(a, b, c) space which divides the surface of reflection into two parts, say T and T' of which T is confined between L and Γ_3.

For other values of c, i.e., $c \geq 1/3$, we get a similar surface of reflection whose section AD by the plane $c = 1$ is shown in Fig. 2. Actually this is the continuation of the previous surface of reflection. But in this case, T is absent because the point B_0 approaches to the line L as c increases from 0 to 1/3. It is numerically verified from the equations representing the locus of B_0.

The above descriptions give the clear idea of the surface of reflection for all values of c.

So for the points (a, b, c), where $c < 1/3$, between the axis of b and T the asymptotic value of I_{15} is given by

$$I_{15} \simeq I \text{ as } t \to \infty \text{ where } I = -\frac{\pi i f_-(\lambda_6)}{[\sigma'(\lambda_6) + b]} e^{i[\{\sigma(\lambda_6) - b\lambda_6\}t - \lambda_6 x]}.$$

Also, for the points (a, b, c), where $c < 1/3$, between T and T', $I_{15} \simeq -I$ as $t \to \infty$. For the points (a, b, c) where $c < 1/3$ in the three-dimensional region bounded by the plane passing through B_0 and parallel to the $(b - c)$ plane, the $(a - c)$ plane and the surface T', $I_{15} \simeq I$ as $t \to \infty$.

Above the plane $c = 1/3$, $I_{15} \simeq \pm I$ as $t \to \infty$ according as the points (a, b, c) lie on the left or right to the surface of reflection. For the points (a, b, c) on the surface T excepting the space curve Γ_3 and on T'

$$I_{15} \simeq \frac{\pi i f_-(\lambda_6)}{2b} e^{i[\{\sigma(\lambda_6) - b\lambda_6\}t - \lambda_6 x]} \text{ as } t \to \infty.$$

Also for the points (a, b, c) on the space curve Γ_3, the asymptotic value of I_{15} is given by

$$I_{15} \simeq \frac{\pi i f_-(\lambda_6)}{6b} e^{i[\{\sigma(\lambda_6) - b\lambda_6\}t - \lambda_6 x]} \text{ as } t \to \infty.$$

From the above calculations it follows that for (a, b, c), on the critical surfaces, the asymptotic values of the integrals become unbounded and the order of the unboundedness is like $t^{1/2}$ where two roots coincide and like $t^{2/3}$ where three roots coincide. Now for (a, b, c) in a region R_n, the asymptotic values of the integrals are bounded leading to waves of constant amplitude. In the following we write down the waves. At first we write down the waves for (a, b, c) in the region R_3:

$$\left.\begin{aligned} \eta &= \eta_1 + \eta_3 + \eta_5 + \eta_r \text{ as } x \to \infty \text{ and } t \to \infty \\ &= \eta_2 + \eta_4 + \eta_6 + \eta_r \text{ as } x \to -\infty \text{ and } t \to \infty \end{aligned}\right\} \quad (8)$$

where

$$\eta_1 = H_+(\lambda_1) e^{i(at - \lambda_1 x)}, \; \eta_3 = H_+(\lambda_3) e^{i(at - \lambda_3 x)},$$

$$\eta_5 = -H_-(\lambda_5) e^{i(at + \lambda_5 x)}$$

$$\eta_2 = -H_+(\lambda_2) e^{i(at - \lambda_2 x)}, \; \eta_4 = H_-(\lambda_4) e^{i(at + \lambda_4 x)},$$

$$\eta_6 = G(\lambda_6) e^{i(at + \lambda_6 x)}$$

$$\eta_r = G(\lambda_6) e^{i[(a - 2b\lambda_6)t - \lambda_6 x]} \text{ and } H_\pm = \frac{\text{i}f_\pm(\lambda)}{2\rho g h[\sigma'(\lambda) - b]},$$

$$G(\lambda) = \frac{\text{i}f_-(\lambda)}{2\rho g h[\sigma'(\lambda) + b]}.$$

The wave system for the case when the values of the parameters a, b, c of the problem are such that the point (a, b, c) lies in the other regions is easy to determine. This is the same wave system as expressed in (8), only the wave corresponding to a pole not occurring in a region being deleted for that region.

The waves η_n ($n = 1$ to 6) are the original waves created by the source as found in the unbounded fluid Pramanik and Majumdar (1984). The wave η_r is an addition to this system due to the existence of the cliff. It is obviously seen that η_r is the reflection of the wave η_6 on the cliff. Among the six waves generated by the source, only the wave η_6 moves towards the cliff. So it must be reflected on reaching the cliff. Now the condition by which it reaches the cliff is obvious and its group velocity is greater than the velocity of source. One can verify that this is the same condition that the points (a, b, c) to the left of the surface of reflection. Thus, occurrence of reflection is physically reasonable. However, it is seen that the amplitude of the reflected wave is the same as the original wave excepting (a, b, c) on the curve Γ_3 where the amplitude of the wave η_r is reduced than the original wave. This seems to be a striking result. It is to be noted that we have dealt with the linear theory in an inviscid fluid. There is no obvious reason for the reduction of the amplitude in the reflected wave η_r.

References

Akylas, T. R., 1984, "On the Excitation of Nonlinear Water Waves by a Moving Pressure Distribution Oscillating at Resonant Frequency," *Phys. Fluids*, Vol. 27, pp. 2803–2807.

Pramanik, A. K., and Majumdar, S. R., 1984, "Small-amplitude Free-Surface Waves Generated by Moving Oscillatory Disturbances," *J. Fluid Mech.*, Vol. 145, pp. 405–415.

Pramanik, A. K., and Majumdar, S. R., 1988, "The Effect of Surface Tension on the Reflection of Gravity Waves from a Vertical Cliff," *Int. J. Engineering Science*, Vol. 26, pp. 549–558.

Analysis of the Interfacial Crack for Anisotropic Materials Under Displacement-Displacement or Traction-Displacement Boundary Conditions

Chien-Ching Ma[6,8] and Jyi-Jiin Luo[7,8]

Introduction

Many engineering structures are comprised of more than one material. The strength of composite materials is influenced by the orientation of existing cracks with respect to the bimaterial interface. A number of solutions for the stress and displacement fields for a crack lying along bimaterial interfaces have been obtained for isotropic materials by Williams (1959) and Rice and Sih (1965). Extensions to anisotropic elasticity have been made by Bogy (1972) and recently by Ting (1986, 1990). All these studies of in-plane problems have shown that the stresses share the inverse square root singularity of the crack and, in addition, exhibit an oscillatory behavior as the crack tip is approached. Recently, Ma and Hour (1989, 1990) investigated the antiplane problems of two dissimilar anisotropic wedges and an inclined crack terminating at a bimaterial interface. They found that the order of the stress singularity is always real for the antiplane anisotropic problems.

[6]Professor.

[7]Graduate Student.

[8]Department of Mechanical Engineering, National Taiwan University, Taipei, Taiwan 10763, Republic of China.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Oct. 28, 1991; final revision, June 3, 1992. Associate Technical Editor: C. F. Shih.

In this study, plane problems for bonded dissimilar half-planes of anisotropic material containing an interfacial crack are considered. The solutions obtained in this paper is valid only for anisotropic bimaterial having monoclinic symmetry with the axis of symmetry being the x_3-axis. Here the problem of displacement prescribed on both crack faces, and the problem of traction prescribed on one face with displacement prescribed on the other, is solved. The problem is solved by application of a generalized Mellin transform in conjunction with the complex stress function. The dependence of the order of the stress singularity on the material constants and boundary conditions is studied in detail. The result shows that the order of stress singularity has reduced dependence on material constants. The full-field solutions in the Mellin transform domain are obtained explicitly. It is very interesting to find that the solutions of the displacement prescribed problems can be obtained from the traction prescribed problems by a simple substitution.

Explicit Solutions in Mellin Transform Domain

The two-dimensional stress-strain relations for a homogeneous anisotropic body are

$$e_{\alpha\beta} = s_{\alpha\beta}^{\gamma\delta}\sigma_{\gamma\delta}. \tag{1}$$

Because of assumed elastic symmetry about $x_3 = 0$ for the plane problem, the six independent material constants are s_{11}^{11}, s_{22}^{11}, s_{12}^{11}, s_{12}^{12}, s_{22}^{12}, s_{22}^{22}. The solution of displacement for the two-dimensional problem has the following form in terms of complex potentials

$$u_r + iu_\theta = e^{-i\theta}\sum_{\alpha=1,2}\{\delta_\alpha\Omega_\alpha'(z_\alpha) + \rho_\alpha\overline{\Omega}_\alpha'(\overline{z}_\alpha)\}, \tag{2}$$

where Ω_α ($\alpha = 1, 2$) are arbitrary analytic functions of the complex variable z_α, $\overline{\Omega}_\alpha$ is complex conjugate and primes denote derivatives with respect to the indicated arguments. The relation between z and z_α is $z = re^{i\theta}$, $z_\alpha = z + \gamma_\alpha\overline{z}$. The complex constants δ_α, ρ_α, and γ_α are defined in terms of the components of the elasticity tensor $s_{\alpha\beta}^{\gamma\delta}$; see Bogy (1972) or Ma and Luo (1992). We now take Mellin transform of r^2 and r times the stress and displacement, respectively.

$$\hat{\sigma}_{\alpha\beta}(s,\theta) = \int_0^\infty \sigma_{\alpha\beta}(r,\theta)r^{s+1}dr, \tag{3}$$

$$\hat{u}_\alpha(s,\theta) = \int_0^\infty u_\alpha(r,\theta)r^s dr, \tag{4}$$

where s is the complex transform parameter. The physical stress and displacement fields are recovered as follows:

$$\sigma_{\alpha\beta}(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{\sigma}_{\alpha\beta}(s,\theta)r^{-s-2}ds, \tag{5}$$

$$u_\alpha(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{u}_\alpha(s,\theta)r^{-s-1}ds, \tag{6}$$

where $\mathrm{Re}(s) = c$ defines the path of integration. The choice of c has to be determined by the regularity of their integrands. Direct use of the transforms with the complex representation of the solution leads to (Bogy, 1972)

$$\hat{u}_r(s,\theta) + i\hat{u}_\theta(s,\theta) = e^{-i\theta}\sum_{\alpha=1,2} \times\left\{\frac{\delta_\alpha\phi_\alpha(s)}{(e^{i\theta}+\gamma_\alpha e^{-i\theta})^{s+1}} + \frac{\rho_\alpha\overline{\phi}_\alpha(s)}{(e^{-i\theta}+\overline{\gamma}_\alpha e^{i\theta})^{s+1}}\right\}, \tag{7}$$

$$\hat{\sigma}_{\theta\theta}(s,\theta) - i\hat{\sigma}_{r\theta}(s,\theta) = -2(s+1)e^{-i\theta}\sum_{\alpha=1,2} \times\left\{\frac{\gamma_\alpha\phi_\alpha(s)}{(e^{i\theta}+\gamma_\alpha e^{-i\theta})^{s+1}} + \frac{\overline{\phi}_\alpha(s)}{(e^{-i\theta}+\overline{\gamma}_\alpha e^{i\theta})^{s+1}}\right\}, \tag{8}$$

where $\phi_\alpha(s)$ is defined as

$$\phi_\alpha(s) = \int_{0(\theta)}^\infty \Omega_\alpha'(z_\alpha)z_\alpha^s dz_\alpha. \tag{9}$$

For convenience, define

$$H_\alpha(s,\theta) = (e^{i\theta}+\gamma_\alpha e^{-i\theta})^{-s-1}, \tag{10}$$

$$T(s,\theta) = [\hat{\sigma}_{\theta\theta}(s,\theta) - i\hat{\sigma}_{r\theta}(s,\theta)]/2(s+1), \tag{11}$$

$$D(s,\theta) = \hat{u}_r(s,\theta) + i\hat{u}_\theta(s,\theta). \tag{12}$$

Then (7) and (8) can be rewritten as

$$D(s,\theta) = e^{i\theta}\sum_{\alpha=1,2}\{\delta_\alpha H_\alpha(s,\theta)\phi_\alpha(s) + \rho_\alpha\overline{H}_\alpha(s,\theta)\overline{\phi}_\alpha(s)\}, \tag{13}$$

$$T(s,\theta) = -e^{-i\theta}\sum_{\alpha=1,2}\{\gamma_\alpha H_\alpha(s,\theta)\phi_\alpha(s) + \overline{H}_\alpha(s,\theta)\overline{\phi}_\alpha(s)\}. \tag{14}$$

We consider an anisotropic bimaterial interface crack, subjected to prescribed displacements at the crack faces $\theta = \pm\pi$ as shown in Fig. 1. Perfect bonding conditions along the interface $\theta = 0$ are ensured by the stress and displacement continuity conditions. It is very interesting to find that the form of solutions for the displacement prescribed problems are very similar to that of the traction prescribed problems solved by Ma and Luo (1992). For convenience, we define the following material constants:

$$\gamma_{d\alpha} = \delta_\alpha/\overline{\rho}_\alpha, \quad \delta_{d\alpha} = \gamma_\alpha/\overline{\rho}_\alpha, \quad \rho_{d\alpha} = 1/\rho_\alpha, \tag{15}$$

and

$$\lambda_d = \frac{\gamma_{d2}\delta_{d1} - \gamma_{d1}\delta_{d2}}{\gamma_{d1} - \gamma_{d2}} = \frac{\rho_{d1} - \rho_{d2}}{\overline{\gamma}_{d1} - \overline{\gamma}_{d2}}, \tag{16}$$

$$\eta_d = \overline{\eta}_d = \frac{\delta_{d1} - \delta_{d2}}{\gamma_{d1} - \gamma_{d2}}, \tag{17}$$

$$\xi_d = \overline{\xi}_d = \frac{\gamma_{d2}\overline{\rho}_{d1} - \gamma_{d1}\overline{\rho}_{d2}}{\gamma_{d1} - \gamma_{d2}}. \tag{18}$$

The subscript d indicates the displacement prescribed problems. It can be proved that η_d and ξ_d are all real values and we also find the following relations:

$$\lambda_d = \lambda\epsilon, \quad \xi_d = \eta\epsilon, \quad \eta_d = \xi\epsilon, \tag{19}$$

where

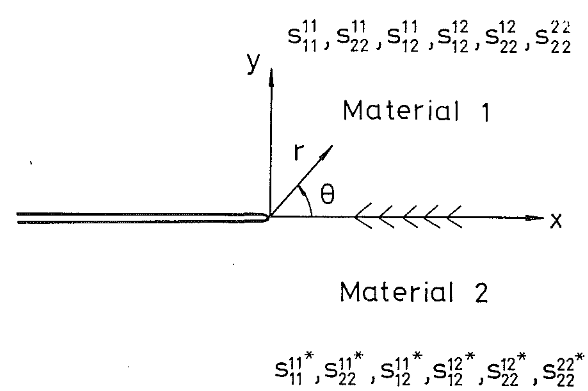

Fig. 1 Configuration of bonded anisotropic dissimilar interface crack

$$\epsilon=\bar{\epsilon}=\frac{1}{\eta\xi-|\lambda|^2}, \quad (20)$$

$$\lambda=\frac{\gamma_2\delta_1-\gamma_1\delta_2}{\gamma_1-\gamma_2}=\frac{\rho_1-\rho_2}{\bar{\gamma}_1-\bar{\gamma}_2}=\frac{S_{22}^{11}}{\gamma_1\gamma_2}[\gamma_1+\gamma_2-\gamma_1\gamma_2(\bar{\gamma}_1+\bar{\gamma}_2)], \quad (21)$$

$$\eta=\bar{\eta}=\frac{\delta_1-\delta_2}{\gamma_1-\gamma_2}=-2\left(S_{11}^{11}-\frac{S_{22}^{11}}{\gamma_1\gamma_2}\right), \quad (22)$$

$$\xi=\bar{\xi}=\frac{\gamma_2\bar{\rho}_1-\gamma_1\bar{\rho}_2}{\gamma_1-\gamma_2}=2(S_{11}^{11}-S_{22}^{11}\bar{\gamma}_1\bar{\gamma}_2). \quad (23)$$

The material constants λ, η, ξ are obtained for the traction prescribed problems analyzed by Ma and Luo (1992). It is very interesting to find that the solutions of the displacement prescribed problems can be obtained from the traction prescribed problems if we perform the following substitution:

Traction	ϕ_α	$T(s)$	G	H	U_α	V_α	γ_α	l	m	η	ξ	λ
Displacement	$\rho_\alpha\phi_\alpha$	$-D(s)$	G_d	H_d	$U_{d\alpha}$	$V_{d\alpha}$	$\gamma_{d\alpha}$	l_d	m_d	η_d	ξ_d	λ_d.

The solutions of the displacement prescribed problem in the Mellin transform domain can be expressed as follows:

$$\bar{\rho}_\alpha\phi_\alpha(s)H_\alpha(s,0)=\frac{p(G_dV_{d\alpha}-H_dU_{d\alpha})}{(-1)^\alpha(\gamma_{d2}-\gamma_{d1})(1-p^2)(l_dp^4+m_dp^2+l_d)}, \quad (24)$$

where

$$p=e^{-i(s+1)\pi}, \quad (25)$$

$$l_d=|\lambda_d+\lambda_d^*|^2-(\eta_d+\xi_d^*)(\xi_d+\eta_d^*), \quad (26)$$

$$m_d=2|\lambda_d+\lambda_d^*|^2-(\eta_d+\xi_d^*)^2-(\xi_d+\eta_d^*)^2, \quad (27)$$

$$\begin{aligned}G_d=&-D(s)[\xi_d-\xi_d^*-(\xi_d+\eta_d^*)p^2]\\&-\bar{D}(s)[-\lambda_d+\lambda_d^*+(\lambda_d+\lambda_d^*)p^2]\\&-D^*(s)(\eta_d^*+\xi_d^*)-\bar{D}^*(s)(-2\lambda_d^*),\end{aligned} \quad (28)$$

$$\begin{aligned}H_d=&-D(s)[\bar{\lambda}_d-\bar{\lambda}_d^*-(\bar{\lambda}_d+\bar{\lambda}_d^*)p^2]\\&-\bar{D}(s)[-\eta_d+\eta_d^*+(\eta_d+\xi_d^*)p^2]\\&-D^*(s)(2\bar{\lambda}_d^*)-\bar{D}^*(s)(-\eta_d^*-\xi_d^*),\end{aligned} \quad (29)$$

$$U_{d\alpha}=\sum_{\beta=1,2}(1-\delta_{\alpha\beta})\gamma_{d\beta}[\eta_d+\xi_d^*+(\xi_d+\eta_d^*)p^2]-(\lambda_d+\lambda_d^*)(1+p^2), \quad (30)$$

$$V_{d\alpha}=\sum_{\beta=1,2}(1-\delta_{\alpha\beta})\gamma_{d\beta}(\bar{\lambda}_d+\bar{\lambda}_d^*)(1+p^2)-[\xi_d+\eta_d^*+(\eta_d+\xi_d^*)p^2], \quad (31)$$

and $\delta_{\alpha\beta}$ is the Kronecker delta. The expressions for $\hat{u}_\alpha$ and $\hat{\sigma}_{\alpha\beta}$ now follow directly from the substitution of (24) into (7)–(8). This completes the formal solution for the transforms of the stress and displacement components. The location of the zeros of the characteristic function $(1-p^2)(l_dp^4+m_dp^2+l_d)=0$ is found to be

$$s=n, \quad (32)$$

or

$$s=n-\frac{1}{2}\pm i\beta_d \quad \text{if} \quad m_d/2l_d\geq 1; \quad (33)$$

$$s=n\pm i\beta_d \quad \text{if} \quad m_d/2l_d\leq -1; \quad (34)$$

$$s=n\pm\sigma \quad \text{if} \quad |m_d/2l_d|<1, \quad (35)$$

where

$$\beta_d=\frac{1}{2\pi}\cosh^{-1}\left|\frac{m_d}{2l_d}\right|=\frac{1}{2\pi}\ln\left|\frac{1+\kappa_d}{1-\kappa_d}\right|, \quad (36)$$

$$\sigma=\frac{1}{2\pi}\cos^{-1}\left(\frac{-m_d}{2l_d}\right), \quad (37)$$

$$\kappa_d=\sqrt{\frac{m_d-2l_d}{m_d+2l_d}}, \quad (38)$$

and n is an integer number. From the condition of the positive definite for the material constant it can be shown numerically that $m_d/2l_d\geq 1$. The similar results as shown in (32) and (33) are also obtained by Ting (1986). It is shown that β_d can be expressed in another form,

$$\beta_d=\frac{1}{\pi}\tanh^{-1}\frac{\eta_d-\xi_d-\eta_d^*+\xi_d^*}{\sqrt{(\eta_d+\xi_d+\eta_d^*+\xi_d^*)^2-4|\lambda_d+\lambda_d^*|^2}}. \quad (39)$$

The order of the power-type stress singularity is $\lambda=s_1+1$, where s_1 denotes the zero of the characteristic function with the largest value in the open strip $-2<\mathrm{Re}(s)<-1$. The order of the stress singularity λ is a complex number and the stress fields are oscillatory in the limit $r\to 0$. The magnitude of the oscillation is depend on the value β_d which is expressed in (36) and depends only on one material parameter κ_d. There are combinations of the material constants that will have the square root singularity, i.e., $\beta_d=0$, should satisfy the following equation:

$$\eta_d-\xi_d=\eta_d^*-\xi_d^*. \quad (40)$$

Homogeneous materials obviously satisfy Eq. (40). For the isotropic case, we have $\eta_d=\mu/(m-1)$, $\xi_d=\mu$ and $\lambda_d=0$. Equation (39) is reduced to

$$\beta_d=\frac{1}{\pi}\tanh^{-1}\frac{\mu(m-2)(m^*-1)-\mu^*(m^*-2)(m-1)}{\mu m(m^*-1)+\mu^*m^*(m-1)}, \quad (41)$$

which is in agreement with the result obtained by Ma and Wu (1990). The largest value of β_d in (41) is $(\ln\sqrt{3})/\pi(\approx 0.175)$, the same as the traction-prescribed boundary conditions.

Next, we consider the interfacial crack problem with the boundary conditions of traction prescribed along one crack face while displacement prescribed on the other crack face. Thus we consider the following boundary conditions:

$$\sum_{\alpha=1,2}\{\gamma_\alpha H_\alpha(s,\pi)\phi_\alpha(s)+\bar{H}_\alpha(s,\pi)\bar{\phi}_\alpha(s)\}=T(s), \quad (42)$$

$$\sum_{\alpha=1,2}\{\delta_\alpha^*H_\alpha^*(s,-\pi)\phi_\alpha^*(s)+\rho_\alpha^*\bar{H}_\alpha^*(s,-\pi)\bar{\phi}_\alpha^*(s)\}=-D^*(s). \quad (43)$$

By using the Cramer's rule and after some algebraic simplifications, we get

$$\phi_\alpha(s)H_\alpha(s,0)=\frac{pQ_\alpha}{(-1)^\alpha(\gamma_2-\gamma_1)Q}, \quad (44)$$

in which Q and Q_α are obtained from the determinant of eight-by-eight matrix. The characteristic equation Q, which presents the dependence of the stress singularity on material constants, is reduced to an explicit simple form as

$$Q=\begin{vmatrix}\eta+\xi p^2 & -\lambda(1+p^2) & 0 & 1-p^2\\ \bar{\lambda}(1+p^2) & -\xi-\eta p^2 & 1-p^2 & 0\\ 1-p^2 & 0 & \lambda_d^*(1+p^2) & -\xi_d^*-\eta_d^*p^2\\ 0 & 1-p^2 & \eta_d^*+\xi_d^*p^2 & -\lambda_d^*(1+p^2)\end{vmatrix}$$

$$
\begin{aligned}
&= (1+p^2)^2\{[\lambda(\eta_d^* + \xi_d^* p^2) - \lambda_d^*(\eta + \xi p^2)\}\\
&\quad \times [\bar{\lambda}(\xi_d^* + \eta_d^* p^2) - \bar{\lambda}_d^*(\xi + \eta p^2)]\}\\
&\quad - \{(1-p^2)^2 + (\xi + \eta p^2)(\eta_d^* + \xi_d^* p^2) - \bar{\lambda}\lambda_d^*(1+p^2)^2\}\\
&\quad \bullet \{(1-p^2)^2 + (\eta + \xi p^2)(\xi_d^* + \eta_d^* p^2) - \lambda\bar{\lambda}_d^*(1+p^2)^2\}. \qquad (45)
\end{aligned}
$$

But Q_α can only be reduced to the determinant of a four-by-four matrix. The results are:

$$
Q_\alpha = \begin{vmatrix}
\xi T(s) - \lambda \bar{T}(s) - D^*(s) & -\lambda(1+p^2) + \Sigma_\beta(1-\delta_{\alpha\beta})\gamma_\beta(\eta + \xi p^2) & 0 & 1-p^2 \\
\bar{\lambda} T(s) - \eta \bar{T}(s) - \bar{D}^*(s) & -\xi - \eta p^2 + \Sigma_\beta(1-\delta_{\alpha\beta})\gamma_\beta \bar{\lambda}(1+p^2) & 1-p^2 & 0 \\
-T(s) - \eta_d^* D^*(s) + \lambda_d^* \bar{D}^*(s) & \Sigma_\beta(1-\delta_{\alpha\beta})\gamma_\beta(1-p^2) & \lambda_d^*(1+p^2) & -\xi_d^* - \eta_d^* p^2 \\
-\bar{T}(s) - \bar{\lambda}_d^* D^*(s) + \xi_d^* \bar{D}^*(s) & 1-p^2 & \eta_d^* + \xi_d^* p^2 & -\bar{\lambda}_d^*(1+p^2)
\end{vmatrix}. \qquad (46)
$$

All the zeros of (45) can be obtained explicitly as shown in the following form:

$$s = n - \frac{1}{2} + \alpha \pm i\beta, \quad n - \frac{1}{2} - \alpha \pm i\beta,$$

$$\text{if } |q^+ - q^- + 16q_o| \le 2\sqrt{q^+ q^-}; \qquad (47)$$

$$s = n - \frac{1}{2} + \alpha \pm \sigma, \quad n - \frac{1}{2} - \alpha \pm \sigma,$$

$$\text{if } \sqrt{q^+} + \sqrt{q^-} < \sqrt{16q_o}; \qquad (48)$$

$$s = n - \frac{1}{2} + i(\beta \pm \tau), \quad n - \frac{1}{2} - i(\beta \pm \tau),$$

$$\text{if } \sqrt{q^+} - \sqrt{q^-} > \sqrt{16q_o}; \qquad (49)$$

$$s = n + i(\beta \pm \tau), \quad n - i(\beta \pm \tau),$$

$$\text{if } \sqrt{q^+} - \sqrt{q^-} < -\sqrt{16q_o}; \qquad (50)$$

$$s = n - \frac{1}{2} \pm i(\vartheta - \varsigma), \quad n \pm i(\vartheta + \varsigma), \quad \text{if } q_o < 0, \qquad (51)$$

where

$$\alpha = \frac{1}{2\pi}\cos^{-1}(\sqrt{q^+/16q_o} - \sqrt{q^-/16q_o}),$$

$$\beta = \frac{1}{2\pi}\cosh^{-1}(\sqrt{q^+/16q_o} + \sqrt{q^-/16q_o}),$$

$$\sigma = \frac{1}{2\pi}\cos^{-1}(\sqrt{q^+/16q_o} + \sqrt{q^-/16q_o}),$$

$$\tau = \frac{1}{2\pi}\cosh^{-1}|\sqrt{q^+/16q_o} - \sqrt{q^-/16q_o}|,$$

$$\vartheta = \frac{1}{2\pi}\sinh^{-1}(\sqrt{-q^+/16q_o} + \sqrt{-q^-/16q_o}),$$

$$\varsigma = \frac{1}{2\pi}\sinh^{-1}(\sqrt{-q^+/16q_o} - \sqrt{-q^-/16q_o}),$$

$$
\begin{aligned}
q_o = -Q|_{p^2=0} &= (\eta\xi - |\lambda|^2)(\eta_d^*\xi_d^* - |\lambda_d^*|^2)\\
&\quad + \eta\xi_d^* + \xi\eta_d^* - \bar{\lambda}\lambda_d^* - \lambda\bar{\lambda}_d^* + 1, \qquad (52)
\end{aligned}
$$

$$q^+ = -Q|_{p^2=1} = [(\eta + \xi)^2 - 4|\lambda|^2][(\eta_d^* + \xi_d^*)^2 - 4|\lambda_d^*|^2], \qquad (53)$$

$$q^- = -Q|_{p^2=-1} = [(\eta - \xi)(\eta_d^* - \xi_d^*) - 4]^2. \qquad (54)$$

Equations (47)–(51) list all the mathematical possibility of the zeros of Eq. (45), but not all zeros are admissible. From the positive definite character of the material constant, it can be shown numerically that the admissible zeros are those expressed in (47), (48), and (49). The order of stress singularity for the mixed boundary condition can then be obtained explicitly. This is the first explicit results for the order of stress singularity of mixed boundary condition for anisotropic interfacial crack. While for the pure traction or displacement prescribed problems Ting (1986) also obtained the explicit results by using Stroh's formulations. For the isotropic interfacial crack subjected to mixed boundary condition, the solutions can be obtained from general results shown in (47)–(51) by setting

$$q_o = \left[1 + \frac{\mu^*(m-1)}{\mu}\right]\left[1 + \frac{\mu^*}{\mu(m^*-1)}\right], \qquad (55)$$

$$q^+ = \left[\frac{\mu^* m m^*}{\mu(m^*-1)}\right]^2, \qquad (56)$$

$$q^- = \left[4 + \frac{\mu^*(m-2)(m^*-2)}{\mu(m^*-1)}\right]^2. \qquad (57)$$

This result of the order of stress singularity for an isotropic case is in agreement with the results obtained by Ting (1986) and Ma and Wu (1990).

Conclusions

The problem of plane deformation for a dissimilar anisotropic interface crack was solved by application of the Mellin transform. The explicit solutions of stresses and displacements are obtained for traction-displacement and displacement-displacement boundary conditions applied on the crack faces. It is very interesting to find that the solutions of the displacement prescribed problems can be obtained from the traction prescribed problems by a simple substitution. The dependence of the order of stress singularity on the material constants and boundary conditions is expressed in explicit closed form. It is shown that the order of stress singularity has reduced dependence on the elastic constants. It needs only one material parameter instead of 12 material constants for displacement-displacement boundary conditions. The reduction in the number of elastic constants may simplify the analysis and investigation of the interface crack problem.

Acknowledgments

The financial support by the National Science Council (Republic of China) through grant NSC 79-0401-E002-37 to National Taiwan University and helpful discussions with professor K. C. Wu are gratefully acknowledged.

References

Bogy, D. B., 1972, "The Plane Solution for Anisotropic Elastic Wedges Under Normal and Shear Loading," ASME JOURNAL OF APPLIED MECHANICS, Vol. 39, pp. 1103–1109.

Ma, C. C., and Hour, B. L., 1989, "Analysis of Dissimilar Anisotropic Wedges Subjected to Antiplane Shear Deformation," *Int. J. Solids Structures*, Vol. 25, pp. 1295–1309.

Ma, C. C., and Hour, B. L., 1990, "Antiplane Problems in Composite Anisotropic Materials with an Inclined Crack Terminating at a Bimaterial Interface," *Int. J. Solids Structures*, Vol. 26, pp. 1387–1400.

Ma, C. C., and Luo, J. J., 1992, "Plane Solutions of Interface Cracks in Anisotropic Dissimilar Media," submitted for publication.
Ma, C. C., and Wu, H. W., 1990, "Analysis of Inplane Composite Wedges under Traction-Displacement or Displacement-Displacement Boundary Conditions," *Acta Mechanica*, Vol. 85, pp. 149-167.
Rice, J. R., and Sih, G. C., 1965, "Plane Problems of Cracks in Dissimilar Media," ASME JOURNAL OF APPLIED MECHANICS, Vol. 32, pp. 418-422.
Ting, T. C. T., 1986, "Explicit Solution and Invariance of the Singularities at an Interface Crack in Anisotropic Composites," *Int. J. Solids Structures*, Vol. 22, pp. 965-983.
Ting, T. C. T., 1990, "Interface Cracks in Anisotropic Bimaterials," *J. Mech. Phys. Solids*, Vol. 38, pp. 505-513.
Williams, M. L., 1959, "The Stresses Around a Fault or Crack in Dissimilar Media," *Bulletin of the Seismological Society of America*, Vol. 49, pp.199-204.

Singularity Eigenvalue Analysis of a Crack Along a Wedge-Shaped Interface

Y. Z. Chen[9] and Norio Hasebe[10]

1 Introduction

Recently, there has been a resurgence of interest in the elastic interface crack problem. Works by Huchinson, Mear, and Rice (1987), Rice (1988), Mukai, Ballarini, and Miller (1990), Hasebe, Okumura, and Nakamura (1990), Toya (1990), and Wu (1990) provide examples of the recent contributions. The interface crack problem between dissimilar materials was first studied by Williams (1959). Williams showed that the stresses at the vicinity of a crack tip possess singularities of type $r^{a-i\epsilon}$, where r is the radial distance from the crack tip and ϵ is a bimaterial constant. The problem of two edge-bonded wedges of dissimilar materials was investigated by Bogy (1971). Bogy used the Mellin transform to investigate the nontrivial solution for the two edge-bonded wedges. He studied the order of the singularity in the case of some particular wedge angle and the material constants changing continuously.

In this paper, singularity eigenvalue analysis of a crack along a wedge-shaped interface is examined. The considered wedges are bonded along one edge and are debonding, or cracking, along another edge (Fig. 1). One wedge has an angle α and the elastic constants μ_1, κ_1 and another wedge has β, μ_2, and κ_2. Two angles are assumed to satisfy $\alpha + \beta = 2\pi$, and α changes from 0 to 2π. The eigenvalue is denoted by $E = a - ib$ in the following analysis. The complex variable function method proposed by Muskhelishvili (1953) is used for the eigenvalue analysis. Comparing with the Mellin transform method, the proposed method is straightforward, and the obtained results and eigenvalues can be directly related to the stress and displacement fields. It is obviously that $\alpha = 0$ or $\alpha = 2\pi$ corresponds to the isotropic case, and the eigenvalue for leading term (abbreviated as ELT) is a real one. Also, it is easily seen that $\alpha = \pi$ corresponds to the conventional interface crack problem, and the ELT is a complex value. Contrary to a previous study, in this paper the angle α is changing continuously and the material constants involved are assigned to be some particular value. Therefore, the change of ELT from a real value ($0 \le \alpha \le \alpha_c$), to a complex value ($\alpha_c \le \alpha \le \alpha_u$), and then to a real value ($\alpha_u \le \alpha \le 2\pi$) can clearly be seen from the obtained numerical results.

[9]Division of Engineering Mechanics, Jiangsu Institute of Technology, Zhenjiang, Jiangsu, 212013, P. R. China.
[10]Department of Civil Engineering, Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya 466, Japan.
Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Sept. 10, 1991; final revision, Apr. 10, 1992. Associate Technical Editor: C. F. Shih.

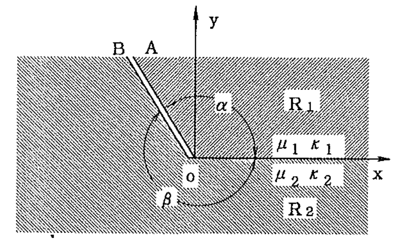

Fig. 1 A crack along a wedge-shaped interface

2 Analysis

It is well known that the complex variable function method proposed by Muskhelishvili (1953) provides a most effective approach to analyze the plane elastic problem. According to this method, the stresses (σ_{xx}, σ_{yy}, σ_{xy}), the resultant force functions (X, Y), and the displacements (u, v) can be described by two complex potentials $\phi(z)$ and $\omega(z)$

$$\sigma_{xx} + \sigma_{yy} = 4\mathrm{Re}[\Phi(z)]$$

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + (z-\bar{z})\overline{\Phi'(z)} + \overline{\Omega(z)} \quad (1)$$

$$P = -Y + iX = \phi(z) + (z-\bar{z})\overline{\phi'(z)} + \overline{\omega(z)} \quad (2)$$

$$2\mu(u+iv) = \kappa\phi(z) - (z-\bar{z})\overline{\phi'(z)} - \overline{\omega(z)} \quad (3)$$

where $\Phi(z) = \phi'(z)$ and $\Omega(z) = \omega'(z)$, μ is the shear modulus of elasticity, $\kappa = 3 - 4\nu$ for the plane strain problem, $\kappa = (3 - \nu)/(1 + \nu)$ for the plane stress problem, and ν is the Poisson's ratio.

We seek the solution of the problem in some region R ($R = R_1 + R_2$, Fig. 1) surrounding by a traction-free interface crack. The elastic constants and the complex potentials are denoted by μ_1, κ_1, $\phi_1(z)$, $\omega_1(z)$ and μ_2, κ_2, $\phi_2(z)$, $\omega_2(z)$ for the regions R_1 and R_2, respectively. From Eqs. (2) and (3) the continuation condition of the resultant force and the displacement along the positive part of real axis gives rise to the following relations:

$$\phi_1^+(x) + \overline{\omega_1^+(x)} = \phi_2^-(x) + \overline{\omega_2^-(x)} \quad (x>0) \quad (4)$$

$$\mu_2(\kappa_1\phi_1^+(x) - \overline{\omega_1^+(x)}) = \mu_1(\kappa_2\phi_2^-(x) - \overline{\omega_2^-(x)}) \quad (x>0), \quad (5)$$

since along the crack faces OA and OB we have

$$z = \bar{z}\exp(2i\alpha) \quad (z\epsilon \mathrm{OA} \text{ or } z\epsilon \mathrm{OB}). \quad (6)$$

Therefore, the traction-free condition along the upper and lower crack faces can be expressed by

$$\phi_1(z) + (\exp(2i\alpha)-1)\bar{z}\,\overline{\phi_1'(z)} + \overline{\omega_1(z)} = 0 \quad (z\epsilon \mathrm{OA}) \quad (7)$$

$$\phi_2(z) + (\exp(2i\alpha)-1)\bar{z}\,\overline{\phi_2'(z)} + \overline{\omega_2(z)} = 0 \quad (z\epsilon \mathrm{OB}). \quad (8)$$

In the following analysis we let the complex potentials take the following expression:

$$\phi_1(z) = p_1 z^{a-ib} + \bar{q}_1 z^{a+ib}$$
$$\omega_1(z) = \bar{s}_1 z^{a+ib} + t_1 z^{a-ib} \quad (9)$$
$$\phi_2(z) = p_2 z^{a-ib} + \bar{q}_2 z^{a+ib}$$
$$\omega_2(z) = \bar{s}_2 z^{a+ib} + t_2 z^{a-ib} \quad (10)$$

where p_1, q_1, s_1, t_1, p_2, q_2, s_2, and t_2 are complex values. In addition, the value $E = a - ib$ (or $a + ib$) will be determined by the condition of a nontrivial solution of the problem and is called the eigenvalue for a crack problem in the bonded wedges.

Substituting Eqs. (9) and (10) in (4), (5), (7), and (8) yields eight equations. Furthermore, after eliminating s_1, t_1, s_2, and t_2 in these equations we get the following equations:

$$(1-e_1)p_1 - e_3 q_1 - (1-e_1 e_2)p_2 + e_3 q_2 = 0 \quad (11)$$