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The Mixed-Mode Fracture Analysis of Multiple Embedded Cracks in a Semi-Infinite Plane by an Analytical Alternating Method

An efficient analytical alternating method is developed in this paper to evaluate the mixed-mode stress intensity factors of embedded multiple cracks in a semi-infinite plane. Analytical solutions of a semi-infinite plane subjected to a point force applied on the boundary, and a finite crack in an infinite plane subjected to a pair of point forces applied on the crack faces are referred to as fundamental solutions. The Gauss integrations based on these point load fundamental solutions can precisely simulate the conditions of arbitrarily distributed loads. By using these fundamental solutions in conjunction with the analytical alternating technique, the mixed-mode stress intensity factors of embedded multiple cracks in a semi-infinite plane are evaluated. The numerical results of some reduced problems are compared with available results in the literature and excellent agreements are obtained. [DOI: 10.1115/1.1493204]

Introduction

The in-plane elasticity problems of a system of cracks embedded in a semi-infinite plane with a free boundary present a particular interest in engineering applications, and have drawn the attention of researchers in elasticity long ago. Among the publications concerning crack problems, the handbook of Murakami [1] provided many results of stress intensity factor (S.I.F.) for edge cracks as well as an internal normal crack and a row of internal cracks in a semi-infinite plane under uniform tension. Dowell and Hills [2] computed the S.I.F. of an inclined internal crack in a semi-infinite plane by distributed edge dislocations. Qian and Hasebe [3] analyzed an oblique edge crack and an internal crack in a semi-infinite plane subjected to concentrated forces. They calculated the S.I.F. by superposition of two fundamental solutions; one is the half-plane with an edge crack subjected to concentrated forces; the other is the half-plane with an edge crack subjected to distributed dislocations on the line of the internal crack. Ioakimidis and Theocaris [4] attempted to deal with a half-plane with arbitrarily distributed multiple curvilinear cracks. They applied the complex variable technique to obtain a series of complex singular integral equations and only numerical results for a single curvilinear edge crack or a single internal crack were presented in the paper for the reason of difficulty in numerical computation. Hu and Chanda [5] analyzed the interactions between a crack and rigid lines in a half-plane. An integral equation approach based on the fundamental solutions due to point loads and point dislocations in a half-plane is utilized to evaluate the S.I.F. of the crack. The purpose of this work is to develop an accurate and efficient method to analyze the S.I.F. of arbitrarily distributed multiple cracks in a semi-infinite plane.

In the analysis of elastic fracture problem, the analytical methods are available only for simple problems with special boundary conditions. Finite element method (FEM) is a powerful numerical technique for fracture analysis, but the accuracy of the solutions computed by FEM is largely mesh-dependent. The Schwartz-Neumann alternating technique [6] caught the highlight to overcome the shortcomings of FEM in dealing with multiple crack problems. Essentially, the alternating method is a linear superposition method. The complicated solution of a crack in a finite body can be obtained by iterating between the solution for the noncrack body and the solution for an infinite body with a crack (or cracks). Various methods used to solve the sub-problems of noncrack body lead to various characteristics of Schwartz-Neumann alternating methods.

The analytical alternating methods utilized analytical solutions for the noncrack configuration, as well as for cracks in an infinite plane. Early application of analytical alternating method involved solving the edge crack problem in a semi-infinite plane [7], and surface crack in 3-D body [8]. The method was used later by O'Douoghue et al. [9] to solve multiple embedded elliptical cracks in an infinite body. Zhang and Hasebe [10] studied the interactions between rectilinear and circumferential crack in an infinite domain. The finite element alternating method (FEAM) [11–13] and boundary element alternating method (BEAM) [14,15] extended the utilizable ability of the alternating technique to complicated cracked geometries. In the absence of cracks, FEM or BEM can easily obtain the numerical solution of the noncrack finite body with a coarser mesh.

In this study, we consider the elastic fracture problem for a semi-finite plane with multiple arbitrarily located cracks subjected to distributed loads on the boundary or to a remote tension. The analytical closed-form solutions of a half-plane and a finite crack in an infinite body are used as the fundamental solutions. An efficient procedure is developed to alternate between these two fundamental solutions. The mixed-mode S.I.F. is computed by analytical alternating method, and the interactive effect of crack tips and boundary are discussed. The numerical results obtained by the proposed technique are first verified by comparison with existing solutions for an infinite plane with two parallel cracks [16–19], or two inclined cracks [20]. The problem of a semi-infinite plane with an internal inclined crack [2] is considered next, and excellent agreements are obtained. Finally, the interaction effects between two inclined cracks and the boundary of the semi-infinite plane are shown and discussed in detail. The proposed methodology in this study is proved to be simple, accurate, and stable to evaluate the stress intensity factor for complex crack problems.

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Analytical Fundamental Solutions

In the literature of Schwartz-Neumann alternating method, various types of analytical solutions have been used to analyze different problems. Polynomial and Chebyshev polynomial distributions have been mostly employed to simulate continuously distributed loads. For discontinuously distributed loads, the point loads or piecewise distributed loads could more precisely describe the large variation of the crack surface loads in the limiting case of two cracks close to one another. For simplicity and versatility, all the analytical solutions presented in this section are point load solutions.

A Finite Crack in an Infinite Plane Subjected to a Pair of Point Forces on the Crack Faces. Figure 1 shows an infinite plane containing a finite crack lying along the *x*-axis. The center of the crack is located at the origin, and the crack length is $2a$. The crack face is subjected to a pair of point forces p or q at $x=b$. From the well-known complex variable method, the complex stress functions are found as

$$\begin{bmatrix} Z_{\mathrm{I}} \\ Z_{\mathrm{II}} \end{bmatrix} = \frac{1}{\pi}\begin{bmatrix} p \\ q \end{bmatrix}\frac{\sqrt{a^2-b^2}}{(z-b)\sqrt{z^2-a^2}} \tag{1}$$

$$\begin{bmatrix} Z'_{\mathrm{I}} \\ Z'_{\mathrm{II}} \end{bmatrix} = \frac{-\sqrt{a^2-b^2}}{\pi}\begin{bmatrix} p \\ q \end{bmatrix}\cdot\left(\frac{z}{(z-b)(z^2-a^2)^{3/2}}+\frac{1}{(z-b)(z^2-a^2)^{1/2}}\right) \tag{2}$$

where $z=x+iy$. The stress intensity factors are

$$\begin{bmatrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}\frac{1}{\sqrt{\pi a}}\sqrt{\frac{a+b}{a-b}} \tag{3}$$

The full field stress distributions for the mode I and II problems can be expressed as follows:

(i) Mode I stress field

$$\begin{aligned}\sigma_{xx}&=\mathrm{Re}\,Z_I-y\,\mathrm{Im}\,Z'_I\\ &=\frac{p}{\pi}\sqrt{a^2-b^2}(r_1r_2)^{-1/2}r_3^{-1}\cos\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)\\ &\quad-\frac{yp}{\pi}\sqrt{a^2-b^2}\left[r_0(r_1r_2)^{-3/2}r_3^{-1}\right.\\ &\quad\times\sin\left(-\theta_0+\frac{3(\theta_1+\theta_2)}{2}+\theta_3\right)\\ &\quad\left.+(r_1r_2)^{-1/2}r_3^{-2}\sin\left(\frac{\theta_1+\theta_2}{2}+2\theta_3\right)\right]\end{aligned} \tag{4a}$$

$$\begin{aligned}\sigma_{yy}&=\mathrm{Re}\,Z_I+y\,\mathrm{Im}\,Z'_I\\ &=\frac{p}{\pi}\sqrt{a^2-b^2}(r_1r_2)^{-1/2}r_3^{-1}\cos\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)\\ &\quad+\frac{yp}{\pi}\sqrt{a^2-b^2}\left[r_0(r_1r_2)^{-3/2}r_3^{-1}\right.\\ &\quad\times\sin\left(-\theta_0+\frac{3(\theta_1+\theta_2)}{2}+\theta_3\right)\\ &\quad\left.+(r_1r_2)^{-1/2}r_3^{-2}\sin\left(\frac{\theta_1+\theta_2}{2}+2\theta_3\right)\right]\end{aligned} \tag{4b}$$

$$\begin{aligned}\tau_{xy}&=-y\,\mathrm{Re}\,Z'_I\\ &=\frac{yp}{\pi}\sqrt{a^2-b^2}\left[r_0(r_1r_2)^{-3/2}r_3^{-1}\cos\left(-\theta_0+\frac{3(\theta_1+\theta_2)}{2}+\theta_3\right)\right.\\ &\quad\left.+(r_1r_2)^{-1/2}r_3^{-2}\cos\left(\frac{\theta_1+\theta_2}{2}+2\theta_3\right)\right]\end{aligned} \tag{4c}$$

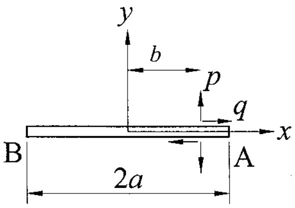

Fig. 1 A finite crack in an infinite plane subjected to point loads

(ii) Mode II stress field

$$\begin{aligned}\sigma_{xx}&=2\,\mathrm{Im}\,Z_{II}+y\,\mathrm{Re}\,Z'_{II}\\ &=\frac{-2q}{\pi}\sqrt{a^2-b^2}(r_1r_2)^{-1/2}r_3^{-1}\sin\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)\\ &\quad-\frac{yq}{\pi}\sqrt{a^2-b^2}\left[r_0(r_1r_2)^{-3/2}r_3^{-1}\right.\\ &\quad\times\cos\left(-\theta_0+\frac{3(\theta_1+\theta_2)}{2}+\theta_3\right)\\ &\quad\left.+(r_1r_2)^{-1/2}r_3^{-2}\cos\left(\frac{\theta_1+\theta_2}{2}+2\theta_3\right)\right]\end{aligned} \tag{5a}$$

$$\begin{aligned}\sigma_{yy}&=-y\,\mathrm{Re}\,Z'_{II}\\ &=\frac{yq}{\pi}\sqrt{a^2-b^2}\left[r_0(r_1r_2)^{-3/2}r_3^{-1}\cos\left(-\theta_0+\frac{3(\theta_1+\theta_2)}{2}+\theta_3\right)\right.\\ &\quad\left.+(r_1r_2)^{-1/2}r_3^{-2}\cos\left(\frac{\theta_1+\theta_2}{2}+2\theta_3\right)\right]\end{aligned} \tag{5b}$$

$$\begin{aligned}\tau_{xy}&=\mathrm{Re}\,Z_{II}-y\,\mathrm{Im}\,Z'_{II}\\ &=\frac{q}{\pi}\sqrt{a^2-b^2}(r_1r_2)^{-1/2}r_3^{-1}\cos\left(\frac{\theta_1+\theta_2}{2}+\theta_3\right)\\ &\quad-\frac{yq}{\pi}\sqrt{a^2-b^2}\left[r_0(r_1r_2)^{-3/2}r_3^{-1}\right.\\ &\quad\times\sin\left(-\theta_0+\frac{3(\theta_1+\theta_2)}{2}+\theta_3\right)\\ &\quad\left.+(r_1r_2)^{-1/2}r_3^{-2}\sin\left(\frac{\theta_1+\theta_2}{2}+2\theta_3\right)\right]\end{aligned} \tag{5c}$$

in which

$$r_0=|z|,\quad \theta_0=\tan^{-1}\left(\frac{y}{x}\right)$$

$$r_1=|z-a|,\quad \theta_1=\tan^{-1}\left(\frac{y}{x-a}\right)$$

$$r_2=|z+a|,\quad \theta_2=\tan^{-1}\left(\frac{y}{x+a}\right)$$

$$r_3=|z-b|,\quad \theta_3=\tan^{-1}\left(\frac{y}{x-b}\right)$$

If the crack face is subjected to an arbitrary distribution of normal traction $p(\xi)$, the stress field can be evaluated by integrating Eq. (4) over the crack length. Similarly, the solution of an arbitrary distribution of shear traction $q(\xi)$ can be achieved by integrating Eq. (5) with respect to the crack length.

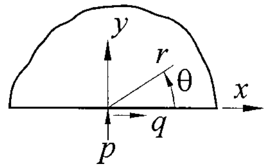

Fig. 2 Semi-infinite plane subjected to point forces applied on the boundary

Semi-Infinite Plane Subjected to Point Forces on the Boundary. Consider a concentrated vertical force with magnitude p acting on the horizontal boundary of a semi-infinite plane as shown in Fig. 2. The distribution of the load along the thickness of the plate is uniform, and p is the load per unit thickness. The solution is given by Timoshenko and Goodier [21]. The stress field expressed in polar coordinate is

$$\sigma_{rr}=-\frac{2p}{\pi r}\sin\theta \tag{6a}$$

$$\sigma_{\theta\theta}=0 \tag{6b}$$

$$\tau_{r\theta}=0 \tag{6c}$$

and the solution in rectangular coordinate can be expressed as

$$\sigma_{xx}=-\frac{2p}{\pi r}\sin\theta\cos^2\theta \tag{7a}$$

$$\sigma_{yy}=-\frac{2p}{\pi r}\sin^3\theta \tag{7b}$$

$$\tau_{xy}=-\frac{2p}{\pi r}\sin^2\theta\cos\theta \tag{7c}$$

A similar solution can be obtained for the case of a horizontal force q applied on the boundary of the semi-infinite plane. The stress field in polar coordinate is given as

$$\sigma_{rr}=-\frac{2q}{\pi r}\cos\theta \tag{8a}$$

$$\sigma_{\theta\theta}=0 \tag{8b}$$

$$\tau_{r\theta}=0 \tag{8c}$$

and the solution in rectangular coordinate can be expressed as

$$\sigma_{xx}=-\frac{2q}{\pi r}\cos^3\theta \tag{9a}$$

$$\sigma_{yy}=-\frac{2q}{\pi r}\sin^2\theta\cos\theta \tag{9b}$$

$$\tau_{xy}=-\frac{2q}{\pi r}\sin\theta\cos^2\theta \tag{9c}$$

For the case of arbitrarily distributed loads, the solution can be obtained by integration, using the solution for point force as presented in (6)–(9).

Analytical Alternating Method for Multiple Cracks in a Semi-Infinite Plane

In this section, an analytical alternating method will be proposed to analyze the S.I.F. of a semi-infinite plane with multiple cracks. In order to illustrate the alternating procedure, a fracture problem is considered as shown in Fig. 3(*a*) in which an arbitrarily distributed loading is applied on the boundary of a semi-infinite plane with n finite cracks. The iteration of superposition can be divided into two parts.

(I) External Alternating Procedure

(a) Solve the noncrack semi-infinite plane under given loads $T_b^{(0)}$ as shown in Fig. 3(*b*) by integration using the solution given by Eqs. (7) and (9). Evaluate the stress distribution $(f_{ci}^{(1)}, i=1,2,\ldots,n)$ in the noncrack body at the crack surfaces of the fictitious cracks. To satisfy the traction-free condition on the crack faces, the sub-problem of Fig. 3(*c*) which is subjected to residual stress $f_{ci}^{(1)}$ at the crack faces, should be superposed.

(b) The sub-problem Fig. 3(*c*) could be further split into two sub-problems as indicated in Figs. 3(*d*) and (*e*). Figure 3(*d*) shows an infinite plane with multiple cracks and subjected to the same traction $f_{ci}^{(1)}$ on the crack faces as shown in Fig. 3(*c*). Evaluation

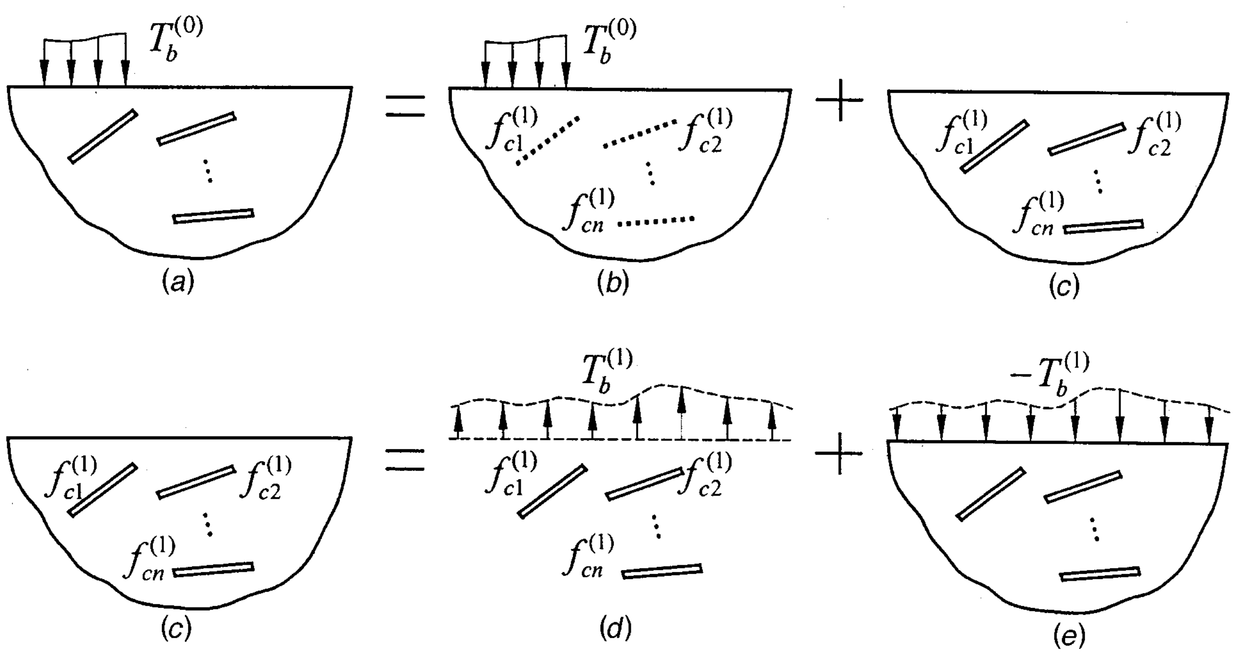

Fig. 3 The external alternating procedure of a semi-infinite plane with multiple cracks

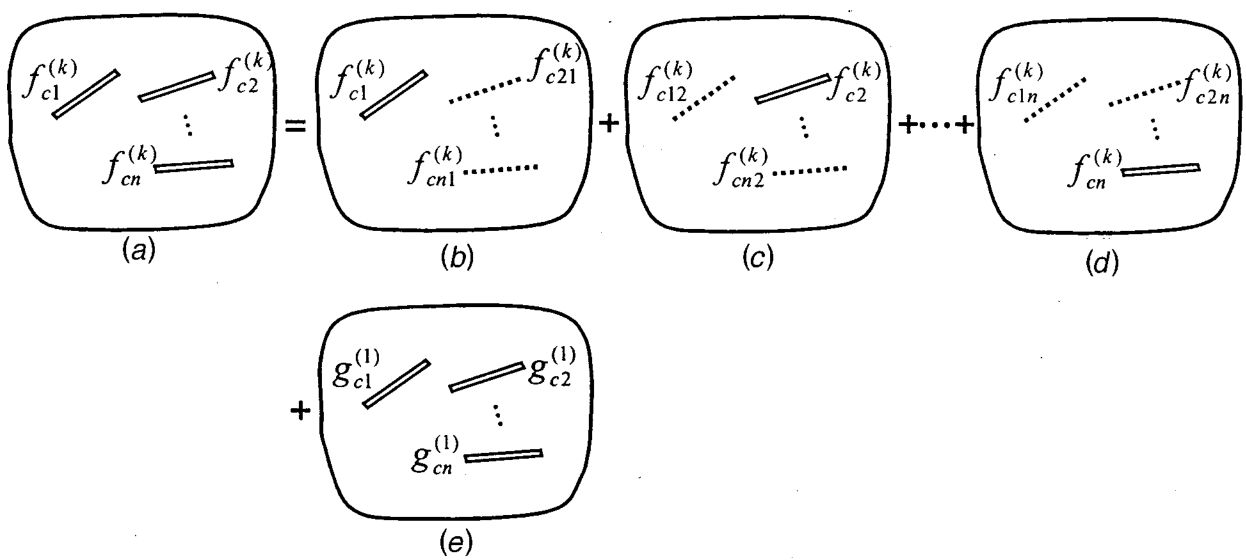

Fig. 4 The internal alternating procedure of the infinite plate with multiple cracks

of the residual stress $T_b^{(1)}$ on the virtual boundary as the original plane by the internal alternating procedure will be described later. The residual boundary stress could be superposed reversely by Fig. 3(e). The problem indicated in Fig. 3(e) is similar to that shown in Fig. 3(a), and the procedure will be iterated.

(c) After several cycles of iteration, the residual traction $f_{ci}^{(k)}$ on the crack faces of the kth cycle of iteration approaches zero; then the solution of Fig. 3(a) could be obtained by the superposition of Figs. 3(b) and (d) for k times successively. The S.I.F. of multiple cracks for the original problem as shown in Fig. 3(a) could be evaluated by considering the problem of an infinite plane with multiple cracks subjected to the crack face loads

$$f_{ci}=\sum_{j=1}^{k} f_{ci}^{(j)}, \quad i=1,2,\ldots,n \tag{10}$$

(d) The convergence criterion in this paper is taken as

$$\frac{f_{ci}^{(k+1)}-f_{ci}^{(k)}}{f_{ci}^{(k)}}<10^{-5}, \quad i=1\sim n \tag{11}$$

(II) Internal Alternating Procedure. In the iteration procedure, the solution of an infinite plate with multiple cracks as shown in Fig. 3(d) can be constructed by the procedure indicated in Fig. 4, in which $f_{ci}^{(k)}$ in Fig. 4(a) is the traction of the ith crack in the kth external alternating procedure. The solution of Fig. 4(a) could be obtained by the following internal procedures:

(a) Consider an infinite plate containing a single crack j, the crack face is subjected to the traction $f_{cj}^{(k)}$, and evaluates the traction $f_{cij}^{(k)}$ on the fictitious crack i.

(b) Superpose the solutions of the problems with single cracks $j(j=1\sim n)$, and the residual stress of crack i for the first cycle of internal iteration becomes

$$g_{ci}^{(1)}=\sum_{\substack{j=1\\ j\neq i}}^{n} f_{cij}^{(k)} \tag{12}$$

For the lth cycle of the internal iteration, we have

$$g_{ci}^{(l)}=\sum_{\substack{j=1\\ j\neq i}}^{n} g_{cij}^{(l-1)}, \quad l=2,3,\ldots \tag{13}$$

(c) After several cycles of iteration, the residual stress $g_{ci}^{(l)}$ approaches zero, the solution for the loading of Fig. 4(a) could be obtained by combining the solutions for the infinite plate with a single crack subjected to the crack face loads g_{ci} as

$$g_{ci}=f_{ci}^{(k)}+\sum_{l=1}^{\infty} g_{ci}^{(l)} \tag{14}$$

(d) The convergence criterion is taken as

$$\frac{g_{ci}^{(l+1)}-g_{ci}^{(l)}}{g_{ci}^{(l)}}<10^{-5}, \quad i=1\sim n \tag{15}$$

Numerical Results and Discussions

To examine the validity of the proposed analytical alternating method, several configurations of cracked bodies and applied loads are analyzed. First, the results of two cracks in an infinite plane are checked with the existing solutions. Next, an inclined crack near the boundary of a semi-infinite plane is considered and excellent agreement is found when compared with the result obtained from the literature. Finally, the problem of two inclined cracks in a semi-infinite plane is considered to examine the interactive effects between the cracks and the external boundary.

For the determination of the S.I.F. of cracks subjected to distributed loads, conventional Gauss's integration cannot give sufficient accuracy due to the existence of singular terms in the integration formula. Consider, for example, a crack in an infinite plane subjected to a distributed normal load $f(\xi)$ on the crack face. Then the mode I S.I.F. of the right-hand crack tip could be determined by the following equation:

$$K_I=\int_{-a}^{a} f(\xi)\,\frac{1}{\sqrt{\pi a}}\,\sqrt{\frac{a+\xi}{a-\xi}}\,d\xi \tag{16}$$

If we consider a uniformly distributed load $(f(\xi)=1,-a<\xi<a)$, 120 integral points are distributed over the crack face (five sections, with 24 Gauss integral points per section) to evaluate the S.I.F. Since a square root singularity will appear in the integration, we find that about 1% numerical error will be induced if compared with the analytical result $(\sqrt{\pi a})$. Now, we change the variable by using

$$u=\sqrt{\frac{a-\xi}{2}}, \quad -a<\xi<a, \quad 0<u<\sqrt{a} \tag{17}$$

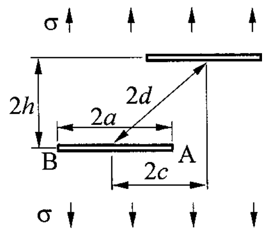

Fig. 5 Two parallel cracks in an infinite plane subjected to a remote tension load

Table 1 Normalized S.I.F. of crack tips A and B for two collinear cracks in an infinite plane ($h=0$)

a/d	F_{IA}		F_{IB}	
	Isida [16]	Present	Isida [16]	Present
0.9	1.45387	1.45387	1.11741	1.11741
0.8	1.22893	1.22893	1.08107	1.08107
0.7	1.13326	1.13326	1.05786	1.05786
0.6	1.08040	1.08040	1.04094	1.04094
0.5	1.04796	1.04796	1.02795	1.02795
0.4	1.02717	1.02717	1.01787	1.01787
0.3	1.01383	1.01383	1.01017	1.01017
0.2	1.00566	1.00566	1.00462	1.00462
0.1	1.00132	1.00132	1.00120	1.00120
0.05	1.00032	1.00032	1.00031	1.00030*

Table 2 The normalized S.I.F. for two overlapping parallel cracks in an infinite plane ($c=0$)

a/d	F_I			F_{II}
	[17][18]	Present	Error (%)	Present
0.2	0.9855[17]	0.9858	0.03	0.0014
0.4	0.9508[17]	0.9505	-0.03	0.0094
0.6	0.9089[17]	0.9092	0.03	0.0246
0.8	0.8727[17]	0.8722	-0.06	0.0431
1.0	0.8319[18]	0.8431	1.3	0.0611
1.25	0.8037[18]	0.8166	1.6	0.0803
2.0	0.7569[18]	0.7734	2.2	0.1166

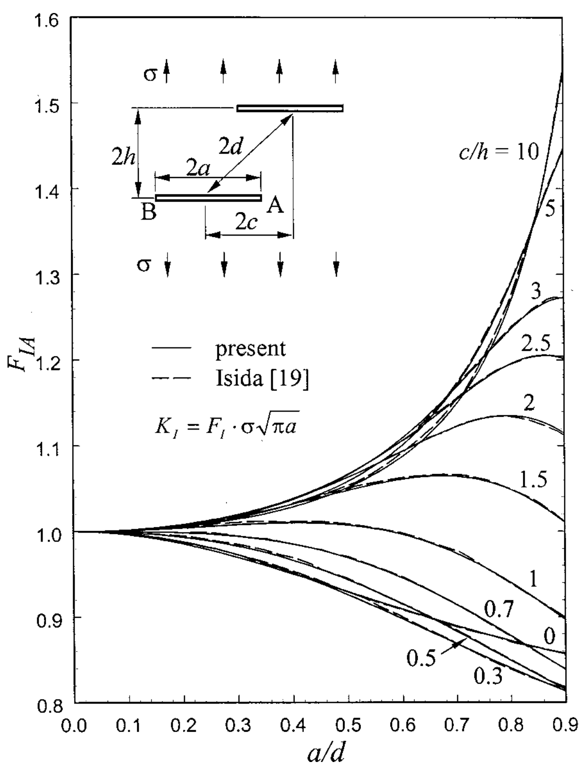

Fig. 6 Comparison of normalized mode I S.I.F. of crack tip A for two parallel cracks in an infinite plane

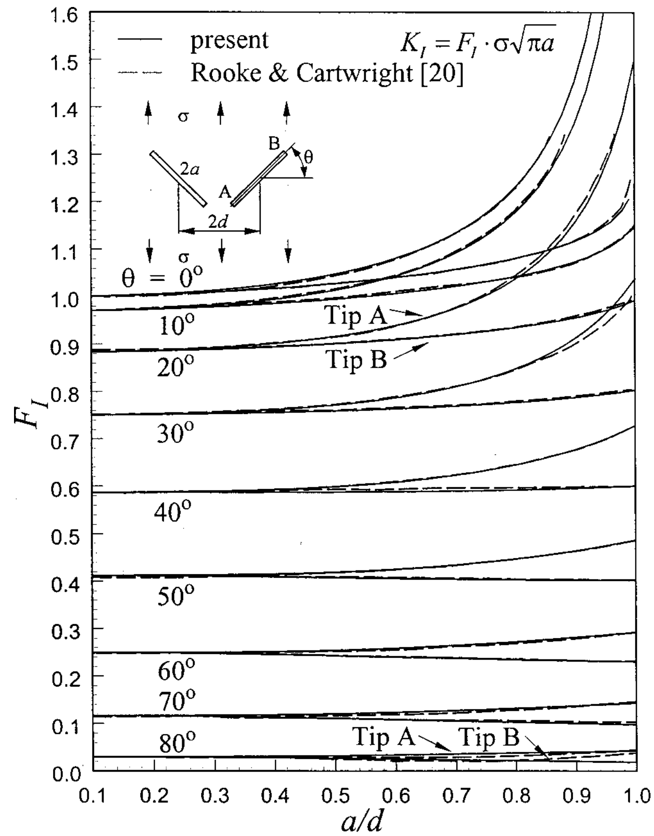

Fig. 7 Comparison of normalized mode I S.I.F. of crack tips A and B for two inclined cracks in an infinite plane

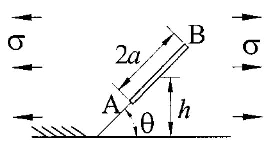

Fig. 8 An inclined crack near the boundary of a semi-infinite plane subjected to remote tension

Table 3 Comparisons of normalized mode I S.I.F. (F_I) for a normal crack ($\theta=90$ deg) near the boundary of a semi-infinite plane

h/a	F_I	
	Tip A	Tip B
1.01	3.641(3.640)*	1.330(1.330)
1.05	2.155(2.155)	1.254(1.254)
1.1	1.758(1.758)	1.211(1.211)
1.2	1.464(1.464)	1.163(1.163)
1.5	1.204(1.204)	1.097(1.097)
2	1.091(1.091)	1.054(1.054)
5	1.011(1.011)	1.009(1.009)
10	1.003(1.003)	1.002(1.002)
99	1.000(1.000)	1.000(1.000)

Then the S.I.F. can be accurately evaluated by the following expression:

$$K_I=\frac{4}{\sqrt{\pi a}}\int_0^{\sqrt{a}} p(a-2u^2)\cdot\sqrt{a-u^2}du \tag{18}$$

The error in numerical result is less than 10^{-4}%, and the accuracy of the solution is vastly improved.

Table 4 (*a*) Comparisons of normalized mode I S.I.F. (F_I) for an inclined crack near the boundary of a semi-infinite plane. (*b*) Comparisons of normalized mode II S.I.F. (F_{II}) for an inclined crack near the boundary of a semi-infinite plane.

h/a	θ = 80°		60°		40°		20°	
	Tip A	Tip B	Tip A	Tip B	Tip A	Tip B	Tip A	Tip B
1.05	1.917(1.917)	1.210(1.210)	1.085(1.085)	0.931(0.931)	0.484(0.487)*	0.537(0.537)	0.105(0.105)	0.176(0.175)*
1.1	1.632(1.632)	1.173(1.173)	1.032(1.032)	0.914(0.914)	0.479(0.479)	0.528(0.528)	0.107(0.107)	0.172(0.172)
1.5	1.162(1.162)	1.066(1.066)	0.871(0.871)	0.840(0.840)	0.456(0.456)	0.482(0.482)	0.115(0.115)	0.150(0.150)
2	1.058(1.058)	1.024(1.024)	0.814(0.814)	0.802(0.802)	0.441(0.441)	0.455(0.455)	0.118(0.118)	0.136(0.136)
3	1.004(1.004)	0.995(0.995)	0.777(0.777)	0.774(0.774)	0.428(0.428)	0.432(0.432)	0.119(0.119)	0.125(0.125)
5	0.981(0.981)	0.979(0.979)	0.760(0.760)	0.759(0.759)	0.419(0.419)	0.420(0.420)	0.118(0.118)	0.120(0.119)*
99	0.970(0.970)	0.970(0.970)	0.750(0.750)	0.750(0.750)	0.413(0.413)	0.413(0.413)	0.117(0.117)	0.117(0.117)

(*a*)

h/a	θ = 80°		60°		40°		20°	
	Tip A	Tip B	Tip A	Tip B	Tip A	Tip B	Tip A	Tip B
1.05	0.351(0.351)	0.154(0.154)	0.638(0.638)	0.402(0.402)	0.611(0.611)	0.480(0.480)	0.369(0.367)*	0.330(0.330)
1.1	0.291(0.291)	0.155(0.155)	0.600(0.600)	0.405(0.405)	0.599(0.599)	0.482(0.482)	0.366(0.366)	0.331(0.331)
1.5	0.196(0.196)	0.164(0.164)	0.489(0.489)	0.421(0.421)	0.546(0.546)	0.492(0.455)*	0.350(0.350)	0.331(0.331)
2	0.180(0.180)	0.168(0.168)	0.457(0.457)	0.428(0.428)	0.521(0.521)	0.495(0.495)	0.339(0.339)	0.330(0.330)
3	0.173(0.173)	0.170(0.170)	0.441(0.441)	0.432(0.432)	0.504(0.504)	0.496(0.496)	0.330(0.330)	0.327(0.327)
5	0.171(0.171)	0.171(0.171)	0.435(0.435)	0.433(0.433)	0.496(0.496)	0.494(0.494)	0.324(0.324)	0.324(0.324)
99	0.171(0.171)	0.171(0.171)	0.433(0.433)	0.433(0.433)	0.493(0.492)*	0.492(0.492)	0.321(0.321)	0.321(0.321)

(*b*)

Example 1: Two Cracks in an Infinite Plane

(a) Two Parallel Cracks in an Infinite Plane. Consider two parallel cracks with equal crack length $(2a)$ in an infinite plane subjected to a remote tension σ as shown in Fig. 5, the distance between the crack centers is $2d$, the horizontal and vertical distance are $2c$ and $2h$, respectively. The normalized stress intensity factors of crack tips A and B are given as

$$\begin{bmatrix} F_I \\ F_{II} \end{bmatrix} = \begin{bmatrix} K_I \\ K_{II} \end{bmatrix} \cdot \frac{1}{\sigma\sqrt{\pi a}} \tag{19}$$

For the case of two collinear cracks $(h=0)$, the numerical results are compared with that obtained by Isida [16]. Excellent agreements are found as shown in Table 1; the only distinct value is symbolized by (*). For the case of two overlapping parallel cracks $(c=0)$, the numerical results are shown in Table 2 and are compared with that obtained by Yokobori et al. [17], and Kamei and Yokobori [18]. Figure 6 shows the normalized mode I S.I.F. of crack tip A for two parallel cracks in an infinite plane for different values of c/h. Excellent agreements are found when compared with the solutions of Isida [19].

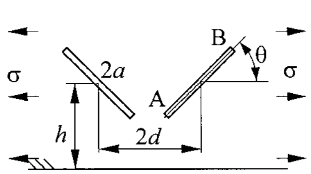

Fig. 9 Two symmetrical inclined cracks near the boundary of the semi-infinite plane

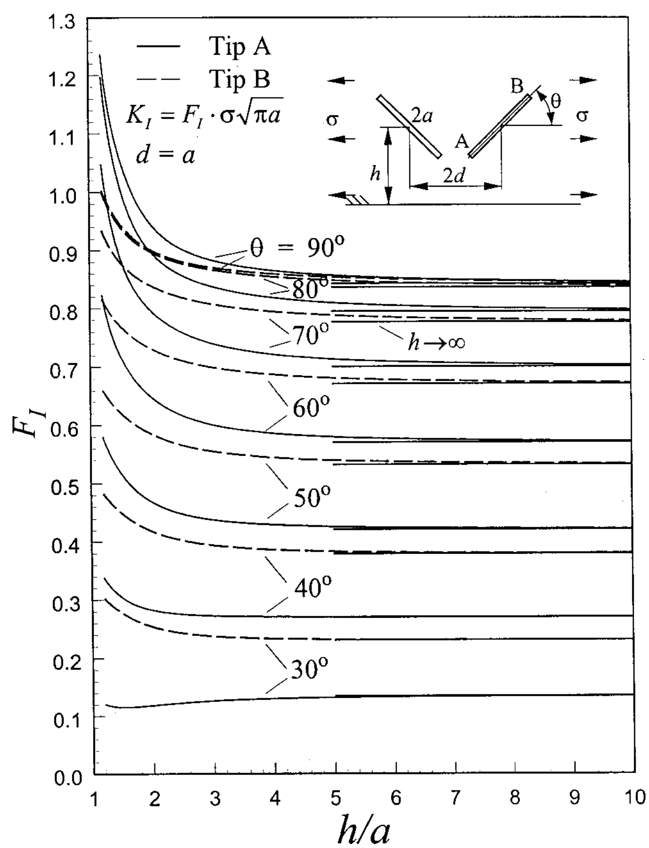

Fig. 10 The normalized mode I S.I.F. of crack tips A and B for two symmetrical inclined cracks near the boundary of a semi-infinite plane

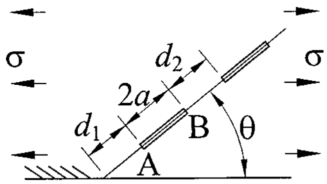

Fig. 11 Two collinear inclined cracks near the boundary of a semi-infinite plane

(b) Two Inclined Crack in an Infinite Plane. Consider two equally inclined cracks in an infinite plane subjected to a remote tensile stress σ, the inclined angle is θ and the horizontal distance between the centers of two cracks is $2d$. Figure 7 shows the normalized mode I S.I.F. of crack tips A and B for different values of θ. Again, excellent agreements are obtained with the solutions of Rooke and Cartwright [20] who also analyzed the same problem.

Example 2: A Crack Near the Boundary of a Semi-Infinite Plane. Consider a finite inclined crack near the boundary of a semi-infinite plane, and subjected to a remote tension σ as shown in Fig. 8, the inclined angle is θ and the vertical distance of the center of the crack from the plane boundary is h. The normalized stress intensity factors (F_I and F_{II}) of crack tips A and B are computed for different values of the vertical distances and inclined angle. The numerical results are compared with those of Nowell and Hills [2], as indicated in Tables 3 and 4, in which the values in brackets are taken from [2], and distinct values are symbolized by (*). Again, excellent agreements are obtained, and the accuracy of present method is established.

Example 3: Two Cracks Near the Boundary of a Semi-Infinite Plane. From the results provided in the previous two examples, the validity and accuracy of the proposed analytical alternating method and numerical scheme are established by comparison with available solutions in the published literature. Fi-

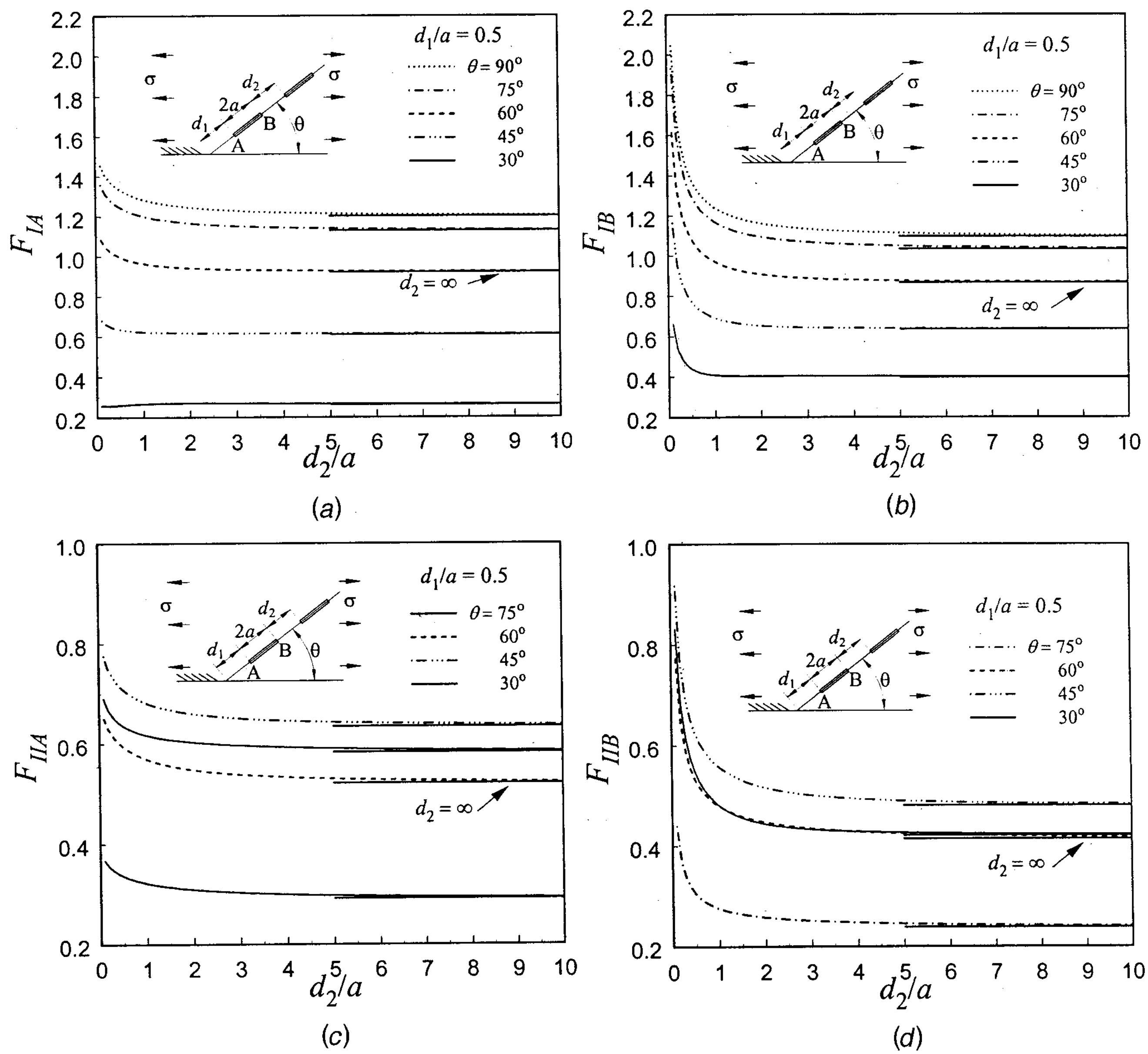

Fig. 12 The normalized S.I.F. for two collinear inclined cracks near the boundary of a semi-infinite plane ($d_1/a=0.5$)

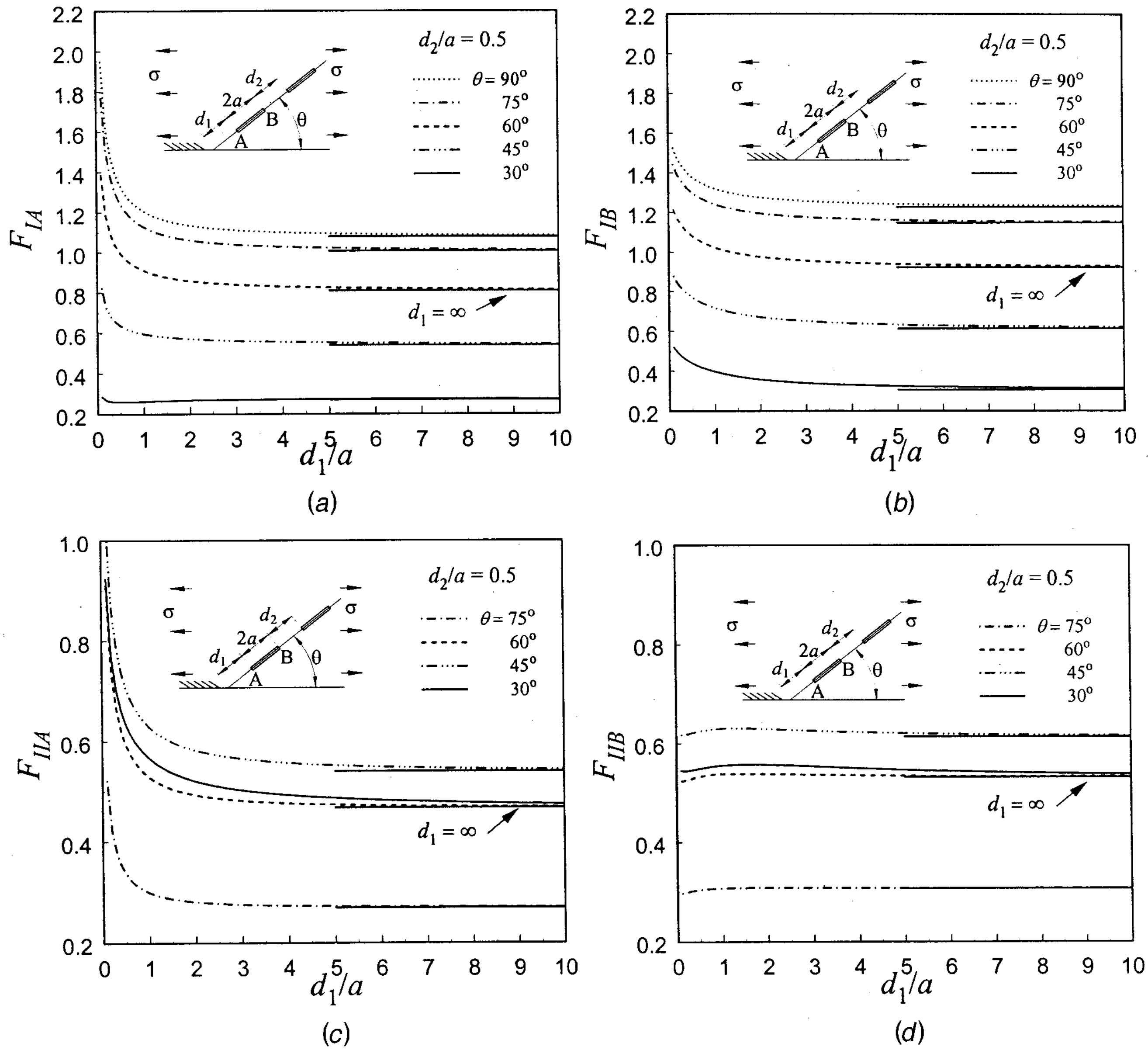

Fig. 13 The normalized S.I.F. for two collinear inclined cracks near the boundary of a semi-infinite plane ($d_2/a = 0.5$)

nally, the results for two finite cracks in a semi-infinite plane will now be discussed in detail. An analysis of this problem does not seem to have been attempted before.

(a) Two Symmetrical Inclined Cracks Near the Boundary of a Semi-Infinite Plane. The problem of two symmetrically inclined cracks near the boundary of a semi-infinite plane, as shown in Fig. 9, is considered; the crack length is $2a$ and the angle of inclination θ. The horizontal distance between the centers of the two cracks is $2d$, and the vertical distance of the centers of the cracks from the boundary is h. The normalized mode I S.I.F. of crack tips A and B are computed for different values of the inclined angle, and for the special case $d = a$. The results are presented in Fig. 10, where the horizontal lines correspond to h tending to infinite, which represents the case of two inclined cracks in an infinite plane. It is worth noting that the results will be strongly influenced by the interactions between the cracks and the half-plane for $h/a < 3$, and the results almost coincide with the solution of two inclined cracks in an infinite plane for $h/a > 10$.

(b) Two Collinear Cracks in a Semi-Infinite Plane. The case of two collinear inclined cracks near the boundary of a semi-infinite plane is considered; the length of each crack is $2a$, and the inclined angle is θ as shown in Fig. 11. The normalized modes I and II S.I.F. of crack tips A and B are first computed for different values of θ, using $d_1/a = 0.5$. The results are presented in Fig. 12, where the horizontal lines represent the case $d_2 \rightarrow \infty$, which is the

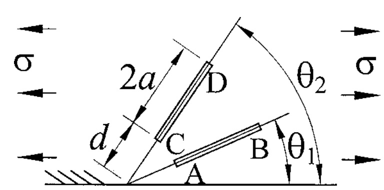

Fig. 14 Two radial cracks near the boundary of a semi-infinite plane

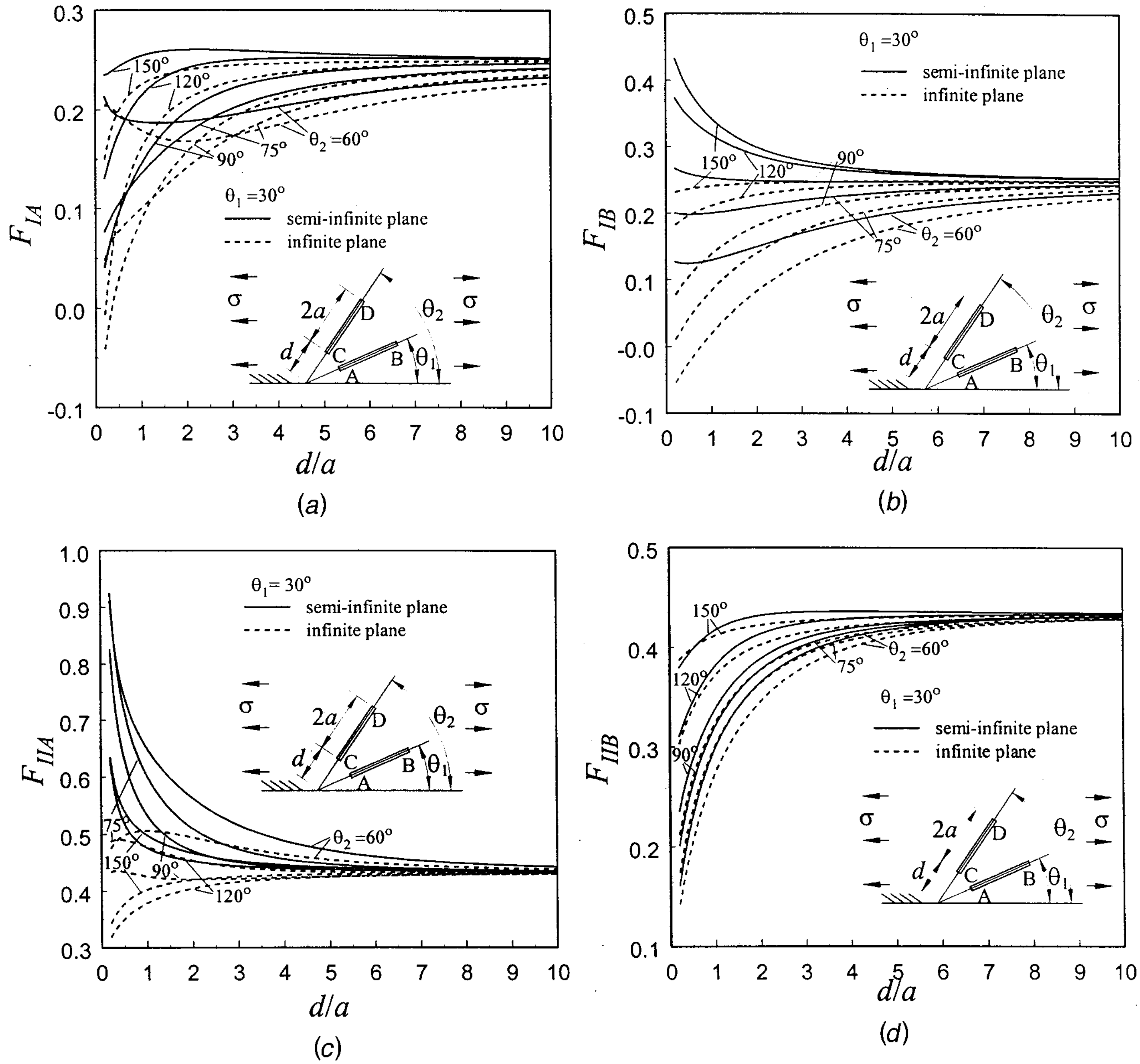

Fig. 15 The normalized S.I.F. of crack tips A and B for two radial cracks near the boundary of a semi-infinite plane

solution for an inclined crack in a semi-infinite plane. Next, the normalized modes I and II S.I.F. of crack tips A and B are computed for different values of θ, using $d_2/a=0.5$. The results are presented in Fig. 13, where the horizontal lines indicate the case $d_1 \rightarrow \infty$, which is the solution for two collinear cracks in an infinite plane. It also follows from Figs. 12 and 13 that the influence of the interaction of cracks (or boundary) on S.I.F. will be small for $d_2/a>10$ (or $d_1/a>10$).

(c) Two Radial Cracks in a Semi-Infinite Plane. The problem of two radial cracks near the boundary of a semi-infinite plane, as shown in Fig. 14, is considered; the radial distance is d and the orientation angles are θ_1 and θ_2. The inclination of the first crack is kept at a constant angle of 30 deg, while that of the second crack is varied from 60–150 deg. The normalized modes I and II S.I.F. for each crack tip versus the ratio of d/a with different orientation angles for the second crack are shown in Figs. 15 and 16. The cases of two radial cracks in an infinite plane under the same loading condition are also plotted for reference. We can see that the effect of the boundary strongly influences the result of S.I.F., especially for the crack tips A and B. It may be noted that as the value of d/a increases, the results of the semi-infinite plane and infinite plane should coincide. The maximum difference between the cases of semi-infinite plane and infinite plane is found to be about 3% for $d/a=10$, decreasing to about 0.8% for $d/a=20$.

(d) A Main Crack and a Collinear Microcrack in a Semi-Infinite Plane. In order to verify the results and the numerical procedure provided in this paper for the interaction of cracks in a semi-infinite plane. The interaction between a main crack and a collinear microcrack in a semi-infinite plane is investigated and the geometrical configuration is shown in Fig. 17(*a*). The length of the main crack is $2a$ and the microcrack is $2b$. The vertical distance of the center of the main crack from the plane boundary is h and the distance between the center of microcrack and tip B of the main crack is d. Rose [22] and Lam et al. [23] considered the problem of a semi-infinite edge crack and a collinear micro-

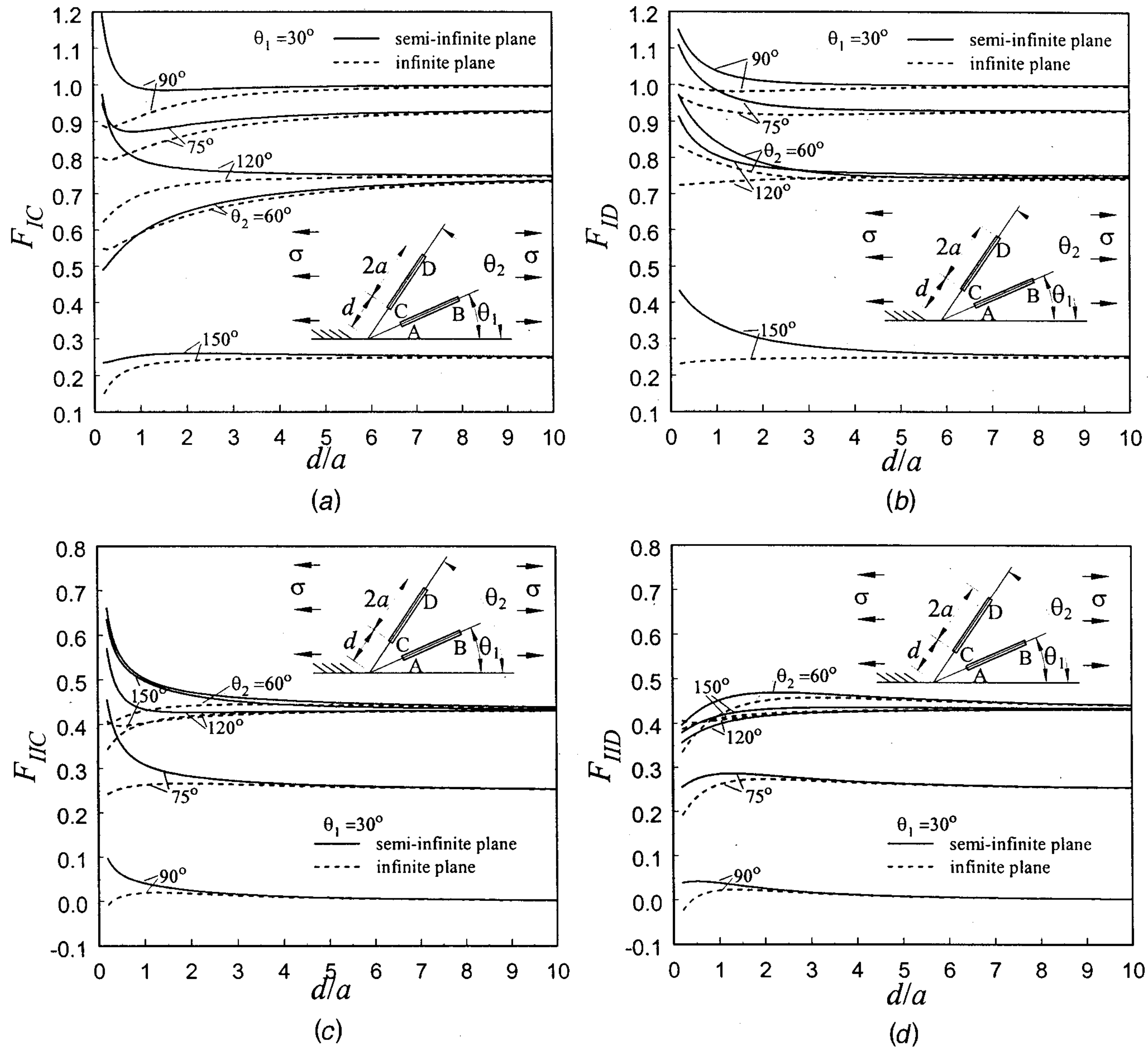

Fig. 16 The normalized S.I.F. of crack tips C and D for two radial cracks near the boundary of a semi-infinite plane

crack in a semi-infinite plane subjected to a remote tension σ as shown in Fig. 17(b). An exact solution of the stress intensity factor for the main crack was obtained by Rose [22] in terms of the complete elliptic integrals. Lam et al. [23] used distributions of edge dislocations to represent the main crack and the microcrack and solved a singular integral equation by an numerical scheme. In order to simulate the problem studied by Rose [22] and Lam et al. [23], we consider the critical case that the length of the main crack is much larger than the microcrack ($b/a=0.01$) and the tip A of the main crack is very close to the semi-infinite plane ($h/a=1.01$). The normalized mode I S.I.F. ($K_I/\sigma\sqrt{\pi a}$) of crack tip B of the main crack are computed for different values of d/b and the result is shown in Table 5. The value of $K_I^*/\sigma\sqrt{\pi a}$ is the normalized S.I.F. of crack tip B without microcrack which is already analyzed in Example 2 and is indicated in the first line of Table 3. The values of K_I/K_I^* represent the effect of the stress amplification due to the microcrack are compared to the results obtained by Rose [22] and Lam et al. [23]. It is shown in Table 5 that good agreement is achieved between the results of present method and available literatures and the accuracy obtained by present method is better than that provided by Lam et al. [23].

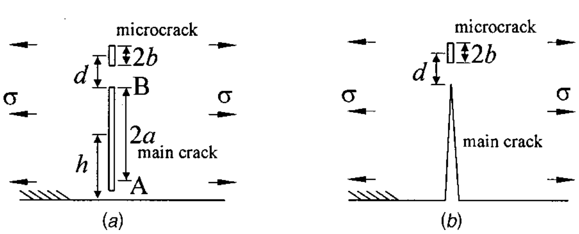

Fig. 17 A main crack and a collinear microcrack in a semi-infinite plane—(a) present, (b) Rose [22]

Conclusion

By using the analytical forms of the fundamental solutions of half-plane and crack, an analytical alternating method is proposed to investigate the in-plane fracture problem of a semi-infinite plane with multiple cracks. An efficient and accurate Gauss's integration method has been successfully developed to analyze the

Table 5 Comparison of the S.I.F. (K_I/K_I^*) at the tip of the main crack

d/b	Rose [22] Exact sol.	Lam et al. [23]	Present (tip B)			
	K_I/K_I^*	K_I/K_I^*	K_I/K_I^*	err(%)※	$\frac{K_I}{\sigma\sqrt{\pi a}}$	$\frac{K_I^*}{\sigma\sqrt{\pi a}}$
1.1	1.652	1.642	1.657	0.28	2.203	1.330
1.2	1.387	1.390	1.389	0.16	1.848	1.330
1.3	1.274	1.279	1.275	0.10	1.696	1.330
1.4	1.209	1.215	1.211	0.13	1.610	1.330
1.5	1.167	1.173	1.169	0.14	1.554	1.330
2.0	1.076	1.081	1.077	0.13	1.433	1.330
3.0	1.030	1.033	1.031	0.07	1.371	1.330

※ $\text{err}(\%) = \frac{(K_I/K_I^*)_{\text{present}} - (K_I/K_I^*)_{\text{Rose}}}{(K_I/K_I^*)_{\text{Rose}}} \cdot 100\%$

mixed-mode stress intensity factors of a semi-infinite plane with arbitrarily located multiple cracks. The numerical results of two cracks in an infinite plane and an inclined crack in a semi-infinite plane are shown to be consistent with the solutions obtained in the literature. In the analysis of multiple cracks in a semi-infinite plane, the interaction between the cracks and the boundary of the semi-infinite plane is investigated, and the results are fully discussed.

The fracture analysis of a cracked body with a single crack has been discussed in many papers. However, due to the complexity of the problem for arbitrarily distributed multiple cracks, the available results are very few in the literature. It is impossible to obtain the analytic solutions for the multiple cracks problem, and is also very difficult to obtain sufficiently accurate results by purely numerical calculations. The analytical alternating method proposed in this study is found to be a simple and accurate technique for the computation of S.I.F. for the multiple cracks problem.

Acknowledgments

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