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Image Singularities of Green's Functions for an Isotropic Elastic Half-Plane Subjected to Forces and Dislocations

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Abstract: Although the Green's function for an isotropic elastic half-space subjected to a line force or a line dislocation is well-known, the physical meaning of the solution is not clear. Green's functions for two-dimensional plane-strain and plane-stress problems of an isotropic elastic half-space with a free or rigidly fixed surface subjected to line forces and line dislocations are reexamined in this study. The results are more explicit when compared with existing solutions in the literature. The Green's function for a half-space consists of four or five Green's functions for an infinite space, the number depending on the boundary condition at the half-space surface and the applied loading. One of the Green's functions in the infinite space has its singularity located in the half-space where the load is applied, and the other image singularities are located outside the half-space with the same distance from the surface as that of the applied load. The nature and magnitude of image singularities have been derived from the principle of superposition and classified according to different loads. The image singularities are found to possess some interesting properties. It is found that the fundamental solutions required to construct all the image singularities of applied forces and dislocations for the half-space are only forces and dislocations and their differentiations in the infinite space. Furthermore, the limiting case of the applied force or dislocation approaching the surface is also discussed in this study.

Key Words: image singularities, isotropic half-plane, forces, dislocations

1. INTRODUCTION

The importance of Green's functions in constructing solutions to boundary value problems has been well recognized. Thus, the fundamental solutions for Green's functions for a half-space have been obtained by different methods. The Green's function for two-dimensional deformations of an anisotropic elastic half-space subjected to a line force and/or a line dislocation inside the half-space has been considered by Willis [1], Barnett and Lothe [2], Suo [3], and Qu and Li [4]. In the earlier study of the Green's function for the half-space with a traction-free boundary condition, the solution is constructed from the Green's function for the infinite space by adding a distribution of forces on the half-space surface so that the net surface traction vanishes. With this approach, the solution is not explicit and the final form requires an integration of the distributed forces on the surface. Some progress has been made in constructing Green's function by Hwu and Yen [5] and Ting [6]. In the theory of linear elasticity, the two-dimensional isotropic antiplane problems can be resolved

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by a superposition of some simple image singularities over the plane. Similar methods can also be applied to the two-dimensional anisotropic plane and antiplane problems; see, for instance, Ting [7]. The image singularities of Green's functions for an anisotropic elastic half-space and bimaterials were discussed in detail by Ting [6]. When an anisotropic half-space is subjected to line forces or line dislocations within the half-space, the image singularities are simply line force and line dislocation outside the half-space. In general, the locations of image singularities of Green's functions for half-spaces are different and depend on elastic constants. As is well-known for isotropic materials, the degenerate case of anisotropic materials, the image singularities are not only concentrated forces and dislocations. An objective of this paper is to find image singularities existing on the image point for isotropic materials and to interpret physical meanings of the Green's function.

There is an interesting group of solutions for the infinite space of linear elastic and isotropic materials, which are referred to as the nuclei of strain solutions. The solutions of many interesting problems may be derived in terms of combinations of these nuclei or by a process of superposition of nuclei. The fundamental solution for three-dimensional deformations of an infinite isotropic elastic body subjected to a point force has been obtained and is known as the Kelvin solution. By differentiation of this solution, a family of additional nuclei can be constructed, such as the double force and double force with moment. Mindlin's [8] results for a point force in the interior of an elastic solid occupying a half-space are based on nuclei of strain. Such solutions may be termed as half-space nuclei of strain. However, some of the nuclei of strain have the form of a line extending from a fixed distance to infinity with constant or variable magnitude. Mindlin and Cheng [9] provided many basic solutions for nuclei of strain in the half-space. The case of an elastic half-space with a fixed boundary has been solved by Phan-Thien [10] using Mindlin's approach. The fundamental solutions for point forces in the interior of one of two elastic half-spaces joined by a sliding contact interface was given by Dundurs and Hetenyi [11]. Vijayakumar and Cormack [12, 13] gave a general approach to derive Green's function for three-dimensional bimaterial elastic media with bonded and sliding interface by dividing the nuclei into independent classes and by representing the displacements and stresses produced by these nuclei in the framework of matrix-vector operations. Further development was done by Carvalho and Curran [14], who obtained the two-dimensional Green's functions of plane strain for elastic bimaterials by reducing the three-dimensional nuclei of strain through an integral procedure.

This paper investigates Green's functions for a line force or a line dislocation in the interior of an isotropic elastic half-space for both plane strain and plane stress cases. The aim is to investigate the Green's function of a half-plane (traction free and rigidly fixed) and to discuss in detail the structure of singularities on the image point. There are eight kinds of image singularities of the half-plane problem for applied force and dislocation. There are force, dislocation, double force without moment, double force with moment, center of extension, center of shear, double center of expansion, and double moment. It is interesting to note that dislocation is one of the image singularities for a traction-free half-plane subjected to concentrated forces, and concentrated force is one of the image singularities for a rigidly fixed half-plane subjected to dislocations. Since all the solutions of higher order image singularities (i.e., double force, double moment, etc.) can be constructed by differentiation of the solutions of force and dislocations, the Green's functions for a traction-free or rigidly fixed half-plane are basically determined only by the solutions of force and dislocation in the infinite plane.

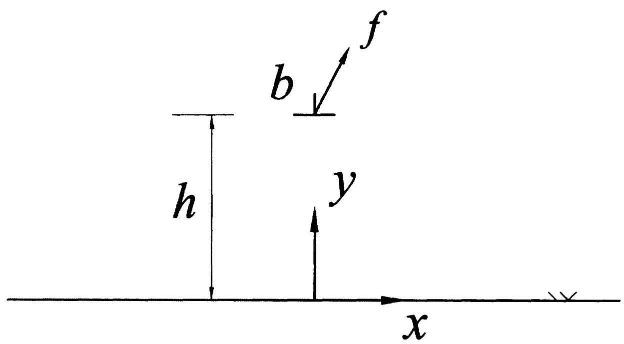

Fig. 1. Half-plane subjected to a body force and dislocation.

2. GREEN'S FUNCTIONS FOR HALF-PLANES

2.1. General Formulation

Consider an isotropic elastic half-space with a straight surface subjected to a line of uniformly distributed force along the z-axis and/or a line dislocation along the z-axis in the interior of the body. The displacements and stresses are independent of the z-axis, and we can consider this problem as two-dimensional. In other words, the original problem can be simplified as a two-dimensional half-plane with a straight-line boundary subjected to a point force and/or a dislocation inside the plane as shown in Figure 1. Both the plane stress and plane strain deformations are investigated in this study.

For the two-dimensional in-plane deformation, the stresses σ_{ij} are independent of z and the equations of equilibrium can be written as

$$\sigma_{jx,x} + \sigma_{jy,y} = 0, \quad j = x, y,$$

where the comma denotes differentiation. If we express stresses in terms of stress functions ϕ_j such that (Ting [7])

$$\sigma_{jx} = -\frac{\partial \phi_j}{\partial y}, \quad \sigma_{jy} = \frac{\partial \phi_j}{\partial x},$$

then the equations of equilibrium are automatically satisfied. It remains to determine the stress functions ϕ_j. The fact that $\sigma_{xy} = \sigma_{yx}$ means that $\phi_{y,y} + \phi_{x,x} = 0$.

The complete Green's functions of stresses and displacements for the half-plane can be written as follows:

$$\phi_j = \phi_j^o + \phi_j^i, \quad u_j = u_j^o + u_j^i, \quad j = x, y,$$

where u_j are displacement fields. The superscripts o and i indicate the force or dislocation which acts on the object point and image point of the infinite plane, respectively. The first terms represent the Green's function for an infinite plane with a concentrated force and a

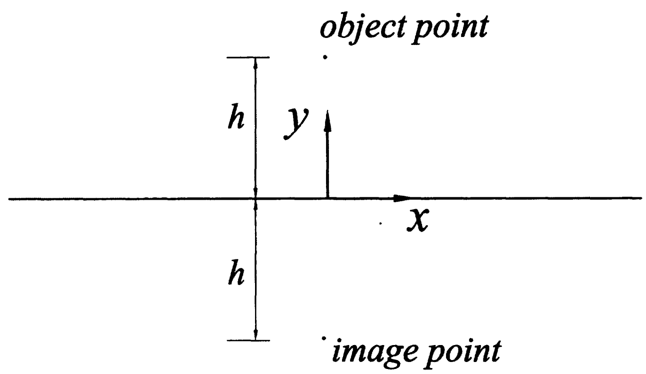

Fig. 2. Geometric configuration of object and image points for a half-plane.

dislocation applied at the object point. The second terms are the disturbed solutions which are added at the image point to satisfy the boundary conditions at $y = 0$. Forces or dislocations acting on the object and image points are referred to the applied singularities and the image singularities, respectively (see Fig. 2). The image point for an isotropic material is always located at the same distance from the plane surface as the object point. The number and the nature of image singularities for different applied singularity and boundary conditions will be discussed in detail in this study. The problem that will be discussed in the following investigation is divided into two categories: traction-free boundary condition and rigidly fixed boundary condition.

2.2. Image Singularities of a Half-Plane Subjected to a Vertical Force (free boundary)

Let the material occupy the half-plane $y \geq 0$. A vertical concentrated force f_y is applied at $x = 0$ and $y = h$ and acts in the positive y-direction as shown in Figure 3. Consider first the case in which the surface $y = 0$ is traction free so that

$$\phi_x\,(x, 0) = \phi_y\,(x, 0) = \text{constant} \equiv 0.$$

It is shown in the above expression that the traction-free boundary condition will be satisfied if the stress functions ϕ_x and ϕ_y vanish at $y = 0$. Hence, it is easier to use stress functions to solve the problem rather than using stresses. We will derive and explain in some detail how we construct the result for image singularity. We first apply a vertical force f_y at the object point $(0, h)$ in an infinite plane and the solutions expressed in terms of stress functions are (reduced from the solution of Ting [7] for an anisotropic case, p. 247)

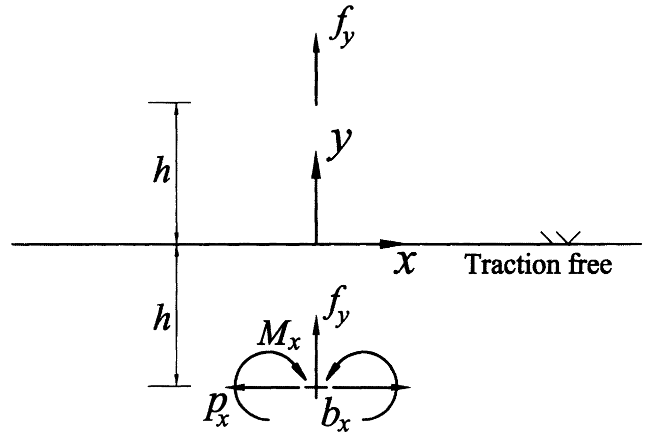

Fig. 3. Image singularities of the Green's function for a half-plane (free boundary) subjected to a vertical concentrated force f_y.

$$
\begin{aligned}
\phi_x^o &= \frac{f_y}{\pi(1+k)}\left[\frac{(y-h)^2}{x^2+(y-h)^2}+\frac{1-k}{2}\ln\sqrt{x^2+(y-h)^2}\right],\\
\phi_y^o &= \frac{-f_y}{\pi(1+k)}\left[\frac{1+k}{2}\tan^{-1}\frac{x}{y-h}+\frac{x(y-h)}{x^2+(y-h)^2}\right],
\end{aligned}
\tag{2.1}
$$

where

$$
k=\begin{cases} 3-4v & \text{for plane strain}\\ \dfrac{3-v}{1+v} & \text{for plane stress}\end{cases},
$$

in which v is the Poisson's ratio. In equation (2.1), there are two special functions, that is, $\ln\sqrt{x^2+(y-h)^2}$ and $\tan^{-1}(x/(y-h))$. To satisfy the condition $\phi_x=\phi_y=0$ at $y=0$, it is therefore necessary to add another solution to remove the boundary stresses, and these solutions must introduce no new singularities in the region $y\geq 0$. The only way is to apply image singularities that have the same special functions that appear in stress functions in (2.1). After examining all of the fundamental solutions for an infinite plane, there are only two types of singularities satisfying the above-mentioned arguments. They are

$$
\begin{aligned}
\phi_x^{f_y} &= \frac{f_y}{\pi(1+k)}\left[\frac{(y+h)^2}{x^2+(y+h)^2}+\frac{1-k}{2}\ln\sqrt{x^2+(y+h)^2}\right],\\
\phi_y^{f_y} &= \frac{-f_y}{\pi(1+k)}\left[\frac{1+k}{2}\tan^{-1}\frac{x}{y+h}+\frac{x(y+h)}{x^2+(y+h)^2}\right],
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
\phi_x^{b_x} &= \frac{\mu b_x}{\pi(1+k)}\left[\frac{2(y+h)^2}{x^2+(y+h)^2}+2\ln\sqrt{x^2+(y+h)^2}\right],\\
\phi_y^{b_x} &= \frac{-\mu b_x}{\pi(1+k)}\left[\frac{2x(y+h)}{x^2+(y+h)^2}\right].
\end{aligned}
\tag{2.3}
$$

Equations (2.2) and (2.3) are stress functions for concentrated force f_y along the y-direction and dislocation with Burger's vector b_x along the x-direction acting on the image point $y = -h$, respectively. These two stress functions are obtained from reducing the solutions of Ting [7]. The magnitude of the force f_y applied at the image point is equal to the vertical force at the object point. When we set $y = 0$ and add (2.1) to (2.2), the stress functions with the term $\tan^{-1}()$ will be eliminated and the term with $\ln()$ can be cancelled by adding (2.3) with

$$b_x = -\frac{(1-k)f_y}{2\mu},$$

where μ is the shear modulus. Add (2.2) and (2.3) to (2.1), and the stress functions on the boundary $y = 0$ take the forms

$$\phi_x = \frac{f_y h}{\pi}\frac{h}{x^2+h^2}, \quad \phi_y = \frac{(1-k)f_y h}{\pi(1+k)}\frac{x}{x^2+h^2}, \tag{2.4}$$

and stresses have the $1/r^3$ and $1/r^2$ type singularity, where r is the radial distance from the image point. Hence, we should superpose another fundamental solution with similar singularity on the image point. The suitable solutions are (Ting [7], p. 306)

$$
\begin{aligned}
\phi_x^{P_x} &= \frac{P_x}{2\pi(1+k)}\left[\frac{(k-1)(y+h)}{x^2+(y+h)^2}+\frac{4x^2(y+h)}{\left(x^2+(y+h)^2\right)^2}\right],\\
\phi_y^{P_x} &= \frac{P_x}{2\pi(1+k)}\left[\frac{(1-k)x}{x^2+(y+h)^2}+\frac{4x(y+h)^2}{\left(x^2+(y+h)^2\right)^2}\right].
\end{aligned}
\tag{2.5}
$$

Equation (2.5) represents the stress function for a double force without moment along the x-direction acting on the image point. The magnitude of the double force can be determined as

$$P_x = -2f_y h.$$

Add (2.5) to (2.4) and the stress function on the boundary is

$$\phi_x = \frac{-2f_y h^2}{\pi(1+k)}\frac{x^2-h^2}{(x^2+h^2)^2}, \quad \phi_y = \frac{-f_y h^2}{\pi(1+k)}\frac{4xh}{(x^2+h^2)^2}. \tag{2.6}$$

Equation (2.6) has $1/r^3$ type of stress singularity. Similarly, we seek a fundamental solution with the stress function

$$\phi_x^{M_x} = \frac{M_x}{\pi} \frac{x^2 - (y+h)^2}{\left(x^2 + (y+h)^2\right)^2}, \quad \phi_y^{M_x} = \frac{M_x}{\pi} \frac{2x\,(x+h)}{\left(x^2 + (y+h)^2\right)^2}. \tag{2.7}$$

Equation (2.7) is the solution for a double moment along the x-direction acting on the image point. This stress function is obtained from differentiating the solution for a concentrated moment applied in an infinite plane which was derived by Ting ([7], p. 309). If we set

$$M_x = \frac{2f_y h^2}{1+k}$$

and add (2.7) to (2.6), then we have

$$\phi_x = 0, \quad \phi_y = 0.$$

The results exactly satisfy the traction-free boundary condition of a half-plane subjected to a vertical concentrated force f_y. Therefore, the stress functions on the image point, $y = -h$, can be specified as

$$\vec{\phi}^{\,i}_{f_y} = \vec{\phi}^{\,f_y} + \vec{\phi}^{\,b_x} + \vec{\phi}^{\,P_x} + \vec{\phi}^{\,M_x}. \tag{2.8}$$

Equation (2.8) indicates that the image singularities consist of vertical concentrated force f_y; dislocation with Burger's vector b_x; double force without moment along the x-direction, P_x; and double moment along the x-direction, M_x. It is noted that each image singularity has its own physical meaning and is shown in Figure 3.

The full field stress distributions due to point loads applied in an isotropic half-plane were presented by Melan [15], Telles and Brebbia [16], and Ma and Huang [17]. For a verification of our solution, we compare the stress σ_{yy} constructed in this study with

$$\begin{aligned}\sigma_{yy} = \frac{f_y}{2\pi}\Bigg\{ & \frac{-(y-h)}{x^2+(y-h)^2} + \frac{(y-h)\left(x^2-(y-h)^2\right)}{2\,(1-\nu)\left(x^2+(y-h)^2\right)^2} - \frac{y+h}{x^2+(y+h)^2} \\ & + \frac{2yh\,(y+h)\left(3x^2-(y+h)^2\right)}{(1-\nu)\left(x^2+(y+h)^2\right)^3} + \frac{((3-4\nu)\,y+h)\left(x^2-(y+h)^2\right)}{2\,(1-\nu)\left(x^2+(y+h)^2\right)^2}\Bigg\},\end{aligned} \tag{2.9}$$

obtained by Ma and Huang [17]. The solution in (2.9) can be rearranged as follows:

$$\sigma_{yy} = -\frac{f_y}{2\pi}\left[\frac{y-h}{x^2+(y-h)^2} - \frac{2(y-h)\left(x^2-(y-h)^2\right)}{(1+k)\left(x^2+(y-h)^2\right)^2}\right] \quad \text{(applied force)} \tag{2.9a}$$

$$- \frac{f_y}{2\pi}\left[\frac{y+h}{x^2+(y+h)^2} - \frac{2(y+h)\left(x^2-(y+h)^2\right)}{(1+k)\left(x^2+(y+h)^2\right)^2}\right] \quad \text{(image force)} \tag{2.9b}$$

$$+ \frac{f_y}{2\pi}\left[\frac{2(k-1)(y+h)\left(x^2-(y+h)^2\right)}{(1+k)\left(x^2+(y+h)^2\right)^2}\right] \quad \text{(image dislocation)} \tag{2.9c}$$

$$+ \frac{f_y}{2\pi}\left[\frac{8h(y+h)^2\left(3x^2-(y+h)^2\right)}{(1+k)\left(x^2+(y+h)^2\right)^3} - \frac{2(k-1)h\left(x^2-(y+h)^2\right)}{(1+k)\left(x^2+(y+h)^2\right)^2}\right]$$
$$\text{(image double force without moment)} \tag{2.9d}$$

$$- \frac{f_y}{2\pi}\left[\frac{8h^2(y+h)\left(3x^2-(y+h)^2\right)}{(1+k)\left(x^2+(y+h)^2\right)^3}\right]. \quad \text{(image double moment)} \tag{2.9e}$$

The Green's function for the traction-free half-plane subjected to a concentrated vertical force as given by (2.9) consists of two parts. The first term on the right-hand side of (2.9a) represents the Green's function for a concentrated force applied in the infinite plane at $(0, h)$; the other terms ((2.9b)–(2.9e)) represent four Green's functions for the infinite plane with obtain loads at $(0, -h)$ which is outside the half-plane. Since $\sigma_{yy} = \phi_{y,x}$, we obtain

$$\sigma_{yy} = \left(\phi^0_{y,x}\right)_{\text{applied singularity}} + \left(\phi^{f_y}_{y,x} + \phi^{b_x}_{y,x} + \phi^{P_x}_{y,x} + \phi^{M_x}_{y,x}\right)_{\text{image singularity}}.$$

The five terms are exactly the same as the terms in (2.9a), (2.9b), (2.9c), (2.9d), and (2.9e).

2.3. Image Singularities of a Half-Plane Subjected to a Vertical Force (fixed boundary)

Now, we consider the problem similar to that discussed in the previous section but changed to the rigidly fixed boundary condition. The boundary conditions are

$$u_x(x, 0) = u_y(x, 0) = 0.$$

Since the boundary conditions are displacement conditions, only the fundamental solution corresponding to the displacement field is needed. The displacement field of an infinite plane subjected to a vertical concentrated force f_y at the object point, $y = h$, is (reduced from the solution of Ting [7], p. 247)

$$\begin{aligned} u_x^o &= \frac{f_y}{2\pi\mu(1+k)} \frac{x(y-h)}{x^2+(y-h)^2}, \\ u_y^o &= \frac{f_y}{2\pi\mu(1+k)} \left[\frac{(y-h)^2}{x^2+(y-h)^2} - k\ln\sqrt{x^2+(y-h)^2} \right]. \end{aligned} \tag{2.10}$$

To eliminate the term with $\ln()$ on the boundary, we consider a negative vertical force $-f_y$ applied at the image point $y = -h$. The displacement field is

$$\begin{aligned} u_x^{-f_y} &= \frac{-f_y}{2\pi\mu(1+k)} \frac{x(y+h)}{x^2+(y+h)^2}, \\ u_y^{-f_y} &= \frac{-f_y}{2\pi\mu(1+k)} \left[\frac{(y+h)^2}{x^2+(y+h)^2} - k\ln\sqrt{x^2+(y+h)^2} \right]. \end{aligned} \tag{2.11}$$

Add (2.11) to (2.10) to obtain

$$u_x = \frac{-f_y h}{\pi\mu(1+k)} \frac{x}{x^2+h^2}, \quad u_y = 0, \tag{2.12}$$

on the boundary $y = 0$. A load with stress singularity of order $1/r^2$ at image point will induce the displacement field as expressed by (2.12). We naturally find the same type of fundamental solution to superpose on the image point, $y = -h$. The corresponding solution is a double force without moment along the x-direction. The displacement field is (reduced from the solution of Ting [7], p. 306)

$$\begin{aligned} u_x^{P_x} &= \frac{P_x}{2\pi\mu(1+k)} \left[\frac{kx}{x^2+(y+h)^2} - \frac{2x(y+h)^2}{\left(x^2+(y+h)^2\right)^2} \right], \\ u_y^{P_x} &= \frac{P_x}{2\pi\mu(1+k)} \frac{x^2(y+h)-(y+h)^3}{\left(x^2+(y+h)^2\right)^2}. \end{aligned} \tag{2.13}$$

The magnitude of the double force without moment is chosen to be

$$P_x = \frac{2f_y h}{k}.$$

By setting $y = 0$ in (2.13) and adding to (2.12), we obtain the following displacement field on the boundary $y = 0$:

$$u_x = \frac{-2f_y h^2}{\pi\mu k(1+k)} \frac{xh}{(x^2+h^2)^2}, \quad u_y = \frac{f_y h^2}{\pi\mu k(1+k)} \frac{x^2-h^2}{(x^2+h^2)^2}. \tag{2.14}$$

A double moment along the x-direction at the image point $y = -h$ is chosen to cancel the displacements expressed in (2.14). The solutions are obtained by differentiating the solution of the displacement field corresponding to a concentrated moment applied in an infinite plane:

$$u_x^{M_x} = \frac{M_x}{\pi\mu}\frac{-x(y+h)}{\left(x^2+(y+h)^2\right)^2}, \quad u_y^{M_x} = \frac{M_x}{2\pi\mu}\frac{x^2-(y+h)^2}{\left(x^2+(y+h)^2\right)^2}, \tag{2.15}$$

with the magnitude

$$M_x = \frac{-2f_y h^2}{k(1+k)}.$$

Setting $y = 0$ in (2.15) and adding (2.15) to (2.14), the displacements on the boundary $y = 0$ are

$$u_x = 0, \quad u_y = 0.$$

Thus, the applied singularities on the image point are

$$\vec{u}_{f_y}^{\,i} = \vec{u}^{\,-f_y} + \vec{u}^{\,P_x} + \vec{u}^{\,M_x}. \tag{2.16}$$

It is concluded that the structure of image singularities is negative vertical point force, $-f_y$; double force without moment along the x-direction, P_x; and double moment along the x-direction, M_x.

2.4. Remarks and the Limiting Case

The image singularity of a half-plane with traction-free or rigidly fixed boundary condition subjected to a vertical concentrated force has been presented in this section. There are some distinctions and characteristics of the image singularity between these two different boundary conditions.

(1) There are four image singularities for the traction-free boundary condition, but for the rigidly fixed boundary condition, there are only three. The extra image singularity for the traction-free boundary is a dislocation. It is also interesting to note that image singularities for applying concentrated force are not only the force system (i.e., concentrated force, double force without moment, and double moment) but also dislocations.
(2) The magnitudes of image singularities are very similar for these two boundary conditions. The magnitude of image singularities for double force and double moment for traction-free boundary is $-k$ times the value for the fixed boundary.
(3) The magnitudes of image singularity with stress singularity of order $1/r^2$ (double force without moment) and $1/r^3$ (double moment) are proportional to h and h^2, respectively. These two higher order singularities will disappear as the applied vertical force f_y approaches the half-plane surface.

For the traction-free boundary, when the applied concentrated force f_y approaches the boundary, the problem reduces to that of a half-plane subjected to a surface concentrated force, the image singularity with high-order terms (i.e., double force without moment and double moment) will disappear as we set $h = 0$ on (2.9d) and (2.9e). Only the concentrated force f_y and dislocation b_x will exist on the image point. Note that when the applied force approaches the boundary, the object point and image point coincide. This indicates that the solution for a half-plane subjected to a vertical concentrated force applied on the boundary, $x = 0$ and $y = 0$, is equal to the sum of the solution for an infinite plane subjected to two vertical point forces and one dislocation applied on the material point $x = 0$ and $y = 0$.

There is only one negative force f_y left on the image point for a rigidly fixed boundary when the applied concentrated force approaches the boundary. On the object point, the concentrated force is f_y, but the magnitude of the image singularity is $-f_y$ on the image point. When these two points are merged into the same point at the plane boundary, the total force is zero on the boundary. This means that the displacements and stresses will be zero in the entire half-plane.

3. IMAGE SINGULARITIES OF A HALF-PLANE SUBJECTED TO OTHER FORCES

Based on procedures similar to those presented in the previous section, we will investigate the image singularities for a half-plane subjected to a horizontal concentrated force or dislocations. Avoiding the tedium of algebraic derivation, we will present only the final results here.

3.1. Half-Plane Subjected to a Horizontal Concentrated Force

3.1.1. Traction-Free Boundary

The solutions of stress functions for a horizontal concentrated force f_x applied on the object point in an infinite plane are

$$\begin{aligned}
\phi_x^o &= \frac{f_x}{\pi(1+k)}\left[\frac{x(y-h)}{x^2+(y-h)^2} - \frac{1+k}{2}\tan^{-1}\frac{x}{y-h}\right], \\
\phi_y^o &= \frac{f_x}{\pi(1+k)}\left[\frac{(y-h)^2}{x^2+(y-h)^2} - \frac{1-k}{2}\ln\sqrt{x^2+(y-h)^2}\right].
\end{aligned} \tag{3.1}$$

The components of image singularities can be obtained by a procedure similar to that used in Section 2.2. The image singularities at $y = -h$ include horizontal concentrated force, f_x; dislocation with vertical Burger's vector, b_y; double force with moment along the x-direction, C_x; and double moment along the y-direction, M_y. The result can be expressed as

$$\vec{\phi}^{\,i}_{f_x} = \vec{\phi}^{\,f_x} + \vec{\phi}^{\,b_y} + \vec{\phi}^{\,C_x} + \vec{\phi}^{\,M_y}. \tag{3.2}$$

The stress functions for image singularities are represented as

$$\phi_x^{f_x} = \frac{f_x}{\pi(1+k)}\left[\frac{x(y+h)}{x^2+(y+h)^2} - \frac{1+k}{2}\tan^{-1}\frac{x}{y+h}\right],$$

$$\phi_y^{f_x} = \frac{f_x}{\pi(1+k)}\left[\frac{(y+h)^2}{x^2+(y+h)^2} - \frac{1-k}{2}\ln\sqrt{x^2+(y+h)^2}\right], \tag{3.3}$$

$$\phi_x^{b_y} = \frac{-2\mu b_y}{\pi(1+k)}\left[\frac{x(y+h)}{x^2+(y+h)^2}\right],$$

$$\phi_y^{b_y} = \frac{-2\mu b_y}{\pi(1+k)}\left[\frac{(y+h)^2}{x^2+(y+h)^2} - \ln\sqrt{x^2+(y+h)^2}\right],\ b_y = \frac{1-k}{2\mu}f_x, \tag{3.4}$$

$$\phi_x^{C_x} = \frac{-C_x}{\pi(1+k)}\left[\frac{1+k}{2}\frac{x}{x^2+(y+h)^2} - \frac{x^3-x(y+h)^2}{\left(x^2+(y+h)^2\right)^2}\right],$$

$$\phi_y^{C_x} = \frac{-C_x}{\pi(1+k)}\left[\frac{1+k}{2}\frac{y+h}{x^2+(y+h)^2} - \frac{x^2(y+h)-(y+h)^3}{\left(x^2+(y+h)^2\right)^2}\right], C_x = 2f_x h, \tag{3.5}$$

$$\phi_x^{M_y} = \frac{M_y}{\pi}\frac{2x(y+h)}{\left(x^2+(y+h)^2\right)^2},$$

$$\phi_y^{M_y} = \frac{M_y}{\pi}\frac{(y+h)^2-x^2}{\left(x^2+(y+h)^2\right)^2}, \quad M_y = \frac{2f_x h^2}{1+k}. \tag{3.6}$$

3.1.2. Rigidly Fixed Boundary

If the boundary is rigidly fixed, the displacement fields for a horizontal force f_x applied at the object point are

$$u_x^o = \frac{-f_x}{2\pi\mu(1+k)}\left[\frac{(y-h)^2}{x^2+(y-h)^2} + k\ln\sqrt{x^2+(y-h)^2}\right],$$

$$u_y^o = \frac{f_x}{2\pi\mu(1+k)}\frac{x(y-h)}{x^2+(y-h)^2}. \tag{3.7}$$

The image singularities for this case include negative horizontal concentrated force, $-f_x$; double force with moment along the x-direction, C_x; and double moment along the y-direction, M_y. The result is

$$\vec{u}_{f_x}^{\,i} = \vec{u}^{-f_x} + \vec{u}^{C_x} + \vec{u}^{M_y}. \tag{3.8}$$

The displacements for image singularities are

$$
\begin{aligned}
u_x^{-f_x} &= \frac{f_x}{2\pi\mu(1+k)}\left[\frac{(y+h)^2}{x^2+(y+h)^2}+k\ln\sqrt{x^2+(y+h)^2}\right], \\
u_y^{-f_x} &= \frac{-f_x}{2\pi\mu(1+k)}\frac{x(y+h)}{x^2+(y+h)^2},
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
u_x^{C_x} &= \frac{C_x}{2\pi\mu(1+k)}\left[\frac{y+h}{x^2+(y+h)^2}-\frac{2x^2(y+h)}{\left(x^2+(y+h)^2\right)^2}\right], \\
u_y^{C_x} &= \frac{-C_x}{2\pi\mu(1+k)}\left[\frac{kx}{x^2+(y+h)^2}+\frac{2x(y+h)^2}{\left(x^2+(y+h)^2\right)^2}\right], \quad C_x=\frac{-2f_x h}{k},
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
u_x^{M_y} &= \frac{M_y}{2\pi\mu}\frac{x^2-(y+h)^2}{\left(x^2+(y+h)^2\right)^2}, \\
u_y^{M_y} &= \frac{M_y}{\pi\mu}\frac{x(y+h)}{\left(x^2+(y+h)^2\right)^2}, \quad M_y=\frac{-2f_x h^2}{k(1+k)}.
\end{aligned} \tag{3.11}
$$

3.2. Half-Plane Subjected to a Dislocation with Burger's Vector along x-Direction

3.2.1. Traction-Free Boundary

The stress functions of a dislocation b_x acting on the object point of an infinite plane are

$$
\begin{aligned}
\phi_x^o &= \frac{\mu b_x}{\pi(1+k)}\left[\frac{2(y-h)^2}{x^2+(y-h)^2}+2\ln\sqrt{x^2+(y-h)^2}\right] \\
\phi_y^o &= \frac{-\mu b_x}{\pi(1+k)}\left[\frac{2x(y-h)}{x^2+(y-h)^2}\right].
\end{aligned} \tag{3.12}
$$

The image singularities for this case are

$$
\vec{\phi}_{b_x}^{\,i}=\vec{\phi}^{\,-b_x}+\vec{\phi}^{\,\gamma_x}+\vec{\phi}^{\,\chi_y}. \tag{3.13}
$$

The image singularities include dislocation with negative horizontal Burger's vector, b_x; center of extension along the x-direction, γ_x; and double center of expansion along the y-direction, χ_y; see Figure 4. The stress functions for image singularities are

$$
\phi_x^{-b_x} = \frac{-\mu b_x}{\pi(1+k)}\left[\frac{2(y+h)^2}{x^2+(y+h)^2}+2\ln\sqrt{x^2+(y+h)^2}\right],
$$

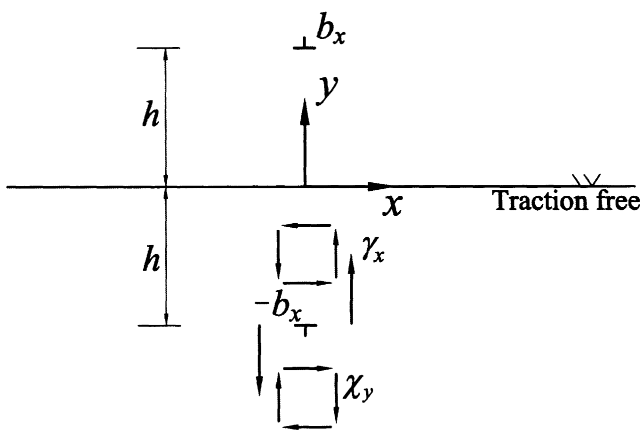

Fig. 4. Image singularities of the Green's function for a half-plane (free boundary) subjected to a dislocation b_x.

$$\phi_y^{-b_x} = \frac{\mu b_x}{\pi(1+k)}\left[\frac{2x(y+h)}{x^2+(y+h)^2}\right], \tag{3.14}$$

$$\phi_x^{\gamma_x} = \frac{2\mu\gamma_x}{\pi(1+k)}\left[\frac{y+h}{x^2+(y+h)^2} - \frac{2x^2(y+h)}{\left(x^2+(y+h)^2\right)^2}\right],$$

$$\phi_y^{\gamma_x} = \frac{-2\mu\gamma_x}{\pi(1+k)}\left[\frac{x}{x^2+(y+h)^2} + \frac{2x(y+h)^2}{\left(x^2+(y+h)^2\right)^2}\right], \quad \gamma_x = 2b_x h, \tag{3.15}$$

$$\phi_x^{\chi_y} = \frac{4\mu\chi_y}{\pi(1+k)}\frac{(y+h)^2-x^2}{\left(x^2+(y+h)^2\right)^2},$$

$$\phi_y^{\chi_y} = \frac{-8\mu\chi_y}{\pi(1+k)}\frac{x(y+h)}{\left(x^2+(y+h)^2\right)^2}, \quad \chi_y = -b_x h^2. \tag{3.16}$$

3.2.2. Rigidly Fixed Boundary

The displacements for a dislocation acting on the object point of an infinite plane are

$$u_x^o = \frac{b_x}{\pi(1+k)}\left[\frac{x(y-h)}{x^2+(y-h)^2} - \frac{1+k}{2}\tan^{-1}\frac{x}{y-h}\right],$$

$$u_y^o = \frac{b_x}{\pi(1+k)}\left[\frac{1-k}{2}\ln\sqrt{x^2+(y-h)^2}+\frac{(y-h)^2}{x^2+(y-h)^2}\right]. \tag{3.17}$$

The image singularities include dislocation with horizontal Burger's vector, b_x; vertical concentrated force, f_y; center of extension along the x-direction, γ_x; and double center of expansion along the y-direction, χ_y. We have

$$\vec{u}^{\,i}_{b_x} = \vec{u}^{\,b_x} + \vec{u}^{\,f_y} + \vec{u}^{\,\gamma_x} + \vec{u}^{\,\chi_y}. \tag{3.18}$$

The complete displacement fields for image singularities are

$$u_x^{b_x} = \frac{b_x}{\pi(1+k)}\left[\frac{x(y+h)}{x^2+(y+h)^2}-\frac{1+k}{2}\tan^{-1}\frac{x}{y+h}\right],$$

$$u_y^{b_x} = \frac{b_x}{\pi(1+k)}\left[\frac{1-k}{2}\ln\sqrt{x^2+(y+h)^2}+\frac{(y+h)^2}{x^2+(y+h)^2}\right], \tag{3.19}$$

$$u_x^{f_y} = \frac{f_y}{2\pi\mu(1+k)}\frac{x(y+h)}{x^2+(y+h)^2},$$

$$u_y^{f_y} = \frac{f_y}{2\pi\mu(1+k)}\left[\frac{(y+h)^2}{x^2+(y+h)^2}-k\ln\sqrt{x^2+(y+h)^2}\right],$$

$$f_y = \frac{2\mu(1-k)}{k}b_x, \tag{3.20}$$

$$u_x^{\gamma_x} = \frac{\gamma_x}{2\pi(1+k)}\left[(1-k)\frac{x}{x^2+(y+h)^2}+\frac{4x(y+h)^2}{\left(x^2+(y+h)^2\right)^2}\right],$$

$$u_y^{\gamma_x} = \frac{\gamma_x}{2\pi(1+k)}\left[(3+k)\frac{y+h}{x^2+(y+h)^2}-\frac{4x^2(y+h)}{\left(x^2+(y+h)^2\right)^2}\right],$$

$$\gamma_x = \frac{-2b_x h}{k}, \tag{3.21}$$

$$u_x^{\chi_y} = \frac{4\chi_y}{\pi(1+k)}\frac{x(y+h)}{\left(x^2+(y+h)^2\right)^2},$$

$$u_y^{\chi_y} = \frac{2\chi_y}{\pi(1+k)}\frac{(y+h)^2-x^2}{\left(x^2+(y+h)^2\right)^2}, \quad \chi_y = \frac{b_x h^2}{k}. \tag{3.22}$$

3.3. Half-Plane Subjected to a Dislocation with Burger's Vector along y-Direction

3.3.1. Traction-Free Boundary

The stress functions for dislocation b_y acting on the object point of an infinite plane are

$$\begin{aligned}
\phi_x^o &= \frac{-2\mu b_y}{\pi(1+k)}\left[\frac{x(y-h)}{x^2+(y-h)^2}\right],\\
\phi_y^o &= \frac{-2\mu b_y}{\pi(1+k)}\left[\frac{(y-h)^2}{x^2+(y-h)^2}-\ln\sqrt{x^2+(y-h)^2}\right].
\end{aligned} \tag{3.23}$$

The image singularities on the image point are

$$\vec{\phi}^{\,i}_{b_y} = \vec{\phi}^{\,-b_y} + \vec{\phi}^{\,\kappa_x} + \vec{\phi}^{\,\chi_x}. \tag{3.24}$$

The image singularities for this case are dislocation with negative vertical Burger's vector, b_y; center of shear along the x-direction, κ_x; and double center of expansion along the x-direction, χ_x. The stress functions for image singularities are

$$\begin{aligned}
\phi_x^{-b_y} &= \frac{2\mu b_y}{\pi(1+k)}\left[\frac{x(y+h)}{x^2+(y+h)^2}\right],\\
\phi_y^{-b_y} &= \frac{2\mu b_y}{\pi(1+k)}\left[\frac{(y+h)^2}{x^2+(y+h)^2}-\ln\sqrt{x^2+(y+h)^2}\right],
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\phi_x^{\kappa_x} &= \frac{2\mu\kappa_x}{\pi(1+k)}\left[\frac{x}{x^2+(y+h)^2}-\frac{2x(y+h)^2}{\left(x^2+(y+h)^2\right)^2}\right],\\
\phi_y^{\kappa_x} &= \frac{-2\mu\kappa_x}{\pi(1+k)}\left[\frac{y+h}{x^2+(y+h)^2}-\frac{2x^2(y+h)}{\left(x^2+(y+h)^2\right)^2}\right], \quad \kappa_x = -2b_y h,
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\phi_x^{\chi_x} &= \frac{8\mu\chi_x}{\pi(1+k)}\frac{x(y+h)}{\left(x^2+(y+h)^2\right)^2},\\
\phi_y^{\chi_x} &= \frac{4\mu\chi_x}{\pi(1+k)}\frac{(y+h)^2-x^2}{\left(x^2+(y+h)^2\right)^2}, \quad \chi_x = b_y h^2.
\end{aligned} \tag{3.27}$$

3.3.2. Rigidly Fixed Boundary

The displacement fields in an infinite plane subjected to a dislocation b_y on the object point $x=0, y=h$, are

$$
\begin{aligned}
u_x^o &= \frac{-b_y}{\pi(1+k)}\left[\frac{1-k}{2}\ln\sqrt{x^2+(y-h)^2}-\frac{(y-h)^2}{x^2+(y-h)^2}\right],\\
u_y^o &= \frac{-b_y}{\pi(1+k)}\left[\frac{x(y-h)}{x^2+(y-h)^2}+\frac{1+k}{2}\tan^{-1}\frac{x}{y-h}\right].
\end{aligned}
\tag{3.28}
$$

The image singularities on the image point include dislocation with vertical Burger's vector, b_y; horizontal point force, f_x; center of shear along the x-direction, κ_x; and double center of expansion along the x-direction, χ_x. We have

$$
\vec{u}^{\,i}_{b_y} = \vec{u}^{\,b_y} + \vec{u}^{\,f_x} + \vec{u}^{\,\kappa_x} + \vec{u}^{\,\chi_x}.
$$

The displacement fields for image singularities are

$$
\begin{aligned}
u_x^{b_y} &= \frac{-b_y}{\pi(1+k)}\left[\frac{1-k}{2}\ln\sqrt{x^2+(y+h)^2}-\frac{(y+h)^2}{x^2+(y+h)^2}\right],\\
u_y^{b_y} &= \frac{-b_y}{\pi(1+k)}\left[\frac{x(y+h)}{x^2+(y+h)^2}+\frac{1+k}{2}\tan^{-1}\frac{x}{y+h}\right],
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
u_x^{f_x} &= \frac{-f_x}{2\pi\mu(1+k)}\left[\frac{(y+h)^2}{x^2+(y+h)^2}+k\ln\sqrt{x^2+(y+h)^2}\right],\\
u_y^{f_x} &= \frac{f_x}{2\pi\mu(1+k)}\frac{x(y+h)}{x^2+(y+h)^2}, \quad f_x = \frac{2(k-1)\mu}{k}b_y,
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
u_x^{\kappa_x} &= \frac{\kappa_x}{2\pi(1+k)}\left[(1-k)\frac{y+h}{x^2+(y+h)^2}-\frac{4x^2(y+h)}{\left(x^2+(y+h)^2\right)^2}\right],\\
u_y^{\kappa_x} &= \frac{\kappa_x}{2\pi(1+k)}\left[(1-k)\frac{x}{x^2+(y+h)^2}-\frac{4x(y+h)^2}{\left(x^2+(y+h)^2\right)^2}\right], \kappa_x = \frac{2b_y h}{k},
\end{aligned}
\tag{3.31}
$$

$$
\begin{aligned}
u_x^{\chi_x} &= \frac{2\chi_x}{\pi(1+k)}\frac{x^2-(y+h)^2}{\left(x^2+(y+h)^2\right)^2},\\
u_y^{\chi_x} &= \frac{4\chi_x}{\pi(1+k)}\frac{x(y+h)}{\left(x^2+(y+h)^2\right)^2}, \quad \chi_x = \frac{b_y h^2}{k}.
\end{aligned}
\tag{3.32}
$$

3.4. Remarks

(1) The number of image singularities for a horizontal concentrated force is the same as that for a vertical concentrated force for traction-free and rigidly fixed boundary conditions. The characteristics of image singularities for these two applied forces are similar.

(2) The number of image singularities for dislocations is different for these two boundary conditions. There are four image singularities when the boundary is rigidly fixed, and three when the boundary is traction free. The extra image singularity for the fixed boundary is a concentrated force.
(3) The higher order image singularities such as double force with moment C_x, center of extension γ_x, and center of shear κ_x, have $O(1/r^2)$ stress singularity and are proportional to h. However, double center of expansion x and double moment M with the stress singularity order $O(1/r^3)$ are proportional to h^2. As the applied singularity on the object point approaches the half-plane surface, that is $h \to 0$, all the higher order image singularities will disappear and only concentrated forces and dislocations will remain.
(4) The magnitudes of higher order singularities mentioned in (3) are very similar for these two boundary conditions. For instance, the magnitudes of γ_x, κ_x, χ_x, and χ_y of traction-free boundary are $-k$ times those of rigidly fixed boundary.
(5) As the applied dislocation approaches the traction-free boundary, we obtain the well-known result that displacements and stresses will vanish in the entire half-plane.

4. REPRESENTATION OF IMAGE SINGULARITIES IN MATRIX FORM

In the previous two sections, we have constructed the complete structure of image singularities for forces or dislocations applied in a half-plane. The image singularities include three different orders of stress singularity which are concentrated force and dislocation with the singular behavior $O(1/r)$, double force without moment, double force with moment, center of extension and center of shear with the singular behavior $O(1/r^2)$, and double center of expansion and double moment with the singular behavior $O(1/r^3)$. However, all the solutions for higher order image singularities (i.e., $O(1/r^2)$ and $O(1/r^3)$) can be obtained by differentiation of the solutions for concentrated force and dislocation. We define the stress function such that

$$\vec{\varphi}^T = \vec{\phi}^T/\mathbf{M},$$

where the superscript T indicates the type of singularity and $\mathbf{M}$ is the magnitude of the singularity. For instance, if $T \equiv f_y$ or $T \equiv b_x$, we have

$$\begin{aligned} \vec{\varphi}^{f_y} &= \left(\varphi_x^{f_y}, \varphi_y^{f_y}\right) = \left(\phi_x^{f_y}, \phi_y^{f_y}\right)/f_y, \\ \vec{\varphi}^{b_x} &= \left(\varphi_x^{b_x}, \varphi_y^{b_x}\right) = \left(\phi_x^{b_x}, \phi_y^{b_x}\right)/b_x. \end{aligned}$$

The stress functions of higher order image singularities can be represented by the solutions of concentrated forces and dislocations as

$$\vec{\varphi}^{P_x} = -\frac{\partial \vec{\varphi}^{f_x}}{\partial x}, \quad \vec{\varphi}^{C_x} = -\frac{\partial \vec{\varphi}^{f_y}}{\partial x},$$

$$
\begin{aligned}
\vec{\varphi}^{\kappa_x} &= -\frac{\partial \vec{\varphi}^{b_x}}{\partial x}, \quad \vec{\varphi}^{\gamma_x} = -\frac{\partial \vec{\varphi}^{b_y}}{\partial x},\\
\vec{\varphi}^{M_x} &= \frac{\partial}{\partial x}\left(\frac{\partial \vec{\varphi}^{f_y}}{\partial x} + \frac{\partial \vec{\varphi}^{f_x}}{\partial y}\right), \quad \vec{\varphi}^{M_y} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{\varphi}^{f_y}}{\partial x} + \frac{\partial \vec{\varphi}^{f_x}}{\partial y}\right),\\
\vec{\varphi}^{\chi_x} &= \frac{\partial}{\partial x}\left(\frac{\partial \vec{\varphi}^{b_y}}{\partial x} + \frac{\partial \vec{\varphi}^{b_x}}{\partial y}\right), \quad \vec{\varphi}^{\chi_y} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{\varphi}^{b_y}}{\partial x} + \frac{\partial \vec{\varphi}^{b_x}}{\partial y}\right).
\end{aligned}
$$

Hence, the fundamental solutions we need for constructing all the image singularities for the half-plane are only the solution of concentrated forces and dislocations in an infinite plane. Finally, all the results of image singularities discussed in the previous two sections can be summarized and represented in the following compact matrix form:

Traction-Free Boundary

$$
\begin{aligned}
\begin{bmatrix} \vec{\phi}^{\,i}_{f_x} \\ \vec{\phi}^{\,i}_{f_y} \\ \vec{\phi}^{\,i}_{b_x} \\ \vec{\phi}^{\,i}_{b_y} \end{bmatrix}
&= \begin{bmatrix} f_x & 0 & 0 & \frac{1-k}{2\mu} f_x \\ 0 & f_y & -\frac{1-k}{2\mu} f_y & 0 \\ 0 & 0 & -b_x & 0 \\ 0 & 0 & 0 & -b_y \end{bmatrix}
\begin{bmatrix} \vec{\varphi}^{f_x} \\ \vec{\varphi}^{f_y} \\ \vec{\varphi}^{b_x} \\ \vec{\varphi}^{b_y} \end{bmatrix}
\\
&+ \begin{bmatrix} 0 & 2f_x h & 0 & 0 \\ -2f_y h & 0 & 0 & 0 \\ 0 & 0 & 0 & 2b_x h \\ 0 & 0 & -2b_y h & 0 \end{bmatrix}
\begin{bmatrix} -\frac{\partial \vec{\varphi}^{f_x}}{\partial x} \\ -\frac{\partial \vec{\varphi}^{f_y}}{\partial x} \\ -\frac{\partial \vec{\varphi}^{b_x}}{\partial x} \\ -\frac{\partial \vec{\varphi}^{b_y}}{\partial x} \end{bmatrix}
\\
&+ \begin{bmatrix} 0 & \frac{2f_x h^2}{1+k} & 0 & 0 \\ \frac{2f_y h^2}{1+k} & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_x h^2 \\ 0 & 0 & -b_y h^2 & 0 \end{bmatrix}
\begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial \vec{\varphi}^{f_y}}{\partial x} + \frac{\partial \vec{\varphi}^{f_x}}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \vec{\varphi}^{f_y}}{\partial x} + \frac{\partial \vec{\varphi}^{f_x}}{\partial y}\right) \\ \frac{\partial}{\partial x}\left(\frac{\partial \vec{\varphi}^{b_y}}{\partial x} + \frac{\partial \vec{\varphi}^{b_x}}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial \vec{\varphi}^{b_y}}{\partial x} + \frac{\partial \vec{\varphi}^{b_x}}{\partial y}\right) \end{bmatrix}.
\end{aligned}
$$

Rigidly Fixed Boundary

$$
\begin{bmatrix} \vec{\phi}^{\,i}_{f_x} \\ \vec{\phi}^{\,i}_{f_y} \\ \vec{\phi}^{\,i}_{b_x} \\ \vec{\phi}^{\,i}_{b_y} \end{bmatrix}
=
\begin{bmatrix}
-f_x & 0 & 0 & 0 \\
0 & -f_y & 0 & 0 \\
0 & \dfrac{2\mu(1-k)}{k} b_x & b_x & 0 \\
-\dfrac{2\mu(1-k)}{k} b_y & 0 & b_y & 0
\end{bmatrix}
\begin{bmatrix} \vec{\varphi}^{f_x} \\ \vec{\varphi}^{f_y} \\ \vec{\varphi}^{b_x} \\ \vec{\varphi}^{b_y} \end{bmatrix}
- \frac{1}{k}
\begin{bmatrix}
0 & 2f_x h & 0 & 0 \\
-2f_y h & 0 & 0 & 0 \\
0 & 0 & 0 & 2b_x h \\
0 & 0 & -2b_y h & 0
\end{bmatrix}
\begin{bmatrix}
-\dfrac{\partial \vec{\varphi}^{f_x}}{\partial x} \\
-\dfrac{\partial \vec{\varphi}^{f_y}}{\partial x} \\
-\dfrac{\partial \vec{\varphi}^{b_x}}{\partial x} \\
-\dfrac{\partial \vec{\varphi}^{b_y}}{\partial x}
\end{bmatrix}
- \frac{1}{k}
\begin{bmatrix}
0 & \dfrac{2f_x h^2}{1+k} & 0 & 0 \\
\dfrac{2f_y h^2}{1+k} & 0 & 0 & 0 \\
0 & 0 & 0 & -b_x h^2 \\
0 & 0 & -b_y h^2 & 0
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial}{\partial x}\left(\dfrac{\partial \vec{\varphi}^{f_y}}{\partial x} + \dfrac{\partial \vec{\varphi}^{f_x}}{\partial y}\right) \\
\dfrac{\partial}{\partial y}\left(\dfrac{\partial \vec{\varphi}^{f_y}}{\partial x} + \dfrac{\partial \vec{\varphi}^{f_x}}{\partial y}\right) \\
\dfrac{\partial}{\partial x}\left(\dfrac{\partial \vec{\varphi}^{b_y}}{\partial x} + \dfrac{\partial \vec{\varphi}^{b_x}}{\partial y}\right) \\
\dfrac{\partial}{\partial y}\left(\dfrac{\partial \vec{\varphi}^{b_y}}{\partial x} + \dfrac{\partial \vec{\varphi}^{b_x}}{\partial y}\right)
\end{bmatrix}
$$

5. CONCLUSIONS

Although the Green's function for an isotropic elastic half-plane is well-known, the physical meaning of the solution is not clear. This study shows that the Green's function for the half-plane consists of several Green's functions for the infinite plane subjected to various forms of line singularities. This is useful information that puts the solution in a different perspective. The image singularities for a half-plane have been derived and discussed in detail in this study. All of the results of image singularities for applied singularity on the object point and different boundary conditions are summarized and listed in Tables 1 and 2. It is interesting to note that image singularities of concentrated forces for a traction-free boundary are not only force systems but also dislocations. In addition, image singularities of dislocations for rigidly fixed boundaries also include concentrated forces. The image singularities for an isotropic half-plane consist of concentrated forces and dislocations with the order of stress singularity $O(1/r)$ and higher order singularities ($O(1/r^2)$ and $O(1/r^3)$). However, the solutions of all

Table 1. Applied singularities (concentrated forces and dislocations) and the corresponding image singularities for traction-free boundary condition.

Image singularity	Applied singularity f_x	f_y	b_x	b_y
Horizontal body force f_x	▲			
Vertical body force f_y		▲		
Dislocation b_x		▲	▲	
Dislocation b_y	▲			▲
Double force without moment along the x-direction P_x		▲		
Double force with moment along the x-direction C_x	▲			
Center of extension along the x-direction γ_x			▲	
Center of shear along the x-direction κ_x				▲
Double center of expansion along the x-direction χ_x				▲
Double center of expansion along the y-direction χ_y			▲	
Double moment along the x-direction M_x		▲		
Double moment along the y-direction M_y	▲			

Table 2. Applied singularities (concentrated forces and dislocations) and the corresponding image singularities for rigidly fixed boundary condition.

Image singularity	Applied singularity f_x	f_y	b_x	b_y
Horizontal body force f_x	▲			▲
Vertical body force f_y		▲	▲	
Dislocation b_x			▲	
Dislocation b_y				▲
Double force without moment along the x-direction P_x		▲		
Double force with moment along the x-direction C_x	▲			
Center of extension along the x-direction γ_x			▲	
Center of shear along the x-direction κ_x				▲
Double center of expansion along the x-direction χ_x				▲
Double center of expansion along the y-direction χ_y			▲	
Double moment along the x-direction M_x		▲		
Double moment along the y-direction M_y	▲			

the higher order singularities can be obtained by differentiating the solutions of concentrated forces and dislocations. Hence, the fundamental solutions we need for constructing the image singularities are only the solution of concentrated forces and dislocations in an infinite plane. The image singularities of a half-plane for applied forces and dislocations are represented in a compact matrix form in terms of the solutions of concentrated forces and dislocations. As the applied concentrated forces or dislocations approach the half-plane surface, the higher order image singularities will disappear and only the image singularities with order $O(1/r)$ (i.e., concentrated forces and dislocations) will remain in the solution. Based on the results found in this study, the image singularities for higher order applied singularities (i.e., applied double force, center of extension, center of shear, etc.) for half-plane can be easily derived.

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