Calculation of Stress Intensity Factors for Elliptical Cracks in Finite Bodies by Using the Boundary Weight Function Method

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In this study, mode I stress intensity factors for a three-dimensional finite cracked body with arbitrary shape and subjected to arbitrary loading is presented by using the boundary weight function method. The weight function is a universal function for a given cracked body and can be obtained from any arbitrary loading system. A numerical finite element method for the determination of weight function relevant to cracked bodies with finite dimensions is used. Explicit boundary weight functions are successfully demonstrated by using the least-squares fitting procedure for elliptical quarter-corner crack and embedded elliptical crack in parallelepipedic finite bodies. If the stress distribution of a cut-out parallelepipedic cracked body from any arbitrary shape of cracked body subjected to arbitrary loading is determined, the mode I stress intensity factors for the cracked body can be obtained from the predetermined boundary weight functions by a simple surface integration. Comparison of the calculated results with some available solutions in the published literature confirms the efficiency and accuracy of the proposed boundary weight function method.

1 Introduction

In the manufacturing process and during service life, a crack may initiate on an internal or external boundary in structure components. The implementation of damage tolerance analysis for the design of structures containing cracks requires the knowledge of stress intensity factors. Considerable effort has been devoted to the computation of the stress intensity factors to assess whether structural failure will occur or not. Stress intensity factors are now available for a wide range of crack configurations and loadings and have been summarized in well-known handbooks (Tada et al., 1973; Rooke and Cartwright, 1976; and Sih 1973). It is still found inadequate with regard to the needs in practical applications because actual structural details are often unique so that ready-made handbook solutions cannot be available. Although there are many numerical techniques that produce accurate results, they will be very expensive when a wide range of loading conditions are involved. There is a great need for simple methods to obtain stress intensity factors for engineering applications with good accuracy. The weight function method provides an alternative, yet more efficient methodology in the analysis of stress intensity factor for cracked bodies.

The weight function concept for two-dimensional elastic crack analysis, first proposed by Bueckner (1970), is a powerful and efficient method for determining the stress intensity factor. Rice (1972) proposed a convenient procedure for weight function determination for plane problems which then became the most prevailing form. He showed that weight function could be determined by differentiating known displacement field with respect to crack length. The weight function serves as a universal function for a given crack geometry and composition and is independent of applied loading. The weight function concept is, in fact, Green's function of the stress intensity factors for a cracked body. The weight function, once obtained from a single simple load case, can then be used to calculate additional stress intensity factors for the same cracked geometry, but with different load conditions.

In recent years, the finite element method applied to fracture mechanics has been well developed. There are several studies that have sought to build up the calculation technique and provide a possible and efficient way to construct the two-dimensional weight functions for finite cracked bodies. Sha (1984) used the stiffness derivative technique coupled with singular crack-tip elements to determine the weight functions, and he obtained the two-dimensional weight function for a single-edge crack by means of the finite element method. Sha and Yang (1985) obtained the two-dimensional weight function for an oblique edge crack by means of the finite element method using the virtual crack extension technique as suggested by Parks (1974) and Hellen (1975). They have extended this method to nonsymmetric mixed mode problems and used a special symmetric mesh in the vicinity of the crack tip such that the stress intensity factors for modes I and II could be determined independently. Recently, Tsai and Ma (1989) and Ma et al. (1990) constructed the explicit form of the two-dimensional mixed mode crack face weight function for finite rectangular plates by using the finite element and curve-fitting technique. These explicit weight functions are expressed in terms of a position coordinate, crack length, specimen width and length, which are certainly more useful in practical applications. However, the aforementioned crack face weight function method can only be used to determine stress intensity factors for specific cracked geometry subjected to mechanical loadings.

In the classical study of thermoelastic crack problem, the theoretical solutions are available only for very few problems in which cracks are contained in infinite media under special thermal loading conditions. For cracked bodies of finite dimension, exact solutions are very difficult to obtain; hence, Wilson and Yu (1979), Hellen and Cesari (1979) employed the finite

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element method to deal with these problems. The method is usually combined with the modified J-integral theory provided by Wilson and Yu (1979). The other prevailing method employed by Emery et al. (1969), Nied (1983), Oliveira and Wu (1987), and Bahr and Balke (1987) is based on the concept of superposition; that is, the thermal loading is replaced by mechanical traction forces which are the same as equivalent internal force at the prospective crack face in the absence of a crack. Previous methods have a common disadvantage that the complicated finite element calculation must be repeated for the same cracked body subjected to different thermal loading. Particularly, when these methods are applied in the transient thermal loading situation, a large amount of numerical calculation will be involved. In a study by Tsai and Ma (1992), the formulation of the thermal weight function was derived from the thermoelastic Betti reciprocal theorem. It was found that the rate of change of the mean stress with respect to the crack length of any arbitrary mechanic loading system is the thermal weight function. Ma and Liao (1996) solved the problem of axial cracks in hollow cylinders subjected to thermal shock by using the thermal weight function method.

Because of the computational efficiency, the virtual crack extension technique suggested by Park (1974) and Hellen (1975) has been widely used in the finite element evaluation of mode I stress intensity factors. The virtual crack extension is a devised algorithm for the efficient calculation of the strain energy release rate of a cracked body. The method described in the foregoing is used by Ma et al. (1990) for evaluating oblique edge crack and oblique central crack weight function on crack faces. There are eight weight functions that should be determined on upper and lower crack faces. Four of them have the square root singular behavior near the crack tip neighborhood, while the other four are nonsingular in crack tip neighborhood. By using the principle of superposition, the stress intensity factor under crack face loading is equivalent to the cracked body with remote loading that produces the same pressure loading on the prospective crack face in the absence of the crack. Hence, for a special specified regular cracked geometry, only the weight functions on the crack faces are needed for evaluating the stress intensity factors. However, for calculating the stress intensity factors of arbitrary cracked geometries, the commonly used crack face weight function concept will not be a suitable method. In this study, we have introduced the boundary weight function method to calculate the stress intensity factors for elliptical cracks in finite bodies subjected to general boundary condition on the boundary.

Most analytic solutions and numerical results for the investigation of cracked bodies are two-dimensional problems. Because of the complexities of the three-dimensional problem in finite crack bodies, no exact solutions of the stress intensity factor are available. Many techniques can be used to calculate the stress intensity factor by finite element method; it usually requires a very fine mesh near the crack tip. Raju and Newman

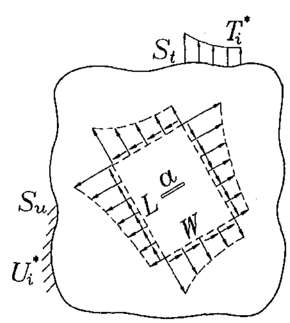


Fig. 1 Configuration for an arbitrary cracked body subjected to general boundary condition with a cut parallelepipedic cracked body containing a crack

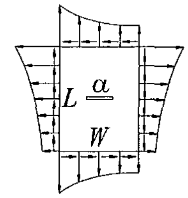


Fig. 2 Geometry of a parallelepipedic cracked body with a crack as indicated in Fig. 1

(1979) used the finite element method to obtain the stress intensity factor for a semi-circular surface crack in a semi-infinite solid and a semi-elliptical surface crack in a plate of finite thickness. Hechmer and Bloom (1977) studied the case of two symmetric corner cracks emanating from a hole in a plate. In this paper, an efficient method which combines finite element with boundary weight functions has been established in determining stress intensity factors in three-dimensional configuration and subjected to arbitrary loadings for finite cracked body. The boundary weight function method has also been investigated by Ma et al. (1995) for two-dimensional case and we will extend this methodology to three-dimensional analogy in this study. Three-dimensional problems are much more difficult than two-dimensional problems, since a weight function exists for every point along the crack front. The stress intensity factor for any specific three-dimensional crack geometry subjected to general prescribed boundary conditions, as shown in Fig. 1, is equal to that of the cut-out parallelepipedic cracked body (Fig. 2) with the same stress distribution on the cut-out parallelepipedic boundary (dash line) in Fig. 1. The stress distribution on the cut-out parallelepipedic boundary can be obtained with accuracy by the ordinary coarse finite element mesh. An efficient finite element method for determining both the stress intensity factors and weight functions for the cracked body of interest has been achieved by combining the singular crack tip elements with the virtual crack extension technique. Once we have obtained the value of boundary weight function for the parallelepipedic cracked geometry, the stress intensity factor under any load can be obtained by a simple surface integration. In this study, boundary weight functions for parallelepipedic cracked bodies are expressed in terms of the nondimensional quantities of position coordinate on the boundary. The cracked geometries to be considered are quarter-corner elliptical crack and embedded elliptical crack. These explicit weight functions are then used to calculate stress intensity factors for some specific crack problems and are compared with known results in the literature.

2 Weight Function Formulations

The two-dimensional mixed mode stress intensity factor can be expressed as a product of the applied load and the weight function as follows:

$$K_I = \int_{S_t} \mathbf{t}^* \cdot \mathbf{h}_{\mathrm{I}} ds + \int_{S_u} \mathbf{u}^* \cdot \mathbf{h}_{\mathrm{I}}^{\mathrm{u}} ds \tag{1}$$

$$K_{II} = \int_{S_t} \mathbf{t}^* \cdot \mathbf{h}_{\mathrm{II}} ds + \int_{S_u} \mathbf{u}^* \cdot \mathbf{h}_{\mathrm{II}}^{\mathrm{u}} ds \tag{2}$$

The decoupled weight function vectors for modes I and II in two-dimensional configuration are represented as follows:

$$\mathbf{H}_{\mathrm{I}} = \frac{H}{2K}\left(K_{\mathrm{II}}^{(2)} \frac{\partial \mathbf{u}^{(1)}}{\partial a} - K_{\mathrm{II}}^{(1)} \frac{\partial \mathbf{u}^{(2)}}{\partial a}\right) \tag{3}$$

$$\mathbf{h}_{\mathrm{I}}^{\mathrm{u}} = -\frac{H}{2K}\left(K_{\mathrm{II}}^{(2)} \frac{\partial \mathbf{t}^{(1)}}{\partial a} - K_{\mathrm{II}}^{(1)} \frac{\partial \mathbf{t}^{(2)}}{\partial a}\right) \tag{4}$$

$$\mathbf{h}_{\mathrm{II}} = \frac{H}{2K}\left(K_{\mathrm{I}}^{(1)}\frac{\partial \mathbf{u}^{(2)}}{\partial a} - K_{\mathrm{I}}^{(2)}\frac{\partial \mathbf{u}^{(1)}}{\partial a}\right) \tag{5}$$

$$\mathbf{h}_{\mathrm{II}}^{u} = -\frac{H}{2K}\left(K_{\mathrm{I}}^{(1)}\frac{\partial \mathbf{t}^{(2)}}{\partial a} - K_{\mathrm{I}}^{(2)}\frac{\partial \mathbf{t}^{(1)}}{\partial a}\right) \tag{6}$$

and

$$K = K_{\mathrm{I}}^{(1)}K_{\mathrm{II}}^{(2)} - K_{\mathrm{I}}^{(2)}K_{\mathrm{II}}^{(1)} \neq 0$$

in which $H = E$ (Young's modulus) for generalized plane stress and $H = E/(1-\nu^2)$ for plane strain, ν being Poisson's ratio. The subscript **I** in $\mathbf{h}_{\mathrm{I}}$ is used to denote the mode I weight function. The boundary has been divided into the part for specified traction S_t and that for specified displacement S_u. Configurations (1) and (2) are geometrically equivalent to the original problem. The corresponding stress intensity factors and displacement vectors will be denoted by $K_{\mathrm{I(II)}}^{(1)}$, $\mathbf{u}^{(1)}$ and $K_{\mathrm{I(II)}}^{(2)}$, $\mathbf{u}^{(2)}$, respectively. Once the weight functions are determined from the solution for any particular load system, the two-dimensional stress intensity factor induced by any other load system can be obtained from Eqs. (1) and (2).

The weight function method also applies to three-dimensional problems of cracked bodies. The developments of the three-dimensional weight functions can be found in the articles of Rice (1989), Bueckner (1987), and Sham (1987). Three-dimensional problems are much more difficult than two-dimensional problems since a weight function exists for every point along the crack front. Consider a plane crack in a solid with a front of smooth shape, both the body and load systems under consideration being symmetrical about the plane of the crack. A three-dimensional weight function may be defined as (Rice, 1972; Labbens et al., 1976)

$$h(P, P') = \frac{H}{2K^*(P')}\frac{\delta U(P)}{\delta A(P')} \tag{7}$$

where P is a load point, P' is a point along the crack front. $\delta A(P')$ is a small extension of the crack surface at P' and $K^*(P')$ is the stress intensity factor at point P' under special loading. This expression is the general definition of three-dimensional weight functions. This is also a unique function of P and P' for a given cracked body geometry and composition, and is completely independent of the way in which the body is loaded. For an arbitrary symmetric load system $t(P)$ applied on the cracked body, the stress intensity factor at P' can be obtained from the surface integration

$$K(P') = \int_s t(P)h(P, P')dA(P) \tag{8}$$

However, by employing the linear superposition method proposed in this study, only the weight functions along the prospective boundaries of the parallelepipedic cracked body are of primary interest for evaluating the stress intensity factor.

Consider a three-dimensional cracked body with general boundary condition applied on the boundary as shown in Fig. 1, which will induce normal and shear stresses along a regular specified parallelepipedic boundary. The stress intensity factors of the original problem is equal to that of the cut-out regular parallelepipedic cracked body subjected to equal stress distribution on the parallelepipedic boundary of Fig. 1. Hence, the problem for the case of a parallelepipedic body with a surface crack, as shown in Fig. 2, is of special interest. Once the weight functions along the parallelepipedic boundary are determined, the stress intensity factors can be easily obtained by simple surface integrations over the distributed stress and predetermined boundary weight functions.

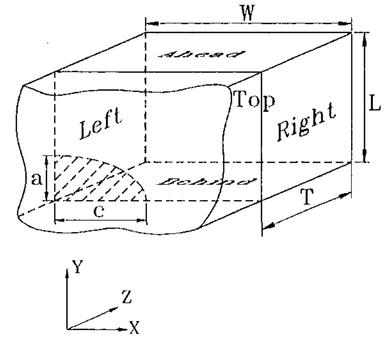


Fig. 3 Configurations of a parallelepipedic cracked body with an elliptical quarter-corner crack

3 Boundary Weight Functions for Parallelepipedic Cracked Body

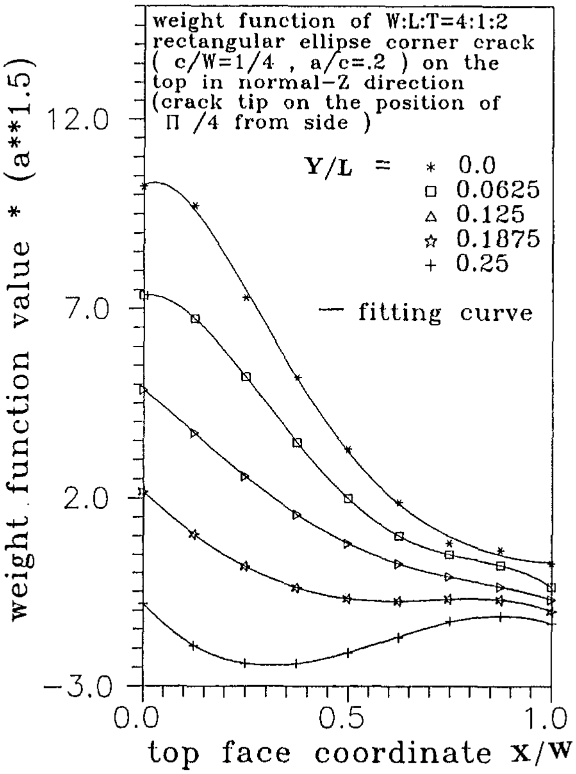
In order to determine the boundary weight functions, reference stress intensity factor together with the displacement field on boundaries must be known. Exact solutions of the displacement field are available for only a very few crack problems. For most cases of practical interest, exact solutions hardly exist. Hence, good approximations to the displacement field for such cases are very useful for constructing the explicit form of weight function. For specimens with finite dimension, the calculations are usually done by finite element method. Because of the invariant characteristics with respect to the loading conditions for a given geometry and constraint conditions, the simplest uniformly distributed loading is used for the finite element evaluation of weight functions for boundaries of interest. The finite element evaluation of the weight functions using the virtual crack extension technique coupled with singular elements is used in this study.

The conventional determination of the two-dimensional weight function by several authors has usually been restricted to crack face only, since the stress intensity factor under crack face loading is equivalent to the cracked body with remote loading that produces the same pressure loading on the prospective crack face in the absence of the crack. However, the crack face weight function will behave singularly near the crack tip, which should be carefully analyzed to get accurate results. In this study, the nonsingular boundary weight functions for a cracked parallelepipedic body are investigated for calculating the stress intensity factor of arbitrary cracked bodies subjected to general boundary conditions. The mesh of the finite element is the standard three-dimensional 20-nodal serendipity element of quadratic form and the elements in the vicinity of the crack tip are modeled with the degenerated quarter-point quadratic element.

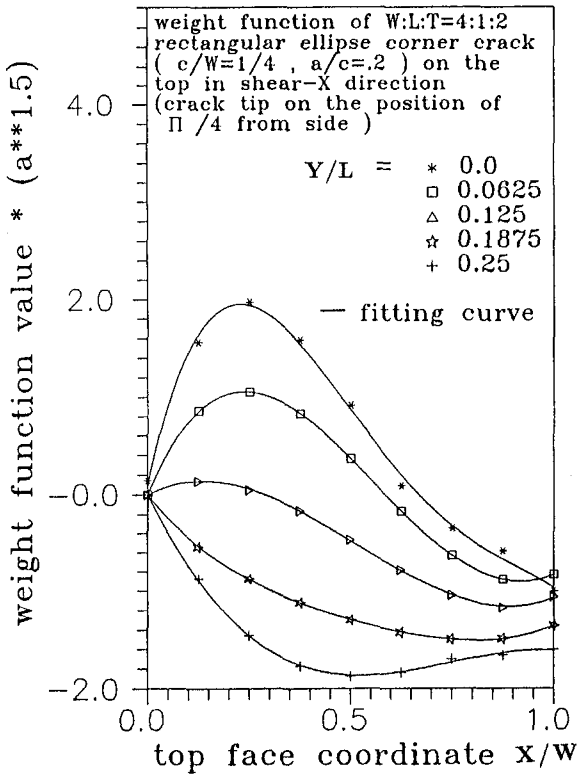
The parallelepipedic cracked body has five boundaries and the boundary weight function expressed as a function of the X-Y plane is taken to have the following polynomial form:

$$\sqrt{a}^3 h(X/l, Y/l) = \sum_{n=1}^{6}\sum_{m=1}^{6} C_{nm}(X/W)^n(Y/L)^m. \tag{9}$$

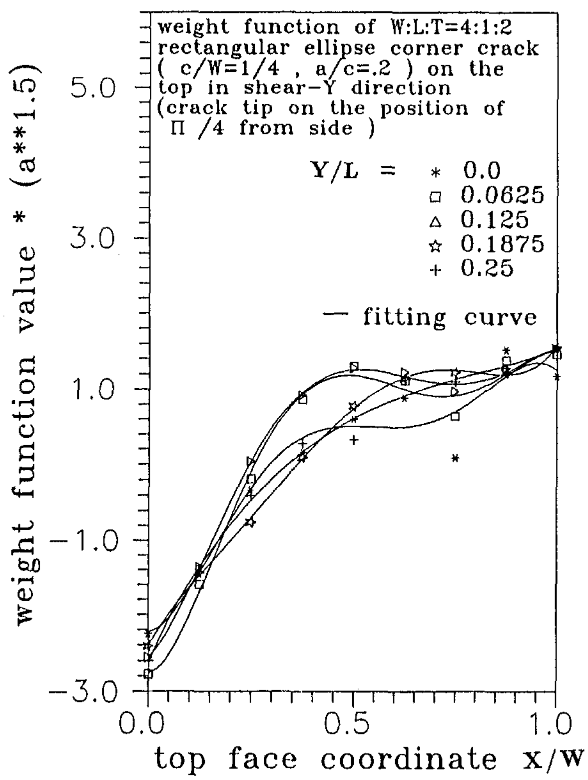
The corresponding boundary weight functions for boundaries on Y-Z and X-Z-planes have a similar form as (9). The corresponding configuration for an elliptical quarter-corner crack in a parallelepipedic body is represented in Fig. 3. Since the weight function is universal and is independent of the loading system, the most simple loading system of uniformly distributed tensile loading applied on the top surface is chosen for the numerical calculation. The stress intensity factor for uniformly distributed tensile loading is obtained by using the nodal-force method proposed by Raju and Newman (1979). The nodal forces normal to the crack plane and ahead of the crack front are used to calculate the stress intensity factor. The advantage of this



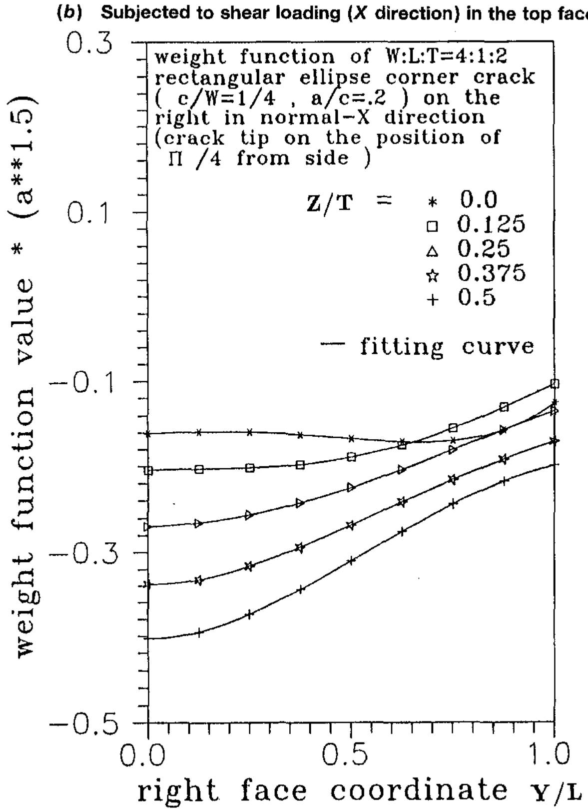
(*a*) Subjected to normal loading (*Z* direction) in the top face



(*b*) Subjected to shear loading (*X* direction) in the top face

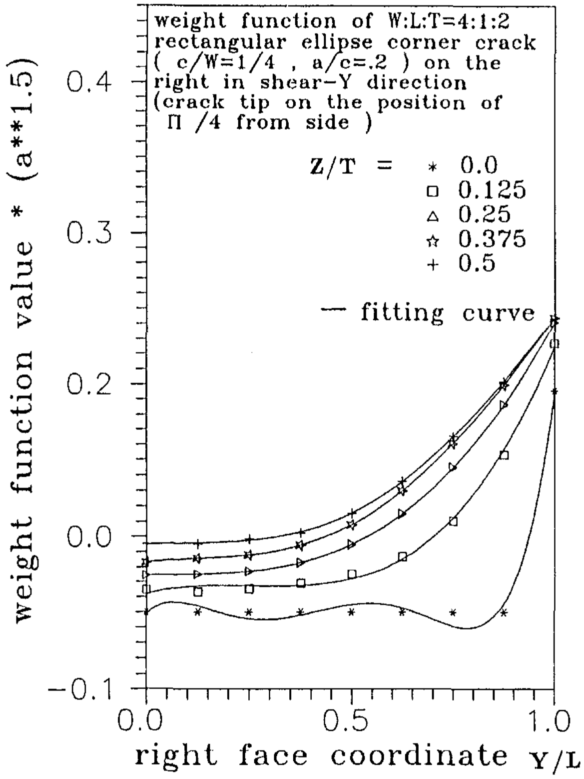


(*c*) Subjected to shear loading (*Y* direction) in the top face

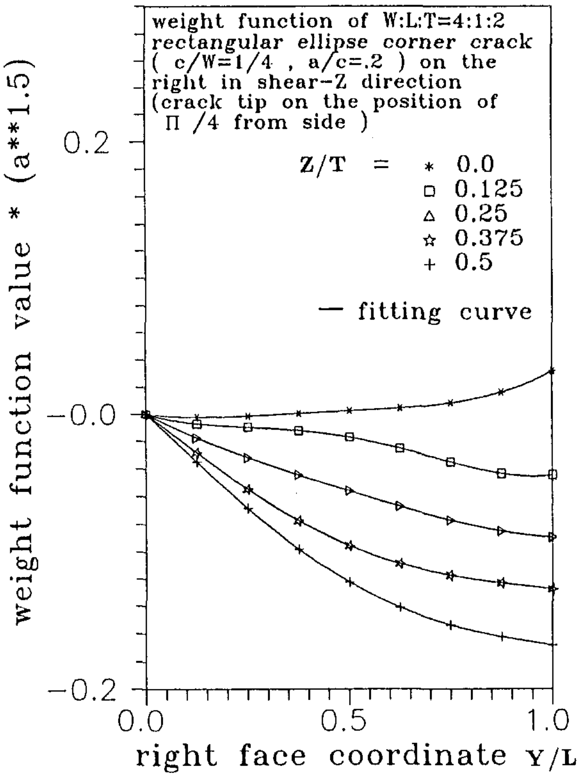


(*d*) Subjected to normal loading (*X* direction) in the right face

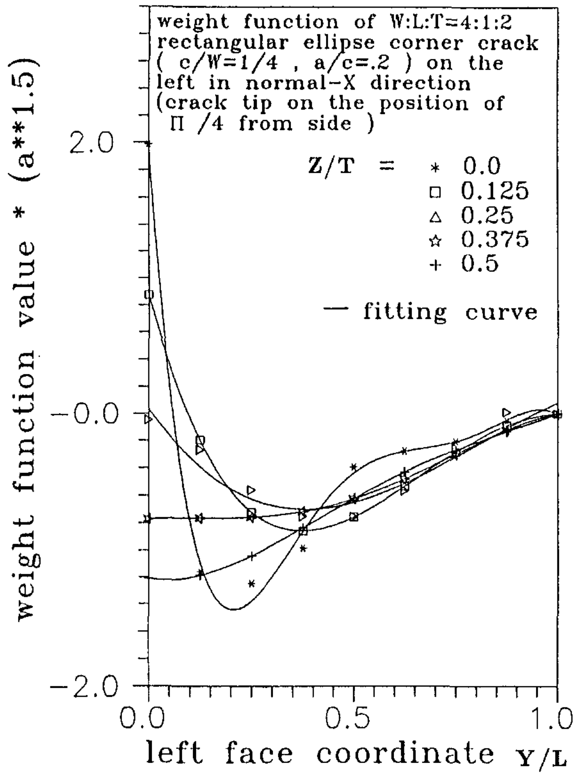
method is that it requires no prior assumption of either plane stress or plane strain condition. The numerical results from the basic data for curve fitting. By using least-squares procedure, the discretized values of the boundary weight functions are approximated with (9). Because of the symmetry, there are only five boundaries, i.e., top side, right side, left side, ahead side, and behind side, that should be analyzed, and the corresponding configurations are shown in Fig. 3. The boundary weight functions for different crack geometries are plotted in Fig. 4. Figures $4(a)-(i)$ represent the boundary weight functions for the top side, right side, and left side of an elliptic quarter-corner crack subjected to symmetrical loading. In these



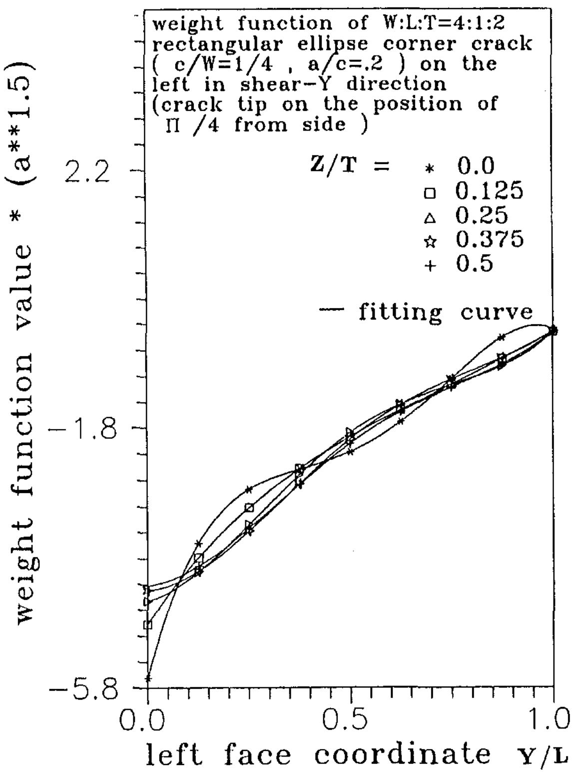
(*e*) **Subjected to shear loading (*Y* direction) in the right face**



(*f*) **Subjected to shear loading (*Z* direction) in the right face**

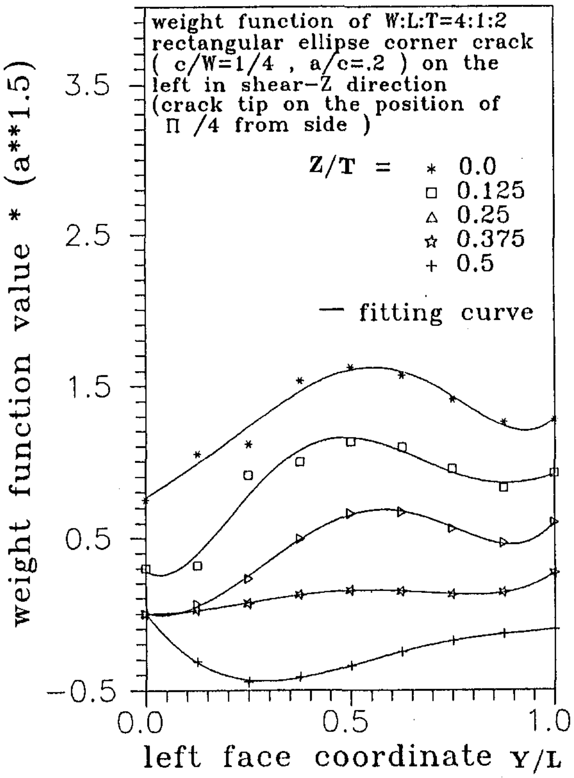


(*g*) **Subjected to normal loading (*X* direction) in the left face**



(*h*) **Subjected to shear loading (*Y* direction) in the left face**

figures, the actual nodal weight functions obtained from finite element analysis are represented by discrete points, and those calculated from the fitted nodal weight functions are plotted as solid lines. The accuracy of the predicted weight functions are checked directly against the finite element results. Excellent agreement between these two results are shown in these figures and this indicates that the fitted weight functions are good approximations to the actual finite element numerical results. The error that results from the process of curve fitting by means of the least-squares method is less than 1 percent, in general. The boundary weight functions are all smooth curves and no singular behavior will occur.



(i) Subjected to shear loading (Z direction) in the left face
Fig. 4 Boundary weight function for an elliptical quarter-corner crack

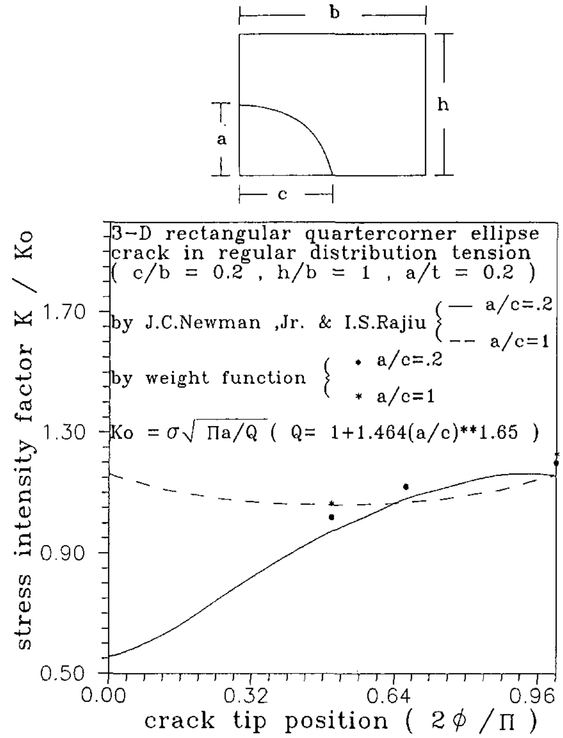


Fig. 5 Comparison of normalized mode I stress intensity factors calculated from the boundary weight function method and other results for a parallelepipedic body containing an elliptical quarter-corner crack subjected to remote tension at top faces

These predetermined explicit weight functions, as expressed in (9), for finite parallelepipedic cracked body along the boundaries are ready to use to evaluate the stress intensity factors for arbitrary cracked geometries. The procedure is rather simple; the first step is to evaluate the stress distribution inside the arbitrary cracked geometry along the cut parallelepipedic boundaries. Since the parallelepipedic boundary is far away from the crack tip, there will be no difficulty in obtaining good numerical results for stress distribution along the cut parallelepipedic boundaries. The next step is to combine the predetermined weight function, Eq. (9), and the obtained stress distribution along the cut parallelepipedic boundaries; accurate stress intensity factors can be evaluated by a simple surface integration according to (8). In order to demonstrate the accuracy and validity of the boundary weight functions obtained in this paper in determining the stress intensity factors, several crack geometries have been considered and the results will be compared with the findings of earlier studies. Figures 5 and 6 show the stress intensity factors along the crack front for an elliptical quarter-corner crack and an embedded elliptical crack, respectively, in a finite body subjected to tension. These results are compared with Newman and Raju (1983), and the error is within 5 percent. The stress intensity factor for two symmetric semielliptical surface cracks at the center of a hole in a finite body subjected to tension is shown in Fig. 7, and a comparison between the present results and those obtained by Newman and Raju (1983) are also shown in the figure. Good agreement between the present results and referenced solutions are noted. Comparison of the stress intensity factors calculated by using the foregoing proposed boundary weight functions with the results obtained by others revealed satisfactory and valid accuracy of the boundary weight function method.

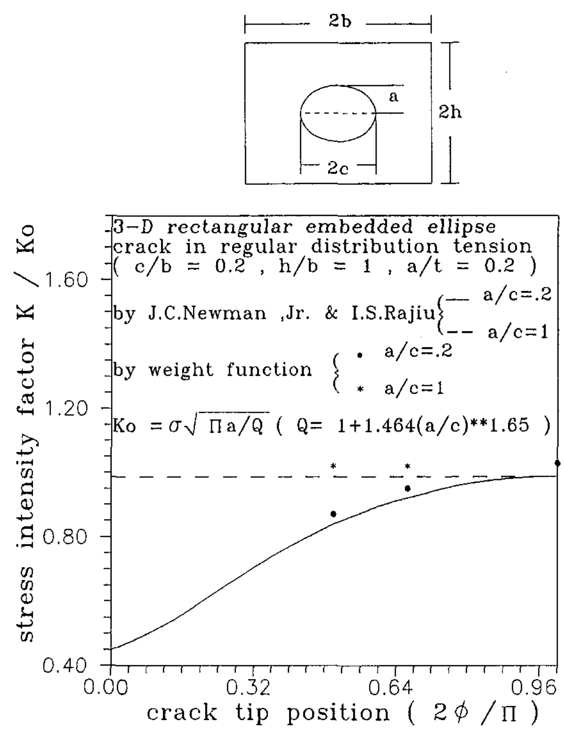


Fig. 6 Comparison of normalized mode I stress intensity factors calculated from the boundary weight function method and other results for a parallelepipedic body containing an embedded elliptical crack subjected to remote tension at top faces

4 Conclusions

Conventional methods for calculating stress intensity factors have a common disadvantage that the complicated finite element calculation must be repeated for the same cracked body subjected to different loadings. For the conventional study on weight function, most investigations are done for the crack face

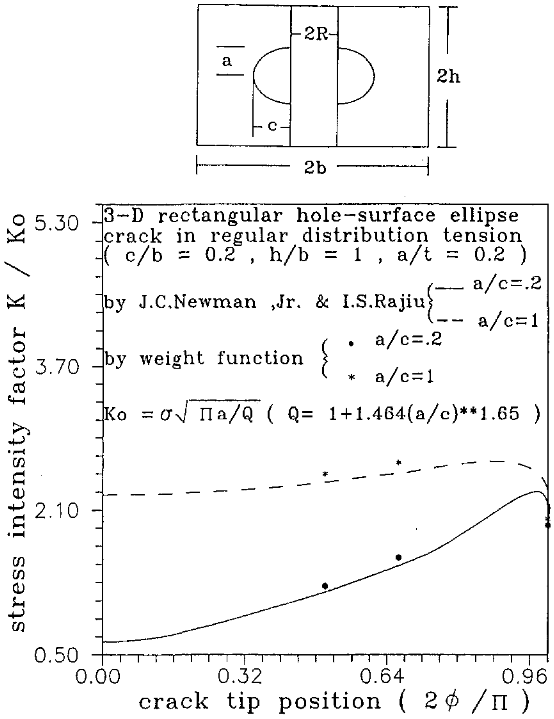


Fig. 7 Comparison of normalized mode I stress intensity factors calculated from the boundary weight function method and other results for a parallelepipedic body containing two symmetric semielliptical cracks at the center of a hole subjected to remote tension at top faces

weight function, which is usually suitable for a given geometry of a cracked body. In this study, the boundary weight function concept proposes calculating the stress intensity factors for arbitrary shape of cracked geometries in three-dimensional configuration and subjected to general boundary conditions. The methods of calculating stress intensity factors based on weight function techniques are efficient and economical, since once the weight function is determined for a given crack geometry, the stress intensity factor for any loading condition can be obtained by a simple integration. Hence, the boundary weight function method is more useful in practical applications. The simplest loading condition of uniform tension and an efficient finite element methodology has been achieved for evaluating the boundary weight functions for parallelepipedic cracked bodies. The boundary weight functions are all smooth curves and no singular behavior will occur.

In order to facilitate the utilization of the discretized nodal boundary weight functions, they are expressed as a function of the boundary for parallelepipedic cracked body. These empirical equations for the boundary weight functions of the parallelepipedic cracked body have been successfully obtained in this study. The stress intensity factors of cracked bodies subjected to arbitrary applied loadings can be obtained very efficiently by combining the stress field on the boundaries of cut parallelepipedic cracked body with the interpolated boundary weight functions. Very satisfactory results of the stress intensity factors are obtained by the proposed boundary weight function method when compared to known solutions of other workers.

Acknowledgments

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