagation crack are evaluated in detail.

1 Introduction

Most of the analyses done regarding cracked bodies are quasi-static. There are numerous situations that the material inertia becomes significant and must be taken into account in the analyses. The question of whether or not inertial effects are significant depends on the loading conditions and the geometrical configuration of the body. Inertial effects can arise either from applying dynamic loading on a cracked solid or from rapid crack propagation. The inherent time dependence of a dynamic fracture process results in mathematical models that are more complex than equivalent quasi-static models. However, there is substantial interest in the dynamic fracture problem due to its importance in many engineering applications. The problem is encountered in impact damage to fan blades, automotive and aircraft windshields. There is also considerable interest in the problem of arrest of a fast running crack, especially in large structures like pipelines, ships, and nuclear reactors.

The main purpose in solving problems concerned with dynamic crack propagation is to determine the dependence of the crack-tip field characterizing parameters on the applied loading and on the configuration of the body. The investigation of a propagation crack in a brittle solid began with the pioneering analysis of Yoffe (1951). She considered a steady-state crack growth problem of a crack of fixed length propagating in an infinite elastic body subjected to a uniform remote tensile loading normal to the crack line. Although this is a physically unrealistic problem, it did at least provide an indication of the influence of crack speed on the stress state of a rapidly propagating crack. Craggs (1960) considered a semi-infinite crack extending at constant speed, with the crack face loading moving with the same speed as the crack tip in such a way that the entire deformation field is constant as seen by an observer moving with the crack tip. The self-similar dynamic crack propagation problem was contributed by Broberg (1960) who was among the first to present detailed analyses of crack propagation as a transient process. He solved the dynamic problem of a crack that suddenly grows from zero length at a constant speed. Baker (1962) subsequently generalized Broberg's solution to include a finite initial crack. Although the aforementioned artificial solutions have no direct application, they also have provided useful insights and continue to be used today for the assessment of dynamic numerical analyses.

In a series of papers, Freund (1972a, 1972b, 1973, 1974) developed important analytical methods for evaluation of the transient stress field of a propagating crack in a two-dimensional geometric configuration under quite general dynamic loading situation. These particular cases analyzed by Freund are also self-similar, but they are solved by means of integral transform methods rather than by direct application to similarity arguments. An indirect analytical approach proposed by Freund based on superposition over a fundamental solution, which opens a way for analysis of certain problems of crack propagation at nonuniform speed. Based on the superposition method proposed by Freund, a series of problems for nonplanar crack propagation in an infinite domain was solved by Ma and Burgers (1986, 1987, 1988) and Ma (1988, 1990). The structure of the near-tip field of crack propagation at nonuniform speeds was discussed in detail by Freund and Rosakis (1992). A representation of the crack-tip field was obtained in the form of an expansion about the crack tip in powers of radial coordinate, with the coefficients depending on the time rates of change of crack-tip speed and stress intensity factor. This representation was used to interpret some experimental observations and some estimates were made of the practical limits of using a stress intensity factor field alone to characterize the local fields.

Most of the solved dynamic fracture problems are regarded as a crack subjected to applying a uniformly distributed dynamic loading on the crack faces or subjected to incident plane waves. For the aforementioned problems, either the direct application of the well-known Wiener-Hopf technique is used or the superposition method proposed by Freund is performed to solve the problems. However, if a crack is subjected to incident nonplanar waves, none of the known methods can be used to obtain the transient solutions. Recently Tsai and Ma (1992) proposed a new fundamental solution to overcome these difficulties. The fundamental problem they considered is an exponentially distributed traction applied on crack faces and the solution is constructed by superposition of the fundamental solution in the Laplace transform domain. Tsai and Ma (1992) used this new fundamental solution to obtain the transient solution for a stationary semi-infinite crack subjected to a suddenly applied dy-

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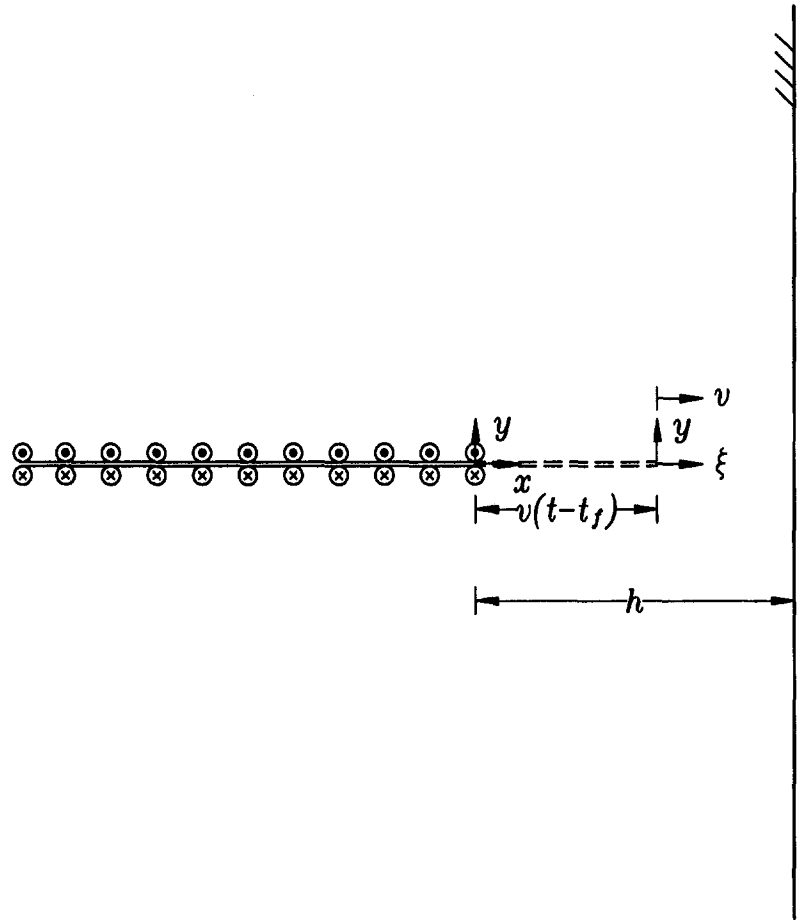

Fig. 1 Coordinate systems of a propagating crack in the configuration with vertical boundary

namic inplane body force in an unbounded medium. This alternative fundamental solution is also successfully applied towards solving more complicated transient problems (Tsai and Ma, 1993a, 1993b; Ma and Chen, 1994) for a subsurface stationary inclined crack subjected to dynamic loadings.

Determination of the dependence of the dynamic stress intensity factor on body configuration and applied loading during rapid crack propagation is a principal objective in dynamic fracture. Analytical results for configurations with boundaries other than the crack faces are rare. It was noted that after the stress waves reflected from the boundaries arrived at the crack tip, the nature of crack propagation changed. The interaction of stress waves with propagating cracks provides a way of altering the crack-tip stress field, and thus provide the basis for the crack branching phenomenon. Dynamic analysis of a mode III crack in a strip was discussed in a paper by Nilsson (1973). He considered the problem of a semi-infinite crack in a strip of finite width subjected to a uniformly distributed static load and propagating with a constant velocity, only the stress intensity factor and its time dependence was determined.

Since the interaction of stress waves with a propagating crack is an important event, we shall investigate it in greater detail in the following. In this study, we will extend the methodology that was used successfully in solving dynamic stationary crack problem to construct the transient solution for crack propagation in the configuration with boundary. Two different cases will be considered in this study in some detail. The first problem considered in this study is a horizontal semi-infinite crack propagating with constant velocity towards a vertical free boundary as shown in Fig. 1. The second problem is a horizontal semi-infinite crack propagates in a strip with finite width as shown in Fig. 2. In analyzing the aforementioned problems, free boundary and propagating crack are involved into the analysis. The interaction of the reflected waves generated from the free boundary and the propagating crack must be taken into account which will make the analysis extremely difficult. A useful fundamental solution is proposed to overcome these difficulties. This proposed fundamental solution is successfully applied towards solving the problem and is demonstrated as an efficient methodology to solve similar problems. Since the stress intensity factor is the key parameter in characterizing dynamic crack grows, we will focus our attention mainly on the determination of the dynamic stress intensity factor.

2 Required Fundamental Solutions

As usual in problems of the type considered here, superposition of solutions plays a significant role. The solutions of the problem considered in this study can be determined by superposition of the following problems. Problem A treats a dynamic uniformly distributed traction acting on semi-infinite crack faces in an unbounded medium at time $t = 0$, at $t = t_f$, a new crack propagates out of the original semi-infinite crack with constant velocity, which induces a traction on the plane that will eventually define the vertical or horizontal half-plane boundary. In problem B, a semi-infinite half-plane is considered in which the boundary is subjected to tractions which are equal and opposite to those on the corresponding planes in problem A. Problem C considers an infinite body containing a propagating semi-infinite crack in which the crack face is subjected to the reflected waves which are generated by the half-plane boundary in problem B. The three fundamental problems A, B, and C, which are

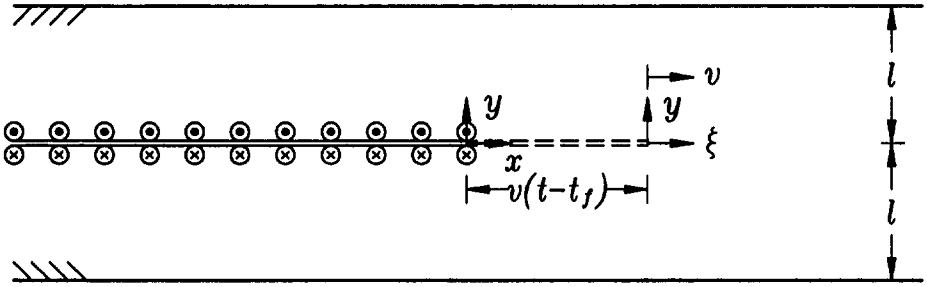

Fig. 2 Configuration and coordinate systems of a propagating crack in a finite strip

superimposed to obtain the solution for propagating crack interaction with stress wave, are shown in Fig. 3. The reflected waves induced from the half-plane boundary as indicated in problem B, can be easily obtained by employing the method of images. While problem A is a well-known problem for a semi-infinite crack propagating in an unbounded medium and many efforts have been devoted to analyze this problem.

Problem C in the above mentioned three fundamental problems is the only one which needs careful analysis. The problem we will deal with is the interaction of the propagating crack with cylindrical reflected waves, which causes the only difficulty in this investigation. For most of the dynamic problems, the propagating waves can be represented in an exponential functional form in the Laplace transform domain of time. The reflected and diffracted waves generated by the half-plane boundary and by the crack can thus be constructed by the superposition method; that is, if the responses toward an applied exponentially distributed traction on boundaries in the Laplace transform domain can be obtained preliminarily. The superposition scheme proposed in this study, unlike usual superposition methods which are performed in the time domain, is performed in the Laplace transform domain.

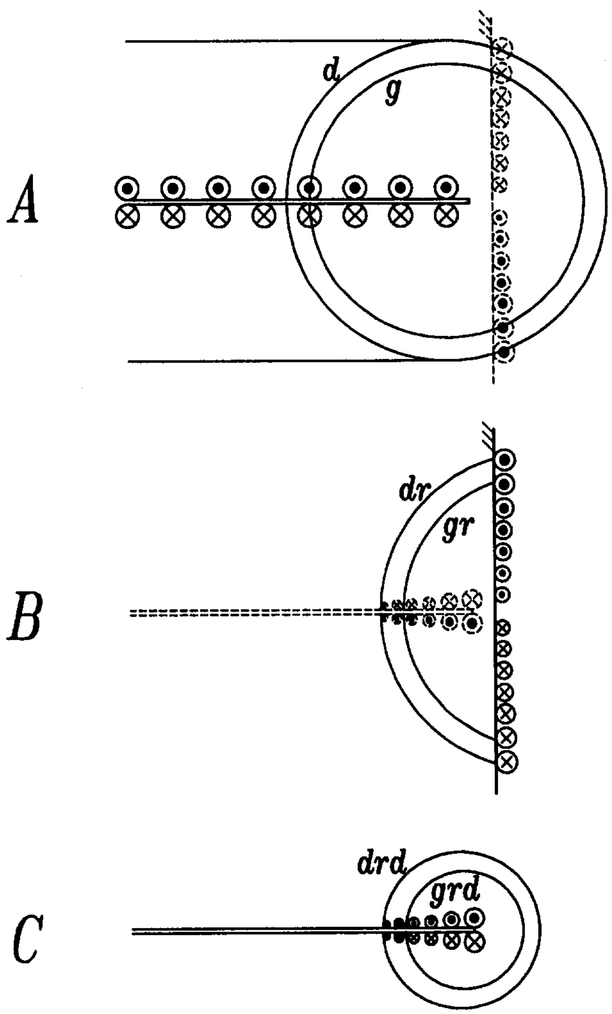

Fig. 3 Three fundamental problems which are superimposed to obtain the solution for constant velocity crack extension and interaction with the stress wave reflected from boundary

Consider the fundamental problem of antiplane deformation for a propagating semi-infinite crack with constant velocity in an unbounded medium. The solution for an exponentially distributed loading applied at the crack faces in the Laplace transform domain will be referred to as the fundamental solution. The problem can be viewed as a half-plane problem with the material occupying the region $y \geq 0$, subjected to the following mixed boundary conditions in the Laplace transform domain

$$\bar{\tau}_{yz}(\xi, 0, s) = e^{s\eta\xi} \quad -\infty < \xi < 0, \tag{2.1}$$

$$\bar{w}(\xi, 0, s) = 0 \quad 0 < \xi < \infty, \tag{2.2}$$

where s is the Laplace transform parameter and η is a constant. The coordinate ξ defined by $\xi = x - vt$ is fixed with respect to the moving crack tip. The overbar symbol is used for denoting the transform on time t. The one-sided Laplace transform with respect to time and the two-sided Laplace transform with respect to ξ are defined by

$$\bar{w}(\xi, y, s) = \int_0^\infty w(\xi, y, t) e^{-st} dt,$$

$$\tilde{w}(\lambda, y, s) = \int_{-\infty}^\infty \bar{w}(\xi, y, s) e^{-s\lambda\xi} d\xi.$$

This fundamental problem can be solved by using the standard transform method and the Wiener-Hopf technique. The governing equation can be represented by the two-dimensional wave equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = b^2 \frac{\partial^2 w}{\partial t^2}, \tag{2.3}$$

where b is the slowness of the shear wave given by

$$b = 1/v_s = \sqrt{\rho/\mu},$$

in which $w(x, y, t)$ is the displacement normal to the xy-plane; v_s is the shear wave speed, and μ and ρ are the respective shear modulus and the mass density of the material. The nonvanishing shear stresses are

$$\tau_{yz} = \mu \frac{\partial w}{\partial y}, \quad \tau_{xz} = \mu \frac{\partial w}{\partial x}. \tag{2.4}$$

In analyzing this problem, it is convenient to express the governing equation in the moving coordinates $\xi - y$ as follows:

$$(1 - b^2 v^2) \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial y^2} + 2(b^2 v) \frac{\partial^2 w}{\partial \xi \partial t} - b^2 \frac{\partial^2 w}{\partial t^2} = 0.$$

This fundamental problem can be solved by the application of integral transforms. Applying the one-sided Laplace transform over time, the two-sided Laplace transform over ξ under the restriction of $\mathrm{Re}(\eta) > \mathrm{Re}(\lambda)$, finally the Wiener-Hopf technique is implemented. The solutions of stresses and displacement for the boundary conditions (2.1) and (2.2) expressed in the transform domain are

$$\bar{\tau}_{yz}(\xi, y, s) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{\alpha_+^*(\lambda)e^{-s(\alpha^* y - \lambda\xi)}}{\alpha_+^*(\eta)(\eta - \lambda)} d\lambda, \quad (2.5)$$

$$\bar{\tau}_{xz}(\xi, y, s) = -\frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{\lambda e^{-s(\alpha^* y - \lambda\xi)}}{\alpha_+^*(\eta)(\eta - \lambda)\alpha_-^*(\lambda)} d\lambda, \quad (2.6)$$

$$\bar{w}(\xi, y, s) = -\frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{e^{-s(\alpha^* y - \lambda\xi)}}{\mu s\alpha_+^*(\eta)(\eta - \lambda)\alpha_-^*(\lambda)} d\lambda, \quad (2.7)$$

where

$$\alpha^*(\lambda) = \sqrt{b + \lambda(1 - bv)}\sqrt{b - \lambda(1 + bv)} = \alpha_+^*(\lambda)\alpha_-^*(\lambda).$$

The corresponding result of the dynamic stress intensity factor in the Laplace transform domain is

$$\bar{K}(s) = \lim_{\xi\to 0} \sqrt{2\pi\xi}\,\bar{\tau}_{yz}(\xi, 0, s) = -\frac{\sqrt{2}\sqrt{1 - bv}}{\sqrt{s}\,\alpha_+^*(\eta)}. \quad (2.8)$$

3 Transient Analysis for Propagating Crack Interaction With Vertical Boundary

Consider a half-plane with a vertical boundary containing a horizontal semi-infinite crack which is stress free and at rest. At time $t = 0$, an antiplane uniformly distributed dynamic loading with magnitude τ_0 is applied at the crack faces of the semi-infinite crack. The time dependence of the loading is represented by the Heaviside step function $H(t)$. At time $t = t_f$, the semi-infinite crack suddenly propagates along the crack-tip line with constant velocity v toward the vertical half-plane boundary as shown in Fig. 1. The transient elastodynamic problem is solved by superposition of the fundamental solutions obtained in the previous section in the Laplace transform domain. The transient solutions are composed of an incident field, reflected field, and diffracted field, which are denoted by superscripts of i, r, and d, respectively. The incident wave is the propagating plane wave in an unbounded medium which is induced by applying a uniformly distributed loading on crack faces. The diffracted wave includes two parts, the first one is induced from the stationary crack tip by the application of a uniformly distributed traction on crack faces and the second one is generated from the propagating crack tip when the crack starts to move. We now focus the analysis on the diffracted field generated by the stationary crack due to the incident plane wave. This problem can also be solved by other methods, but the following analysis will be performed by the proposed superposition method in this study to indicate how the transient solution to be constructed. The incident field of the plane wave expressed in the Laplace transform domain can be obtained as follows:

$$\bar{\tau}_{yz}^i(x, 0, s) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{\tau_0}{s\lambda} e^{s\lambda x} d\lambda. \quad (3.1)$$

The applied traction on the crack face as indicated in (3.1), has the functional form $e^{s\lambda x}$. Since the solutions of applying traction $e^{s\eta x}$ on crack faces have been solved in Section 2 by setting $v = 0$, the diffracted field generated from the stationary semi-infinite crack can be constructed by superimposing the incident wave traction that is equal to (3.1). When we combine (2.5) (by setting $v = 0$) and (3.1), the solution of diffracted wave for $\bar{\tau}_{yz}^d$ and $\bar{\tau}_{xz}^d$ in the Laplace transform domain can be expressed as follows:

$$\bar{\tau}_{yz}^d(x, y, s) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{\tau_0}{s\lambda} \times \left\{\frac{1}{2\pi i}\int_{\Gamma_{\eta_2}} \frac{(b + \eta_2)^{1/2}}{(\lambda - \eta_2)(b + \lambda)^{1/2}} e^{-s(\alpha y - \eta_2 x)} d\eta_2\right\} d\lambda$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{\tau_0\sqrt{b + \lambda}}{s\lambda\sqrt{b}} e^{-s(\alpha y - \lambda x)} d\lambda, \quad (3.2)$$

$$\bar{\tau}_{xz}^d(x, y, s) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{-\tau_0}{s\sqrt{b - \lambda}\sqrt{b}} e^{-s(\alpha y - \lambda z)} d\lambda. \quad (3.3)$$

By using the Cagniard-de Hoop method of Laplace inversion, the diffracted stress field in time domain is obtained as follows:

$$\tau_{yz}^d(x, y, t) = \frac{\tau_0}{\pi\sqrt{b}}\int_{br}^{t} \mathrm{Im}\left[\frac{\sqrt{b + \lambda^+}}{\lambda^+}\frac{\partial\lambda^+}{\partial t}\right] dt, \quad (3.4)$$

$$\tau_{xz}^d(x, y, t) = \frac{-\tau_0}{\pi\sqrt{b}}\int_{br}^{t} \mathrm{Im}\left[\frac{1}{\sqrt{b - \lambda^+}}\frac{\partial\lambda^+}{\partial t}\right] dt, \quad (3.5)$$

where

$$\lambda^+ = -\frac{t}{r}\cos\theta + i\left(\frac{t^2}{r^2} - b^2\right)^{1/2}\sin\theta,$$

$$r = (x^2 + y^2)^{1/2}, \quad \theta = \cos^{-1}(x/r).$$

Equations (3.4) and (3.5) can be integrated and further simplified as ($\theta < \pi/2$)

$$\tau_{yz}^d(x, y, t) = \frac{\tau_0}{\pi}\left\{2\cos(\theta/2)\sqrt{\frac{t}{br} - 1} - \tan^{-1}\sqrt{\frac{t/br - 1}{1 - \sin\theta}} - \tan^{-1}\sqrt{\frac{t/br - 1}{1 + \sin\theta}}\right\}, \quad (3.6)$$

$$\tau_{xz}^d(x, y, t) = -\frac{2\tau_0}{\pi}\sin(\theta/2)\sqrt{\frac{t}{br} - 1}. \quad (3.7)$$

The corresponding stress intensity factor in the Laplace transform domain is

$$\bar{K}^d(s) = \frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{\tau_0}{s\lambda}\left[\frac{-\sqrt{2}}{\sqrt{s}(b + \lambda)^{1/2}}\right] d\lambda = \frac{\tau_0\sqrt{2}}{s^{3/2}\sqrt{b}}. \quad (3.8)$$

The dynamic stress intensity factor induced by diffracted d wave expressed in time domain will be

$$K^d(t) = 2\tau_0\sqrt{\frac{2t}{\pi b}}. \quad (3.9)$$

The results expressed in (3.6), (3.7), and (3.9) are the well-known solutions for a semi-infinite crack in an unbounded medium subjected to a uniformly distributed loading on crack faces. The dynamic stress intensity factor shown in (3.9) increases from zero as the square root of the time measured from the instant the uniformly distributed loading applied on crack faces. At time $t = t_f$, the dynamic stress intensity factor reaches its critical value and the crack starts to propagate with constant velocity v. The transient full-field analysis for a propagating crack just mentioned above has also been solved by Ma and

Burgers (1988) by using the method proposed by Freund (1972b). In their investigations, the transient solution for constant speed crack propagation is obtained by determining a fundamental solution for a concentrated force appearing through the moving crack tip, and then building up the general solution by superposition. In this study, a more direct and simple methodology will be used to solve this problem. We consider the transient problem of a semi-infinite crack propagates at $t = t_f$ with uniformly distributed loading applied only on the original crack face $-\infty < x < 0$. The applied uniform stress τ_0 on the original crack face written in the Laplace transform domain for the moving coordinate system will have the following form:

$$\bar{\tau}_{yz}(\xi, 0, s) = \frac{1}{2\pi i} \int_{\Gamma_\lambda} \frac{-\tau_0 d}{s\lambda(\lambda - d)} e^{s\lambda(\xi - vt_f)} d\lambda, \quad (3.10)$$

in which $d = 1/v$ is the slowness of the crack velocity and $\xi = x - v(t - t_f)$. The applied traction on crack faces as expressed in (3.10), has the functional form $e^{s\lambda\xi}$. Since the Laplace transform solutions of applying traction $e^{s\eta\xi}$ on crack faces have been solved in the previous section, the diffracted field generated from the propagating crack tip can be constructed by superimposing the fundamental solution and the stress distribution in (3.10). The results of shear stresses expressed in the Laplace transform domain will be

$$\bar{\tau}^g_{yz}(\xi, y, s) = \frac{1}{2\pi i} \int_{\Gamma_{\eta_1}} \frac{-\tau_0 d}{s\eta_1(\eta_1 - d)} e^{-s\eta_1 vt_f} \times \left\{ \frac{1}{2\pi i} \int_{\Gamma_{\eta_2}} \frac{\alpha^*_\ddagger(\eta_2)}{(\eta_1 - \eta_2)\alpha^*_\ddagger(\eta_1)} e^{-s[\alpha^* y - \eta_2 \xi]} d\eta_2 \right\} d\eta_1, \quad (3.11)$$

$$\bar{\tau}^g_{xz}(\xi, y, s) = \frac{1}{2\pi i} \int_{\Gamma_{\eta_1}} \frac{-\tau_0 d}{s\eta_1(\eta_1 - d)} e^{-s\eta_1 vt_f} \left\{ \frac{1}{2\pi i} \times \int_{\Gamma_{\eta_2}} \frac{-\eta_2}{\alpha^*_\pm(\eta_2)(\eta_1 - \eta_2)\alpha^*_\ddagger(\eta_1)} e^{-s[\alpha^* y - \eta_2 \xi]} d\eta_2 \right\} d\eta_1. \quad (3.12)$$

The exact transient solutions for a propagating crack at unbounded medium in time domain can be obtained by inverting the Laplace transform domain of (3.11) and (3.12). The results are

$$\tau^g_{yz}(\xi, y, t) = \frac{\tau_0}{\pi\sqrt{b}} \int_{t_m}^{t} \mathrm{Im}\left\{ \frac{\alpha^*_\ddagger(\lambda^+)}{\lambda^+} \frac{\partial \lambda^+}{\partial \tau} \right\}_{t=\tau} d\tau - \frac{\tau_0}{\pi\sqrt{d}} \int_{t_m}^{t-t_f} \mathrm{Im}\left\{ \frac{\alpha^*_\ddagger(\lambda^+)}{\lambda^+ - d} \frac{\partial \lambda^+}{\partial \tau} \right\}_{t=\tau} d\tau, \quad (3.13)$$

$$\tau^g_{xz}(\xi, y, t) = \frac{-\tau_0}{\pi\sqrt{b}} \int_{t_m}^{t} \mathrm{Im}\left\{ \frac{1}{\alpha^*_\pm(\lambda^+)} \frac{\partial \lambda^+}{\partial \tau} \right\}_{t=\tau} d\tau + \frac{\tau_0}{\pi\sqrt{d}} \int_{t_m}^{t-t_f} \mathrm{Im}\left\{ \frac{\lambda^+}{(\lambda^+ - d)\alpha^*_\pm(\lambda^+)} \frac{\partial \lambda^+}{\partial \tau} \right\}_{t=\tau} d\tau, \quad (3.14)$$

where

$$\lambda^+ = \frac{-(\xi t + b^2 v y^2) + iy\sqrt{t^2 - b^2[y^2 + (\xi + vt)^2]}}{\xi^2 + (1 - b^2v^2)y^2},$$

$$\frac{\partial \lambda^+}{\partial t} = \frac{-\xi + iy\{t^2 - b^2[y^2 + (\xi + vt)^2]\}^{-1/2} \times [t(1 - b^2v^2) - b^2 v\xi]}{\xi^2 + (1 - b^2v^2)y^2},$$

$$t_m = \frac{b\{bv\xi + [\xi^2 + (1 - b^2v^2)y^2]^{1/2}\}}{1 - b^2v^2}.$$

It is very interesting to see that the full-field solution of a crack propagating with constant speed subjected to loads applied on the original crack faces as shown in (3.13) and (3.14) can be expressed by two terms, each term having its own physical meaning. The first term represents the solution due to applying the uniform loading on the original and new crack faces for a crack which begins to grow at constant speed at time t_f after the loading is applied. The functional form of the first term is the same as the solution to the problem with a uniform loading applied on the original and new crack faces with no delay time. The only dependence on delay time is through the definition of ξ. This problem corresponds to Baker's problem (Baker, 1962) in mode III. The second term represents the solution for a crack which starts to propagate at time t_f for constant speed with loading applied uniformly on the new crack faces only. The results shown in (3.13) and (3.14) are also found to be the same as that obtained by Ma and Burgers (1988) from different methodology.

The dynamic stress intensity factor for a propagating crack at infinite medium can also be constructed by a similar manner. The result in the Laplace transform domain can be obtained from (2.8) and (3.10) and is expressed as follows:

$$K^g(s) = \frac{1}{2\pi i} \int_{\Gamma_\lambda} \frac{-\tau_0 d}{s\lambda(\lambda - d)} e^{-s\lambda vt_f} \left\{ \frac{-\sqrt{2}\sqrt{1 - bv}}{\sqrt{s}\alpha^*_\ddagger(\lambda)} \right\} d\lambda. \quad (3.15)$$

The inversion Laplace transform of (3.15) will have the following form:

$$K^g(t) = 2\sqrt{\frac{2}{\pi}}\, \tau_0 \sqrt{1 - bv} \left[\sqrt{\frac{t}{b}} - \sqrt{\frac{t - t_f}{d}} \right]. \quad (3.16)$$

After some later time, the diffracted wave (d wave) generated from the stationary crack and the shear wave (g wave) radiated out from the propagating crack will be reflected from the free vertical half-plane which will be indicated as the dr and gr wave, respectively. The solutions for reflected waves generated from the free boundary can be constructed by employing the method of images, which can be easily obtained from the solution of d wave and g wave just constructed previously, the result for dr wave in the fixed coordinate x–y system will be

$$\bar{\tau}^{dr}_{yz}(x, y, s) = \frac{-1}{2\pi i} \int_{\Gamma_\lambda} \frac{\tau_0\sqrt{b - \lambda}}{s\sqrt{b}\lambda} e^{-s[\alpha y - \lambda(x - 2h)]} d\lambda. \quad (3.17)$$

The solution expressed in the moving coordinate ξ–y system will become

$$\bar{\tau}^{dr}_{yz}(\xi, y, s) = \frac{-1}{2\pi i} \int_{\Gamma_\lambda} \frac{\tau_0\sqrt{b - \bar{\lambda}}}{s\bar{\lambda}\sqrt{b}(1 - \lambda v)} e^{-s[\alpha^* y - \lambda(\xi - vt_f - 2h)]} d\lambda, \quad (3.18)$$

where

$$\bar{\lambda} = \frac{\lambda}{1 - \lambda v}.$$

The reflected dr wave will arrive at the propagating crack tip at later time and will induce additional contribution on dynamic stress intensity factor. The induced stress intensity factor by dr wave can be obtained by setting $y = 0$ in (3.18) and the fundamental solution expressed in (2.8). The result written in the Laplace transform domain will have the following form:

$$\bar{K}^{dr}(s)=\frac{1}{2\pi i}\int_{\Gamma_\lambda}\frac{\tau_0\sqrt{b-\lambda}}{s\lambda\sqrt{b}(1-\lambda v)}\,e^{-s\lambda(2h+vt_f)}\left\{-\frac{\sqrt{2}\sqrt{1-bv}}{\sqrt{s}\alpha_{\ddagger}^{*}(\lambda)}\right\}d\lambda$$

$$=-\frac{\sqrt{2}\tau_0\sqrt{1-bv}}{\sqrt{b}s^{3/2}}\frac{1}{2\pi i}\int_{\Gamma_\lambda}\frac{\alpha_{\pm}^{*}(\lambda)}{\sqrt{1-\lambda v}\alpha_{\ddagger}^{*}(\lambda)\lambda}\,e^{-s\lambda(2h+vt_f)}d\lambda. \quad (3.19)$$

The inversion Laplace transform of (3.19) will become

$$K^{dr}(t)=\frac{-2\sqrt{2}\tau_0\sqrt{1-bv}}{\sqrt{b}\pi^{3/2}}\int_{t_d}^{t}\sqrt{t-\tau}\,\mathrm{Im}\left\{\frac{\alpha_{\pm}^{*}(\lambda^{+})}{\sqrt{1-\lambda^{+}v}\alpha_{\ddagger}^{*}(\lambda^{+})\lambda^{+}}\frac{\partial\lambda^{+}}{\partial\tau}\right\}_{t=\tau}d\tau$$

$$=\frac{2\sqrt{2}\tau_0\sqrt{1-bv}}{\sqrt{b}\pi^{3/2}}\int_{t_d}^{t}\frac{\sqrt{(t-\tau)[(1+bv)\tau-b(2h+vt_f)](2h+vt_f)}}{\tau\sqrt{(2h+vt_f-\tau v)}[b(2h+vt_f)+\tau(1-bv)]}d\tau, \quad (3.20)$$

where

$$\lambda^{+}=\frac{t}{2h+vt_f},$$

$$t_d=\frac{b(2h+vt_f)}{1+bv}.$$

The value of t_d denotes the time for a wave (d wave) generated from the stationary crack, reflected from the vertical boundary and arrives at the moving crack tip. In a similar way, the wave (g wave) generated from the propagating crack at time t_f as shown in (3.13) and (3.14) will also be reflected from the vertical free boundary. The reflected gr wave expressed in the Laplace transform domain for the moving coordinate system will have the following form:

$$\bar{\tau}_{yz}^{gr}(\xi,y,s)=\frac{-1}{2\pi i}\int_{\Gamma_\lambda}\frac{\tau_0\alpha_{\pm}^{*}(\lambda)}{s\sqrt{1-2\lambda v}}\times\left[\frac{1}{\lambda\sqrt{b}}+\frac{e^{-st_f}}{(\lambda-d)\sqrt{d}}\right]e^{-s[\alpha^{*}y-\lambda(\xi-2h)]}d\lambda, \quad (3.21)$$

$$\bar{\tau}_{xz}^{gr}(\xi,y,s)=\frac{1}{2\pi i}\int_{\Gamma_\lambda}\frac{\tau_0\lambda}{s\alpha_{\ddagger}^{*}(\lambda)\sqrt{1-2\lambda v}}\times\left[\frac{1}{\lambda\sqrt{b}}+\frac{e^{-st_f}}{(\lambda-d)\sqrt{d}}\right]e^{-s[\alpha^{*}y-\lambda(\xi-2h)]}d\lambda. \quad (3.22)$$

The reflected gr wave will also arrive at the propagating crack tip at some later time. The induced stress intensity factor by gr wave in the Laplace transform domain is

$$\bar{K}^{gr}(s)=\frac{1}{2\pi i}\int_{\Gamma_\lambda}\frac{\tau_0\alpha_{\pm}^{*}(\lambda)}{s\sqrt{1-2\lambda v}}\times\left[\frac{1}{\lambda\sqrt{b}}+\frac{e^{-st_f}}{(\lambda-d)\sqrt{d}}\right]e^{-2s\lambda h}\left\{-\frac{\sqrt{2}\sqrt{1-bv}}{\sqrt{s}\alpha_{\ddagger}^{*}(\lambda)}\right\}d\lambda. \quad (3.23)$$

The Laplace inverse transform of (3.23) in time domain can be expressed as follows:

where

$$t_p=\frac{2bh}{1+bv}.$$

For the numerical investigation, we consider a semi-infinite crack subjected to a uniformly distributed loading τ_0 at time $t=0$. At some delay time $t_f/bh=0.5$, the crack start to propagate out of the original semi-infinite crack with constant velocity toward the vertical free boundary and the loading is applied only on the original crack faces. The pattern of wave fronts and the position of the crack tip for $t>t_f$ is shown in Fig. 4. The dimensionless dynamic stress intensity factors (solid lines) for various value of crack propagation velocity are shown in Fig. 5, which account the contribution of reflected dr and gr waves on the calculation of stress intensity factor. The dot lines in Fig. 5 represent the solution for a semi-infinite crack propagating in an infinite medium and the difference of solid and dot lines is the influence due to the vertical free boundary. Where K_0 represents the dynamic stress intensity factor for the crack starts to propagate at time t_f. It shows clearly from this figure that the vertical free-boundary effect will increase the dynamic stress intensity factor due to the reflected waves (dr and gr waves) generated from the free boundary. However, the influence of the first few reflected waves on the dynamic stress intensity factor is very small.

4 Transient Analysis for Propagating Crack in a Strip

Consider an elastic strip in the region of the x, y-plane with $-\infty<x<\infty$, $-l<y<l$, contains a semi-infinite crack as shown in Fig. 2. At time $t=0$, a uniformly distributed antiplane dynamic loading with magnitude τ_0 is applied at the crack faces, the crack starts to propagate with constant velocity v at time t_f in the x-direction along the symmetry line $y=0$ of the strip. This transient problem will be solved in a similar way as we have proposed in the last section for analyzing the problem of the interaction of a propagating crack with a vertical boundary. In this problem, the plane wave induced from the uniformly

$$K^{gr}(t)=\frac{2\sqrt{2}\tau_0\sqrt{1-bv}}{\pi^{3/2}}\left\{\sqrt{\frac{h}{b}}\int_{t_p}^{t}\frac{\sqrt{(t-\tau)[(1+bv)\tau-2bh]}}{\tau\sqrt{v(bv-1)\tau^2+(1-3bv)h\tau+2bh^2}}d\tau\right.$$

$$\left.+\sqrt{\frac{h}{d}}\int_{t_f+t_p}^{t}\frac{\sqrt{(t-\tau)[(1+bv)(\tau-t_f)-2bh]}}{(\tau-2hd-t_f)\sqrt{v(bv-1)(\tau-t_f)^2+h(1-3bv)(\tau-t_f)+2bh^2}}d\tau\right\}, \quad (3.24)$$

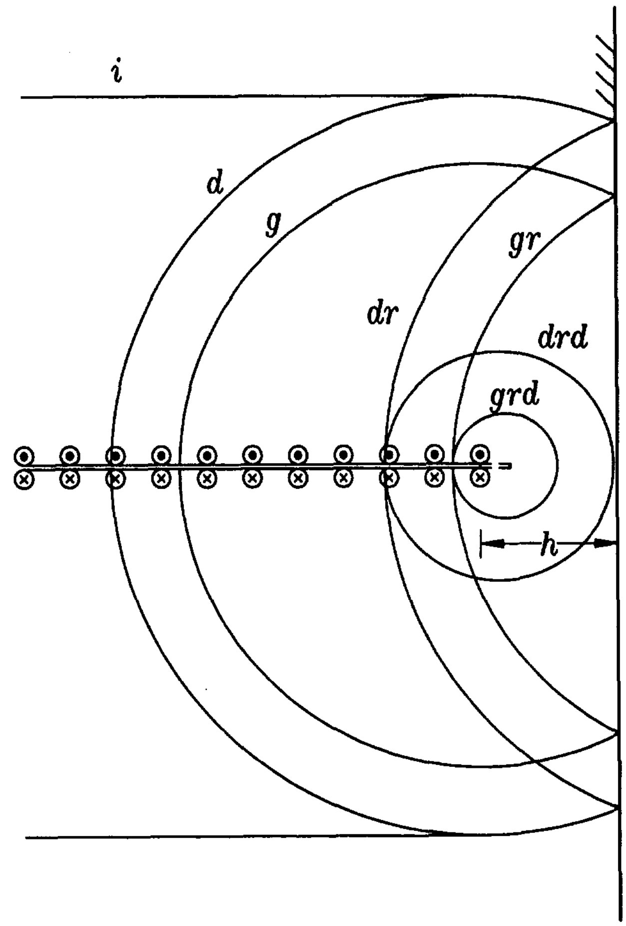

Fig. 4 Wave fronts of the incident, reflected and diffracted waves in the configuration with vertical boundary

distributed loading and the diffracted waves generated from the stationary crack tip and from the propagating crack tip will be reflected from horizontal boundaries, these waves will interact with the propagating crack at some later time. The pattern of wave fronts for $t > t_f$ is indicated in Fig. 5. The major difficulty in analyzing this problem will be the one that deals with the interaction of reflected waves with the propagating crack and the superposition technique of the fundamental solutions in the Laplace transform domain will be used. Before the time that the i, d, and g waves reflected from the horizontal boundary of the strip, the problem can be considered as a semi-infinite crack propagating in an unbounded medium and the solutions have been analyzed and presented in the previous section. In the following analysis, the solution is valid for the case that crack starts to propagating before the waves (i.e., i wave and d wave) generated from the stationary crack returns to the stationary crack (i.e., $t_f < 2bl$).

The diffracted d wave generated from the stationary crack tip will be reflected from the horizontal traction-free boundary to interact with the moving crack tip at later time. The solution of diffracted d wave in the Laplace transform domain is expressed in (3.2) and the reflected dr wave from the horizontal boundary can be constructed by applying the method of images, the result will be

$$\bar{\tau}^{dr}_{yz}(x, y, s) = \frac{-1}{2\pi i}\int_{\Gamma_\lambda} \frac{\tau_0\alpha_+(\lambda)}{s\lambda\sqrt{b}}\, e^{s[\alpha(y-2l)+\lambda x]}\, d\lambda. \quad (4.1)$$

If we express the solution in (4.1) in the moving coordinate $\xi-y$ system, where $\xi = x + vt_f - vt$, the result will become

$$\bar{\tau}^{dr}_{yz}(\xi, y, s) = -\frac{1}{2\pi i}\int_{\Gamma_\lambda} \frac{\tau_0\alpha_+(\bar{\lambda})}{s\bar{\lambda}\sqrt{b}(1-\lambda v)}\, e^{s[\alpha^*(y-2l)+\lambda(\xi - vt_f)]}\, d\lambda, \quad (4.2)$$

where

$$\bar{\lambda} = \frac{\lambda}{1-\lambda v}.$$

The induced dynamic stress intensity factor by dr wave can be obtained by setting $y = 0$ in (4.2) and the fundamental solution expressed in (2.8). The result for the stress intensity factor expressed in the Laplace transform domain will have the following form:

$$\bar{K}^{dr}(s) = -\frac{2\sqrt{2}\tau_0\sqrt{1-bv}}{\sqrt{b}s^{3/2}}\,\frac{1}{2\pi i}\int_{\Gamma_\lambda}\frac{1}{\lambda\sqrt{1-\lambda v}}\, e^{-s(2\alpha^* l+\lambda v t_f)}\, d\lambda$$

$$= -\frac{2\sqrt{2}\tau_0\sqrt{1-bv}}{\sqrt{b}s^{3/2}\pi} \times \int_{t_d}^{\infty} \mathrm{Im}\left\{\frac{1}{\lambda_d^+\sqrt{1-\lambda_d^+ v}}\,\frac{\partial\lambda_d^+}{\partial t}\right\} e^{-st}\, dt, \quad (4.3)$$

where

$$\lambda_d^+ = \frac{vt_f t - 4b^2 v l^2 + i2l\sqrt{t^2 - b^2[(vt - vt_f)^2 + 4l^2]}}{v^2 t_f^2 + 4l^2(1-b^2v^2)},$$

$$t_d = \frac{-b^2v^2 t_f + \sqrt{b^2v^2t_f^2 + 4l^2b^2(1-b^2v^2)}}{1-b^2v^2}.$$

The dynamic stress intensity factor expressed in time domain will be

$$K^{dr}(t) = -\frac{4\sqrt{2}\tau_0\sqrt{1-bv}}{\pi^{3/2}\sqrt{b}} \times \int_{t_d}^{t} \sqrt{t-\tau}\,\mathrm{Im}\left\{\frac{1}{\lambda_d^+\sqrt{1-\lambda_d^+ v}}\,\frac{\partial\lambda_d^+}{\partial\tau}\right\}_{t=\tau} d\tau. \quad (4.4)$$

The value of t_d denotes the time required for diffracted d wave generated from the stationary crack, reflected from the horizontal boundary and arrives at the moving crack tip.

Finally, we consider the contribution from the reflected gr wave which is originally generated from the propagating crack tip. The full-field analysis for this diffracted g wave has already been solved and discussed in the previous section. The solution expressed in the Laplace transform domain will be

$$\bar{\tau}^{g}_{yz}(\xi, y, s) = \frac{1}{2\pi i}\int_{\Gamma_\lambda}\frac{\tau_0\alpha_+^*(\lambda)}{s} \times \left[\frac{1}{\lambda\sqrt{b}} - \frac{e^{-st_f}}{(\lambda-d)\sqrt{d}}\right] e^{-s(\alpha^* y - \lambda\xi)}\, d\lambda. \quad (4.5)$$

The reflected gr wave generated from the horizontal boundary can be easily constructed from (4.5) by the image method and the result is

$$\bar{\tau}^{gr}_{yz}(\xi, y, s) = \frac{-1}{2\pi i}\int_{\Gamma_\lambda}\frac{\tau_0\alpha_+^*(\lambda)}{s} \times \left[\frac{1}{\lambda\sqrt{b}} - \frac{e^{-st_f}}{(\lambda-d)\sqrt{d}}\right] e^{s[\alpha^*(y-2l)+\lambda\xi]}\, d\lambda. \quad (4.6)$$

This reflected gr wave will interact with the propagating crack which will induce additional contribution on the dynamic stress intensity factor. The induced stress intensity factor by gr wave can be constructed by superimposing the fundamental solution in (2.8) and the stress distribution induced by gr wave

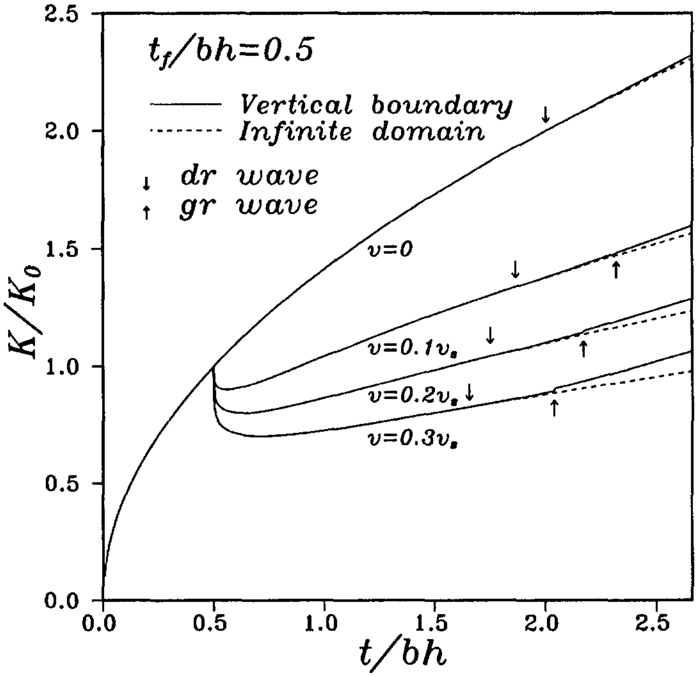

Fig. 5 Transient response of the dynamic stress intensity factor for a propagating crack in the configuration with vertical boundary

on the propagating crack face. The result is obtained in the Laplace transform domain as follows:

$$\bar{K}^{gr}(s) = -\frac{2\sqrt{2}\tau_0\sqrt{1-bv}}{s^{3/2}}\frac{1}{2\pi i}$$

$$\times\int_{\Gamma_\lambda}\left[\frac{1}{\lambda\sqrt{b}} - \frac{e^{-st_f}}{(\lambda-d)\sqrt{d}}\right]e^{-2s\alpha^* l}d\lambda. \quad (4.7)$$

We deform the path of integration to a path along which the integral can be recognized as a one-sided Laplace transform. The desired path of integration in the λ-plane is obviously defined by the equation

$$2l\alpha^* = 2l(b^2 - \lambda^2 + b^2v^2\lambda^2 - 2b^2v\lambda)^{1/2} = t. \quad (4.8)$$

The foregoing equation can be solved for λ to yield

$$\lambda_p^+ = \frac{-2b^2vl + i\sqrt{(1-b^2v^2)t^2 - 4l^2b^2}}{2l(1-b^2v^2)}.$$

Along this new contour, the variable t is set as the new variable, which leads to

$$\bar{K}^{gr}(s) = -\frac{2\sqrt{2}\tau_0\sqrt{1-bv}}{\pi}\left\{\frac{1}{s^{3/2}}\int_{t_p}^{\infty}\mathrm{Im}\left[\frac{1}{\lambda_p^+\sqrt{b}}\frac{\partial\lambda_p^+}{\partial t}\right]e^{-st}dt - \frac{e^{-st_f}}{s^{3/2}}\int_{t_p}^{\infty}\mathrm{Im}\left[\frac{1}{(\lambda_p^+ - d)\sqrt{d}}\frac{\partial\lambda_p^+}{\partial t}\right]e^{-st}dt\right\}, \quad (4.9)$$

where

$$t_p = \frac{2lb}{\sqrt{1-b^2v^2}}.$$

The inverse Laplace transform of (4.9) to time domain can be expressed as

$$K^{gr}(t) = -\frac{4\sqrt{2}\tau_0\sqrt{1-bv}}{\pi^{3/2}}\left\{\int_{t_p}^{t}\sqrt{t-\tau}\,\mathrm{Im}\left[\frac{1}{\lambda_p^+\sqrt{b}}\frac{\partial\lambda_p^+}{\partial\tau}\right]_{t=\tau}d\tau - \int_{t_p}^{t-t_f}\sqrt{t-t_f-\tau}\,\mathrm{Im}\left[\frac{1}{(\lambda_p^+-d)\sqrt{d}}\frac{\partial\lambda_p^+}{\partial\tau}\right]_{t=\tau}d\tau\right\}$$

$$= \frac{8\sqrt{2}\tau_0\sqrt{1-bv}lv}{\pi^{3/2}}\left\{\sqrt{b^3}\int_{t_p}^{t}\frac{\tau\sqrt{(t-\tau)[(1-b^2v^2)\tau^2-4l^2b^2]}}{[(1-b^2v^2)\tau^2-4l^2b^2](\tau^2-4b^2l^2)}d\tau - \frac{1}{\sqrt{d}}\int_{t_p}^{t-t_f}\frac{\tau\sqrt{(t-t_f-\tau)[(1-b^2v^2)\tau^2-4l^2b^2]}}{[(1-b^2v^2)\tau^2-4l^2b^2](v^2\tau^2+4l^2)}d\tau\right\}. \quad (4.10)$$

The value of t_p denotes the time needed for the diffracted g wave generated from the propagating crack at time t_f, reflected from the horizontal boundary and reach the propagating crack tip.

For the numerical calculation of the transient stress intensity factor, we consider a semi-infinite crack subjected to a uniformly distributed loading τ_0 on the stationary crack faces at time $t = 0$. At nondimensional delay time $t_f/bl = 0.5$, the crack start to propagate with a constant speed from the semi-infinite crack and the dynamic loading is applied only on the original crack faces. The pattern for the incident, reflected and diffracted wave fronts for $t > t_f$ is shown in Fig. 6. The dynamic stress intensity factors for stationary crack and for various value of crack propagation velocity

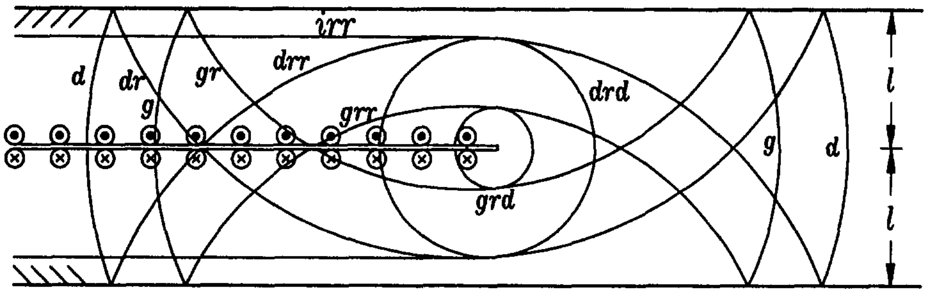

Fig. 6 Wave fronts of the incident, reflected and diffracted waves for a propagating crack in a finite strip

are shown in Fig. 7, which account the contribution from the reflected *dr* and *gr* waves after the crack grows. The stress intensity factor is normalized by K_0 which is the stress intensity factor at $t/bl = 0.5$. The dot lines in Fig. 7 represent the solution for a semi-infinite crack propagating in an infinite medium, and the influence of the horizontal boundary on the stress intensity factor can be evaluated from the difference of solid and dot lines. It also indicates that the reflected waves will increase the value of the dynamic stress intensity factor and the influence is much larger than the case of the vertical boundary discussed in the previous section.

5 Conclusions

The phenomena of crack propagation, arrest, and branching are important subjects in the areas of dynamic fracture analysis. The interaction of reflected waves with the moving crack subjected to dynamic loading had only been discussed in experimental works. Experimental results indicated that the reflected waves dominate the stability of crack propagation. It is very important to have the analytical results to investigate this important event. But it seems difficult to obtain the analytical solutions by using the well-known conventional method.

We have proposed a powerful superposition methodology and a useful fundamental solution is constructed in this study. The fundamental solution is the problem of applying an exponentially distributed traction on the propagating crack face and the solution is determined by superposition of the fundamental solution in the Laplace transform domain. The dynamic crack propagation with constant velocity in a configuration with boundary is investigated. The orientations of crack face respected to the boundary of half-plane are horizontal and vertical types. We only focus our attention on the interaction of first two reflected waves from the boundary with the moving crack. An explicit result of the dynamic stress intensity factor is obtained in closed form and numerical results are evaluated in detail. The numerical results show that the stress intensity factors induced by reflected waves from horizontal boundary are more significant than that of vertical boundary. When reflected waves generated from horizontal boundary return to the moving crack tip, the stress intensity factor will generally increase rapidly.

There still have many unanswered questions in dynamic fracture and this work may provide a useful technique for further investigation in more complicated dynamic fracture problems especially on the crack propagation event. The proposed method in this study has already been extended to solve more difficult

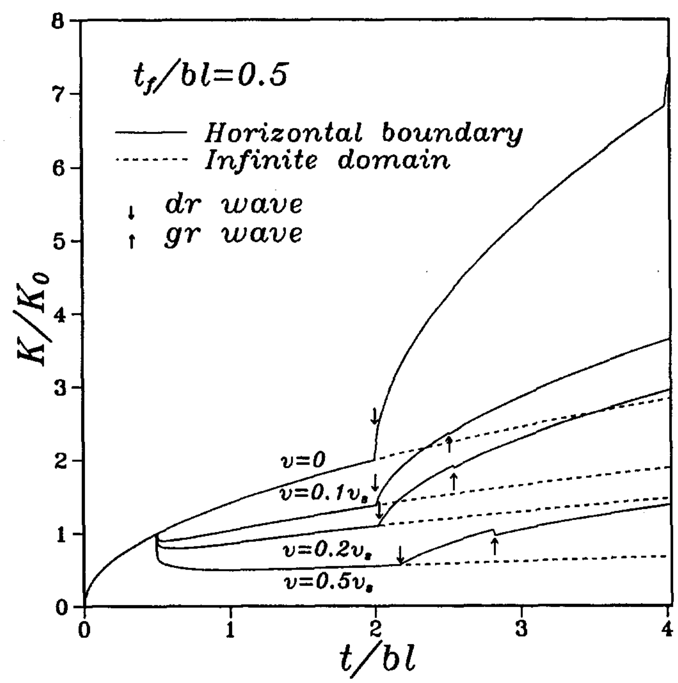

Fig. 7 Transient response of the dynamic stress intensity factor for a propagating crack in a finite strip

inplane problem of crack propagation with boundary effect, the results will be shown in a future paper.

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