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Study of the Fluid Flow in the Elliptical Duct by the Method of Characteristics

This study develops a mathematical model to determine the properties of laminar flow in the elliptical duct. With some assumptions, the nonlinear governing equations of the air in the elliptical duct are transformed into the hyperbolic type. The method of characteristics is then applied. Numerical results are obtained by using the finite difference method and the uniform interval scheme. The air properties in the elliptical duct are analyzed. The local Nusselt number and the heat transfer coefficient along the duct are studied. The numerical results are compared and show good agreement with the available data.

Introduction

The circumferential length of the elliptical duct is longer than that of the circular duct if the cross-sectional areas are the same. Hence, when the heat exchanger requires good cool effect and the space is an important factor in design, the elliptical duct is used practically.

The governing equations for the air flow in the elliptical duct are nonlinear. Bodoia and Osterle (1961) transformed equations to get the appropriate solutions by numerical methods. Lundgren et al. (1964) studied the pressure drop due to the entrance region in ducts with arbitrary cross sections. Igbal et al. (1972) used variational method for the steady fully developed laminar flow. Dunwoody (1962) investigated the steady, viscous and fully developed laminar flow with constant wall temperature for the entrance region. Gilbert et al. (1973) used the slug flow model and ignored the influence of the viscosity for the forced laminar and steady flow to determine the heat transfer on the boundary. Abdel-Wahed et al. (1984) investigated experimentally the laminar developing and fully developed flow with the ratio of the major axis length to the minor axis length being 2.

The purpose of this study is to develop a mathematical model to analyze the air properties and to determine the local Nusselt number and the heat transfer coefficient of the air in the elliptical duct.

Mathematical Approach

The coordinate system for an elliptical duct is shown in Fig. 1. For a laminar flow field, the gradients of the velocity components in the y and z directions are negligible compared with the gradient of the velocity component in the x direction. The energy generated by the friction force is assumed much smaller than all other terms.

At the location far from the duct wall, the speed and the temperature gradients and the second-order differential of the

heat transfer and the viscosity are very small; therefore, their effects are negligible. Large variation exists only at the location near the duct wall. The air temperature at the duct wall is assumed to be equal to the temperature of the duct wall.

The assumed dimensionless temperature is used to simplify the second-order terms in the differential equations. The exponential form assumed for the temperature distribution is

$$\theta = \frac{T(x,y,z) - T_w(x)}{T_i(x) - T_w(x)} = \left\{ 1 - \left[\left(\frac{y}{a} \right)^2 + \left(\frac{z}{b} \right)^2 \right]^{d_1} \right\} \quad (1)$$

where T_i is the highest air temperature at the duct center and T_w is the air temperature at the duct wall. The exponential form assumed for the velocity distribution is

$$\frac{u}{u_m} = \frac{u_i}{u_m} \left\{ 1 - \left[\left(\frac{y}{a} \right)^2 + \left(\frac{z}{b} \right)^2 \right]^{d_2} \right\} \quad (2)$$

where u_i is the highest air velocity at the duct center in the x direction and u_m is the air mean velocity. The air is assumed to be an ideal gas.

The nondimensional variables used are

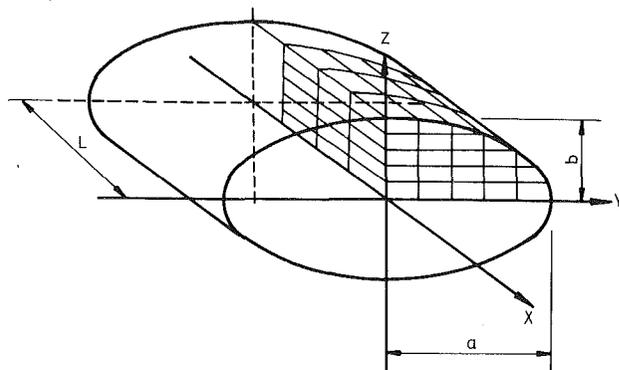


Fig. 1 Coordinates of the elliptical duct

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$$\begin{aligned} \rho^* &= \frac{\rho}{\rho_a}; & T^* &= \frac{(T - T_a)}{(T_{in} - T_a)} \\ t^* &= \frac{t}{(D_h/u_{in})}; & x^* &= \frac{x}{D_h} \\ y^* &= \frac{y}{D_h}; & z^* &= \frac{z}{D_h} \\ u^* &= \frac{u}{u_{in}}; & v^* &= \frac{v}{u_{in}} \\ w^* &= \frac{w}{u_{in}}; & c^* &= \frac{c}{c_a} \end{aligned} \quad (3)$$

Using Eqs. (1), (2) and (3), the column vector L_i of nondimensional forms of the continuity, momentum and energy equations can be obtained as

$$\begin{bmatrix} \rho^* \delta_{1i} & \rho^* \delta_{2i} & \rho^* \delta_{3i} & U_i^* & 0 \\ \rho^* U_i^* & 0 & 0 & \frac{RT}{u_{in}^2} \delta_{1i} & \frac{\rho^* R (T_{in} - T_a)}{u_{in}^2} \delta_{1i} \\ 0 & \rho^* U_i^* & 0 & \frac{RT}{u_{in}^2} \delta_{2i} & \frac{\rho^* R (T_{in} - T_a)}{u_{in}^2} \delta_{2i} \\ 0 & 0 & \rho^* U_i^* & \frac{RT}{u_{in}^2} \delta_{3i} & \frac{\rho^* R (T_{in} - T_a)}{u_{in}^2} \delta_{3i} \\ \frac{\rho^* RT}{c_v T_a} \delta_{1i} & \frac{\rho^* RT}{c_v T_a} \delta_{2i} & \frac{\rho^* RT}{c_v T_a} \delta_{3i} & 0 & \left(\frac{T_{in} - T_a}{T_a} \right) \rho^* U_i^* \end{bmatrix} \begin{bmatrix} \frac{\partial u^*}{\partial x_i^*} \\ \frac{\partial v^*}{\partial x_i^*} \\ \frac{\partial w^*}{\partial x_i^*} \\ \frac{\partial \rho^*}{\partial x_i^*} \\ \frac{\partial T^*}{\partial x_i^*} \end{bmatrix} = \begin{bmatrix} 0 \\ [G]^* \\ 0 \\ 0 \\ [F]^* \end{bmatrix} \quad (4)$$

where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$

$$[G]^* = \frac{\mu u_i^*}{\rho_a \mu_{in} D_h} \left\{ -\alpha_2 (\alpha_2 - 1) \left[\left(\frac{y}{a} \right)^2 + \left(\frac{z}{b} \right)^2 \right]^{d_2 - 1} \times \left[\frac{4y^2}{a^4} + \frac{4z^2}{b^4} \right] - \alpha_2 \left[\left(\frac{y}{a} \right)^2 + \left(\frac{z}{b} \right)^2 \right]^{d_2 - 1} \left[\frac{2}{a^2} + \frac{2}{b^2} \right] \right\} \quad (5)$$

and

$$[F]^* = \frac{k (T^* - T_w^*) (T_{in} - T_a)}{\rho_a c_v T_a \mu_{in} D_h} \left\{ -\alpha_1 (\alpha_1 - 1) \times \left[\left(\frac{y}{a} \right)^2 + \left(\frac{z}{b} \right)^2 \right]^{d_1 - 1} \left[\frac{4y^2}{a^4} + \frac{4z^2}{b^4} \right] - \alpha_1 \left[\left(\frac{y}{a} \right)^2 + \left(\frac{z}{b} \right)^2 \right]^{d_1 - 1} \left[\frac{2}{a^2} + \frac{2}{b^2} \right] \right\} \quad (6)$$

The method of characteristics is applied by using the arbitrary functions $\omega_1, \omega_2, \omega_3, \omega_4$ and ω_5 to combine these nondimensional equations (Rudinger, 1969; Huang, 1992). Let the five coefficient vectors of the combined equation $L_1 \omega_1 + L_2 \omega_2 + L_3 \omega_3 + L_4 \omega_4 + L_5 \omega_5 = 0$ be related. Then, there exists a common normal vector for these coefficient vectors as

$$\begin{cases} \left[\rho^* \delta_{1i} \omega_1 + \rho^* U_i \omega_2 + \frac{\rho^* RT}{c_v T_a} \delta_{1i} \omega_5 \right] N_i = 0 \\ \left[\rho^* \delta_{2i} \omega_1 + \rho^* U_i \omega_3 + \frac{\rho^* RT}{c_v T_a} \delta_{1i} \omega_5 \right] N_i = 0 \\ \left[\rho^* \delta_{3i} \omega_1 + \rho^* U_i \omega_4 + \frac{\rho^* RT}{c_v T_a} \delta_{1i} \omega_5 \right] N_i = 0 \\ \left[U_i^* \omega_1 + \frac{RT}{u_{in}^2} (\delta_{1i} \omega_3 + \delta_{2i} \omega_5 + \delta_{3i} \omega_4) \right] N_i = 0 \\ \left[\frac{\rho^* RT (T_{in} - T_a)}{u_{in}^2} (\delta_{1i} \omega_2 + \delta_{2i} \omega_3 + \delta_{3i} \omega_4) + \left(\frac{T_{in} - T_a}{T_a} \right) \rho^* U_i \omega_5 \right] N_i = 0 \end{cases} \quad (7)$$

If a four-dimensional normal vector and the velocity vector, respectively, are

Nomenclature

a = semi-major axis length of ellipse
 b = semi-minor axis length of ellipse
 Bi = Biot no. = ht_w/k
 c = sonic speed
 c_v = specific heat at constant volume
 D_h = hydraulic diameter of duct
 Gz = Graetz no. = $RePrD_h/x$
 h = local heat transfer coefficient
 k = thermal conductivity
 N = normal vector
 Nu = Nusselt no. = hD_h/k
 Pr = Prandtl no. = $C_p \mu / k$
 R = ideal gas constant
 Re = Reynolds no. = $\rho u D_h / \mu$
 T = temperature
 t = time; thickness

U = velocity vector
 u = velocity component in x direction
 v = velocity component in y direction
 w = velocity component in z direction
 x = distance along x coordinate
 y = distance along y coordinate
 z = distance along z coordinate
 γ = ratio of specific heat at constant pressure to that at constant volume
 ρ = density
 μ = viscosity
 θ = nondimensional temperature distribution
 ω = arbitrary function

α_1 = exponent of temperature distribution
 α_2 = exponent of velocity distribution

Subscripts

a = condition at room temperature and standard pressure (303 K and 101325 Pa)
 i = component or vector
 in = condition at inlet of duct
 m = mean value of cross section
 t = condition at center of duct
 w = condition at wall

Superscript

* = nondimensional quantity

Table 1 The data of the elliptical duct

Semi-major axis length (<i>a</i>)	0.015469 m
Semi-minor axis length (<i>b</i>)	0.0077349 m
Hydraulic diameter (<i>D_h</i>)	0.020 m
Ratio of <i>a</i> to <i>b</i>	2.0
Length of the duct (<i>l</i>)	3.0 m
Thickness of the duct (<i>t_w</i>)	<i>t_w</i> < 0.01 <i>k/h</i> (<i>Bi</i> < 0.01)

$$N_i = (N_1, N_2, N_3, N_4) \quad (8)$$

and

$$U_1^* = (u^*, v^*, w^*, 1) \quad (9)$$

the solution for ω_i not all equal to zero is

$$(U_i^* N_i)^3 = 0 \quad (10)$$

or

$$U_i^* N_i = \pm \frac{c}{u_{in}} \quad (11)$$

where the sonic speed *c* is

$$c = (\gamma RT)^{0.5} \quad (12)$$

Each *N_i* defines a vector space which is normal to the normal vector and is also called the characteristic vector space. Using Eq. (10), two independent equations for each *N_i* obtained are

$$\omega_1 + \frac{R[T^*(T_{in} - T_a) + T_a]}{c_v T_a} \omega_5 = 0 \quad (13)$$

and

$$\omega_2 N_1 + \omega_3 N_2 + \omega_4 N_3 = 0 \quad (14)$$

Since Eqs. (13) and (14) are independent to each other, the simplest method is to consider ω_1, ω_5 , and $\omega_2, \omega_3, \omega_4$ separately. When one of the two sets is considered, the other set is chosen to be zero as follows:

(a) $\omega_i = (\omega_1, 0, 0, 0, \omega_5)$

From Eq. (13), we obtain

$$\omega_i = \left\{ \frac{-R[T^*(T_{in} - T_a) + T_a]}{c_v T_a}, 0, 0, 0, 1 \right\} \quad (15)$$

(b) $\omega_i = (0, \omega_2, \omega_3, \omega_4, 0)$

From Eq. (14), the three variables ω_2, ω_3 and ω_4 have only one restriction. The vector of $\omega_2 i + \omega_3 j + \omega_4 k$ is just perpendicular to the vector of $N_1 i + N_2 j + N_3 k$. The characteristic directions are arbitrarily selected as long as they are satisfied with the physical meaning and the simplification of the equations to reduce the computer time.

Using Eq. (11), we can obtain

$$\omega_i = \left(c^{*2}, \gamma N_1 c^* \frac{u_{in}}{c_a}, \gamma N_2 c^* \frac{u_{in}}{c_a}, \gamma N_3 c^* \frac{u_{in}}{c_a}, 1 \right) \quad (16)$$

When ω_i in Eq. (15) is used, only one compatibility equation which is independent of *N_i* is

$$(1 - \gamma) \left[T^* \left(\frac{T_{in} - T_a}{T_a} \right) + 1 \right] D_u \rho^* + \rho^* \left(\frac{T_{in} - T_a}{T_a} \right) D_u T^* = \frac{u_{in}}{c_a} [F]^* \quad (17)$$

along the direction

$$u = \left(u^* \frac{u_{in}}{c_a}, v^* \frac{u_{in}}{c_a}, w^* \frac{u_{in}}{c_a}, \frac{u_{in}}{c_a} \right) \quad (18)$$

When $\omega_i = (0, \omega_2, \omega_3, \omega_4, 0)$ and the normal vector *N_i* = (1, 0, 0, -*u*^{*}) are selected to match the restriction of Eq. (10), $\omega_i = (0, 0, 1, 0, 0)$ is chosen to match the restriction of Eq. (14). The compatibility equation is

Table 2 The initial data and the input data of air properties

Inlet velocity	2.0 m/s
Inlet temperature	383 K
Inlet pressure	101329 Pa
Assumed outlet velocity	1.415 m/s
Assumed outlet temperature	303 K
Assumed outlet pressure	101325 Pa
Room temperature	303 K
Viscosity	2.002×10^{-6} kg/m·s
Thermal conductivity	0.02709 w/m·k
Original exponent of the temperature distribution (α_1)	4.0
Original exponent of the velocity distribution (α_2)	6.0

Table 3 The size of the grid

Δx	0.15 m	Δx^*	7.50
Δy	0.003094 m	Δy^*	0.1547
Δz	0.001547 m	Δz^*	0.07735
Δt	0.0005 s	Δt^*	0.0005

$$\rho^* D_u v^* + \frac{R[T^*(T_{in} - T_a) + T_a]}{u_{in}^2} D_{\omega_1} \rho^* + \frac{\rho^* R(T_{in} - T_a)}{u_{in}^2} D_{\omega_1} T^* = 0 \quad (19)$$

If *N_i* = (1, 0, 0, -*u*^{*}) is chosen and $\omega_i = (0, 0, 0, 1, 0)$ is selected, the compatibility equation is

$$\rho^* D_u w^* + \frac{R[T^*(T_{in} - T_a) + T_a]}{u_{in}^2} D_{\omega_2} \rho^* + \frac{\rho^* R(T_{in} - T_a)}{u_{in}^2} D_{\omega_2} T^* = 0 \quad (20)$$

When ω_i in Eq. (16) is used, *N_i* = (1, 0, 0, -*u*^{*} - *c*^{*}) is chosen to satisfy the restriction of Eq. (11). Then Eq. (16) becomes

$$\omega_i = \left(c^{*2}, \gamma c^* \frac{u_{in}}{c_a}, 0, 0, 1 \right) \quad (21)$$

The compatibility equation is

$$\gamma \rho^* c^* D_L u^* + \gamma \rho^* c^{*2} D_{\omega_1} v^* + \gamma \rho^* c^{*2} D_{\omega_2} w^* + \frac{c_a}{u_{in}} c^{*2} D_L \rho^* + \rho^* \left(\frac{T_{in} - T_a}{T_a} \right) \frac{c_a}{u_{in}} D_L T^* = \frac{\gamma c^* u_{in}}{c_a} [G]^* + \frac{u_{in}}{c_a} [F]^* \quad (22)$$

along the direction

$$L = \left(u^* \frac{u_{in}}{c_a} + c^*, v^* \frac{u_{in}}{c_a}, w^* \frac{u_{in}}{c_a}, 1 \right) \quad (23)$$

When ω_i in Eq. (16) is used and *N_i* = (-1, 0, 0, *u*^{*} - *c*^{*}) is chosen, ω_i becomes

$$\omega_i = \left(c^{*2}, -\gamma c^* \frac{u_{in}}{c_a}, 0, 0, 1 \right) \quad (24)$$

The compatibility equation is

$$-\gamma \rho^* c^* D_L u^* + \gamma \rho^* c^{*2} D_{\omega_1} v^* + \gamma \rho^* c^{*2} D_{\omega_2} w^* + \frac{c_a}{u_{in}} c^{*2} D_L \rho^* + \rho^* \left(\frac{T_{in} - T_a}{T_a} \right) \frac{c_a}{u_{in}} D_L T^* = -\frac{\gamma c^* u_{in}}{c_a} [G]^* + \frac{u_{in}}{c_a} [F]^* \quad (25)$$

along the direction

$$L' = \left(u^* \frac{u_{in}}{c_a} - c^*, v^* \frac{u_{in}}{c_a}, w^* \frac{u_{in}}{c_a}, 1 \right) \quad (26)$$

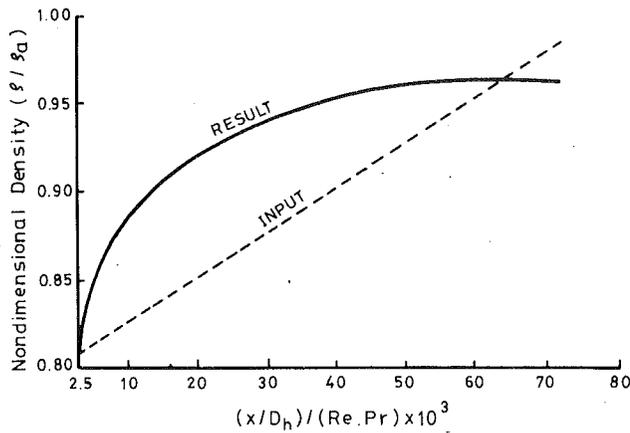


Fig. 2 Air mean density along the duct

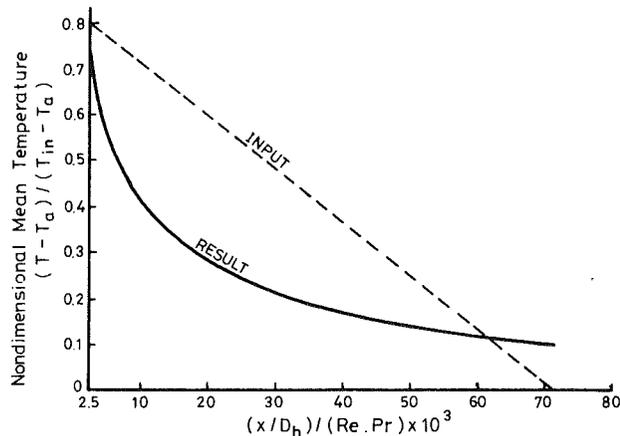


Fig. 3 Air mean temperature along the duct

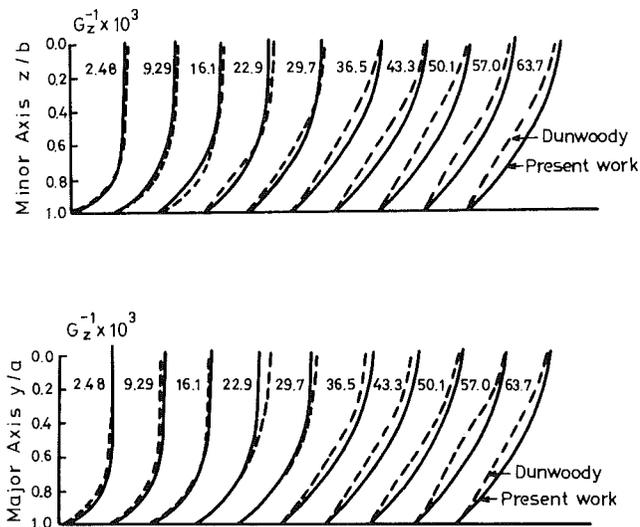


Fig. 4 Air temperature along the duct

Results

The data used in the analysis are shown in Table 1. If the duct is very thin and Biot number is less than 0.01, the wall temperature can be assumed equal to the air temperature which does not generate error more than 5 percent (Days and Crawford, 1980). Using the method of characteristics, the boundary values and the initial conditions of the elliptical duct should be given. The temperature on the surface of the duct is assumed isothermal. The variations of initial air temperature, pressure and velocity are all assumed linearly decreased along the duct. The initial data and the input air properties are shown in Table

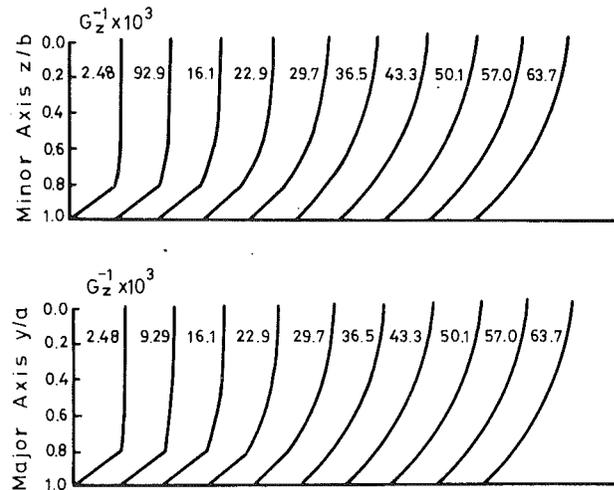


Fig. 5 Air velocity along the duct

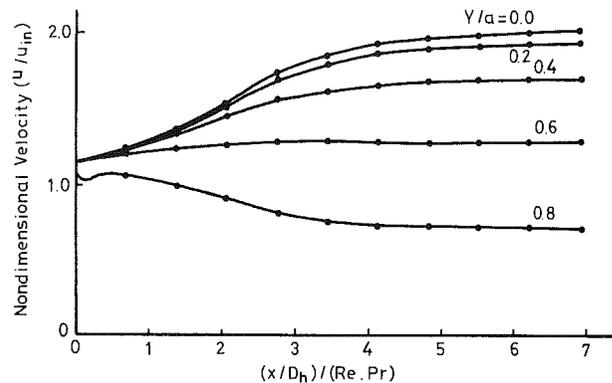


Fig. 6 Air velocity at the major axis along the duct

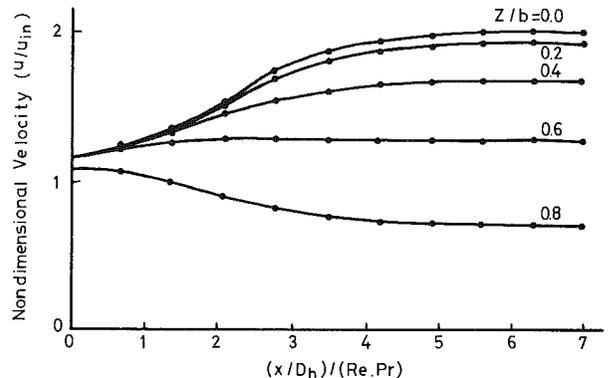


Fig. 7 Air velocity at the minor axis along the duct

2. The time unit is 5×10^{-4} s for each grid. The size of grid and its dimensionless values are shown in Table 3.

The nondimensional assumed data and the result of the air mean density and air mean temperature along the duct are shown in Figs. 2 and 3, respectively. The air temperature distribution at each cross section is shown in Fig. 4. The coordinates are expressed in the dimensionless reciprocal Graetz number Gz^{-1} . The dotted lines in Fig. 4 are data obtained by Dunwoody (1962) used for comparison. The nondimensional air velocity distribution along the duct is shown in Fig. 5. The velocities at the major axis and the minor axis along the duct are shown in Figs. 6 and 7, respectively.

Calculated local Nusselt numbers along the major axis and the minor axis for the isothermal elliptical duct are compared with the results of Dunwoody (1962) and Abdel-Wahed et al.

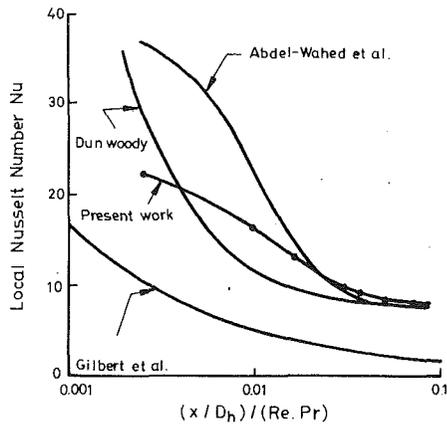


Fig. 8 Local Nusselt number along the major axis for the isothermal elliptical duct

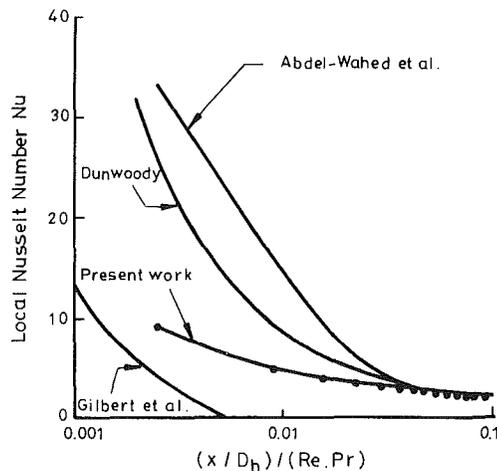


Fig. 9 Local Nusselt number along the minor axis for the isothermal elliptical duct

(1984) as shown in Figs. 8 and 9, respectively. The heat transfer coefficient for the duct obtained from the gradient of the temperature distribution along the wall is shown in Fig. 10.

Discussion

The initial values of α_1 and α_2 are chosen arbitrarily equal to 4 and 6, respectively, to reduce the computer time. These values are revised by the calculated results of the air temperature and velocity until the final results of nondimensional values are converged with the deviation less than 0.001.

The characteristic directions are not unique. Hence, the selection of the characteristic directions has significant effect on the accuracy and the convergent speed of the result. Two factors are considered for the selection of the characteristic directions. One factor is to calculate data of locations easily from locations with known data along the characteristic directions. It is related to the consideration of the arrangement of the grid points. The other factor is that the coefficients and their difference in the compatibility equations should be in the same order of magnitude to avoid the error in solving the simultaneous equations. The characteristic directions should be spread for the proper selection.

The size of the grid and the error are related. The grid points in the y and z directions are relatively less. The speed of convergence is slower and the permissible range for convergence is larger.

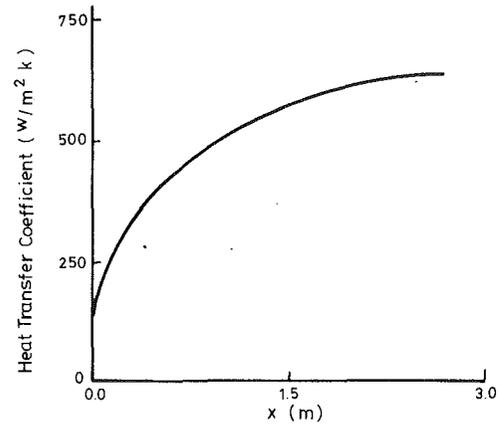


Fig. 10 Heat transfer coefficient along the duct

The calculated results of the local Nusselt numbers along the major axis and minor axis for the isothermal elliptical duct are comparable with the results obtained by Abdel-Wahed et al. (1984), Dunwoody (1962), and Gilbert et al. (1973), as shown in Figs. 8 and 9, respectively. The calculated results agree very well with those obtained by Dunwoody (1962) and Abdel-Wahed et al. (1984) if the reciprocal Graetz number is increased. Therefore, the assumptions, chosen characteristic directions, arrangement and the density of the grids used in this analysis provide very good calculated results.

Conclusion

The calculated results of the three-dimensional flow in the elliptical duct are compared and show good agreement with the available data. The governing equations of the air flow were derived in general form. It is believed that the mathematical model can be extended to analyze the ducts with arbitrary cross sections if the assumed distribution functions are reasonable.

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