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Analytical expressions for relay feed back responses

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Abstract

In this work, a systematic approach to derive analytical expressions for relay feed back responses is proposed. It is based on the observation that a relay feedback test consists of a series of step inputs and a stable limit cycle implies a convergent infinite series. The second order plus dead time processes with different damping coefficients are used to illustrate the derivation. Analytical expressions for typical transfer functions up to 5th order are tabulated and a general expression for nth order system is also given. These analytical expressions are useful to identify unknown system parameters and subsequently for autotuning.

Keywords: Modeling; Relay feedback; Second order process; Identification; Autotuning

1. Introduction

Chemical processes are complex and non-linear in nature due to interactions between inputs and outputs. With increase in complexity of the process, it becomes difficult to control the loops properly. Most of the chemical industries use PID controllers. Proper tuning of PID loops need proper model structure process parameters. Relay feedback is one promising tool for identification of process models. Relay feedback is important as it is used in autotuning of controller parameters. Ultimate gain (*Ku*) and ultimate frequency (*pu*) are used to approximate the process model parameters. But it has been found that those models are not very accurate. A theory of relay control systems based on the concepts of transfer functions and frequency characteristics is presented by Tsypkin [1] that helps to develop computational methods to analyze general properties of relay control systems. Li et al. [2] proposed a method to estimate process gain by using information from two relay experiments. Chang et al. [3] derived transfer functions of first order plus dead time (FOPDT) systems from relay feedback tests with increased accuracy using autotuning variation (ATV) method. But these methods use frequency domain parameters (*Ku* and *pu*), which are derived from describing functions and carry only approximate information of process at ultimate frequency. A method, to derive FOPDT type of systems, was proposed by Wang et al. [4] using multiple method identification and a single relay test. In one separate attempt, Majhi and Atherton [5] proposed a technique to identify plant parameters. But the method needs a correct initial guess and convergence is not guaranteed. Kaya and Atherton [6] described another method (A-locus) to identify low order process parameters from relay auto-tuning response. Closed-form solutions are not available for general transfer functions. Recent progresses in relay feedback is described elaborately by Yu [7]. Leva and Lovera [8] attempted to extract model structure information using singular value decomposition technique. Luyben [9] discussed shape factor in relay feed back for stable and unstable FOPDT systems and explained their identification procedure. Thyagarajan and Yu [10] proposed improved auto-tuning using shape factor of relay feedback response. They discussed identification methods for FOPDT, second order plus dead time (SOPDT) and higher order processes. Though, time domain model equations for relay output is derived for FOPDT systems [10], no exact expressions are available for relay output of SOPDT and higher order processes. So, this paper gives the first approach to derive analytical expressions for relay output of different processes. The remainder of this paper is organized as follows. Section 2 gives the derivation of analytical expressions.

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Results are discussed in Section 3 and a detailed discussion from this study and application of these analytical equations are presented in Section 4. The conclusion is drawn in Section 5.

2. Analytical expressions

According to Astrom and Hagglund [11], when a relay output lags behind the input by π radians, the closed-loop system oscillates around set-point with a period of pu (Fig. 1A). If a relay of magnitude 'h' is inserted in a feedback loop, the input $u(t)$ becomes 'h'. As the output $y(t)$ starts increasing after a time delay of 'D', the relay output switches to opposite direction and becomes $u(t)=-h$. With a phase lag of $-\pi$, a limit cycle of amplitude 'a' is formed and the process variable crosses the set-point. From the principle harmonic approximation of the oscillations, the ultimate gain (Ku) can be approximated [11] as $\mathrm{Ku}=4h/\pi a$ and ultimate frequency (ω_u) thus becomes $\omega u=2\pi/pu$ (where pu is the period of oscillation). Since the shapes of input and output are far from sinusoidal response, the approximation in Ku and ω_u can be off by 2–35%. Let us use SOPDT systems with different time delay to time constant ratios ($D/\tau = 0.01-10.0$) and different damping coefficients ($\xi = 0.2-5.0$) values to illustrate potential deviations from sinusoidal responses (Fig. 2). It can be seen that, at the lower left corner, the oscillation is far from sinusoidal response.

Mathematical models are developed to represent relay responses produced by different systems. The relay output consists of a series of step changes in manipulated variable (with opposite sign). Hence, the stabilized output is a sum of infinite terms of step responses due to those step changes. For systems with time delay, D, the actual relay output lags the input by an amount D. The inputs and outputs can be synchronized by shifting behind the output by an amount D as shown in Fig. 1B, and, in doing this, the time delay D can be eliminated from the expression for relay responses as will be shown later. The shifted version of a typical relay feedback response provides the basis for the derivation.

It is assumed that the relay response is formed by n-number of step changes, of opposite directions ($\pm u$), in input. Each period is one half cycle or from time, $t=0$ to time, $t=pu/2$. In Fig. 3, in the first interval, as time changes from $t=0$ to $t=D$, the response y_1 is produced due to the first step change (u_1). Again, in the second interval, time progressing from D to $D+pu/2$, response y_2 results due to combined effects of step changes u_1 and u_2. Similarly, the effect of u_1, u_2 and u_3 produces y_3 during the third time interval ($D+pu/2$ to $D+pu$). Two half periods ($pu/2$) are of special interests in Fig. 3. The even values of n result in ascending half period (y_{2n}) while the odd values of n formulate the descending half periods (y_{2n+1}). It is interesting to note that the generalized response term (y_n) slowly forms a convergent series. Let us use second order sys-

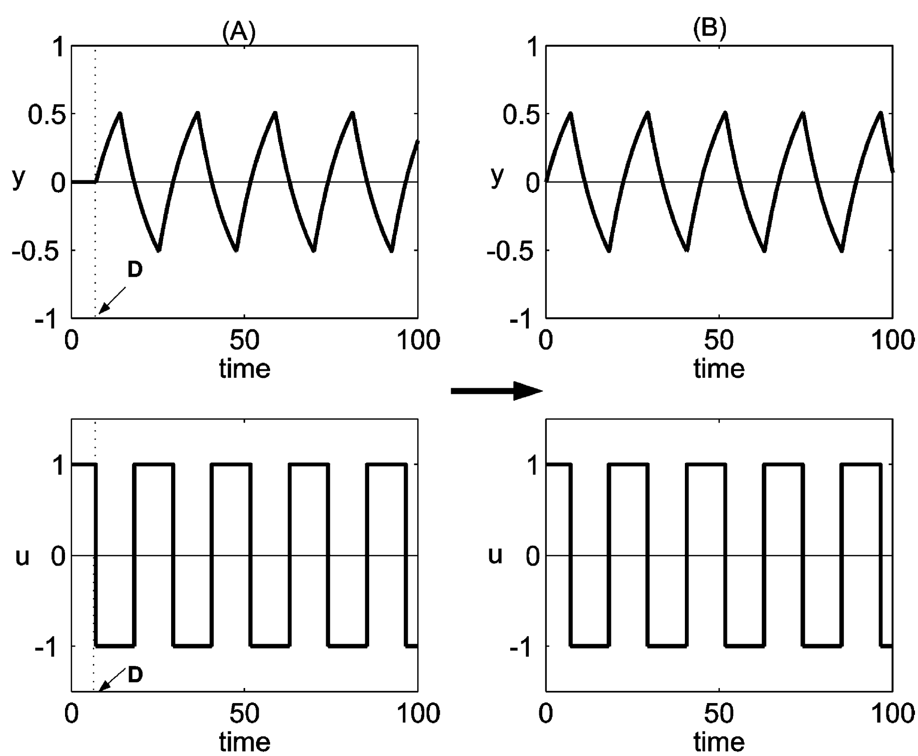


Fig. 1. Schematic representation for the development of analytical expressions of relay feedback responses. A1 is original relay response. Input (u) is shifted by D in A2. Input (u) and output (y) are shifted by $-D$ in A3.

tems to illustrate the derivation as they are rich in system dynamics.

2.1. Overdamped system

The transfer function of an overdamped SOPDT system with damping coefficient greater than one can be expressed as: $G(s)=\frac{k_p e^{-Ds}}{(\tau_1 s+1)(\tau_2 s+1)}$ where k_p is the steady state gain, τ_1 and τ_2 are process time constants with $\tau_1 > \tau_2$, and D is the time delay. The original step response of an overdamped SOPDT can be given by:

$$y = k_p\left[1 - a_1 e^{-(t-D)/\tau_1} + b_1 e^{-(t-D)/\tau_2}\right]$$

where a_1 and b_1 are given by

$$a_1 = \frac{\tau_1}{\tau_1-\tau_2} \quad \text{and} \quad b_1 = \frac{\tau_2}{\tau_1-\tau_2}$$

Under the shifted version (Figs. 1B or 3), the first segment of the relay response (y_1) is simply the step response without time delay in the time index.

$$y_1 = k_P\left[1 - a_1 e^{-t/\tau_1} + b_1 e^{-t/\tau_2}\right] \tag{1}$$

At second instant, the time is reset to zero at the initial point. The step response (relay output) can be given by [i.e., introducing a time shift by D amount in the Eq. (1)]

$$y_2 = k_p\left[1-a_1 e^{-\frac{t+D}{\tau_1}} + b_1 e^{-\frac{t+D}{\tau_2}}\right] - 2k_p\left[1 - a_1 e^{-\frac{t}{\tau_1}} + b_1 e^{-\frac{t}{\tau_2}}\right]$$

Here the first term represents the effect of the first step change (occurred D time earlier) and the second term shows the effect of the second step input, switching to the opposite direction. The above Equation can be simplified as

$$y_2 = k_p\left\{[1-2] - a_1 e^{-\frac{t}{\tau_1}}\left(e^{-\frac{D}{\tau_1}} - 2\right) + b_1 e^{-\frac{t}{\tau_2}}\left(e^{-\frac{D}{\tau_2}} - 2\right)\right\} \tag{2}$$

The relay response at the third interval, is the result of three step changes, lags by an amount $D+pu/2$ from input. After introducing a time shift of $D+pu/2$ in Eq. (1), the net effect becomes:

$$y_3 = k_p\left\{\left[1 - a_1 e^{-\frac{t+D+\frac{pu}{2}}{\tau_1}} + b_1 e^{-\frac{t+D+\frac{pu}{2}}{\tau_2}}\right] -2\left[1 - a_1 e^{-\frac{t+\frac{pu}{2}}{\tau_1}} + b_1 e^{-\frac{t+\frac{pu}{2}}{\tau_2}}\right] + 2\left[1 - a_1 e^{-\frac{t}{\tau_1}} + b_1 e^{-\frac{t}{\tau_2}}\right]\right\}$$

which can be easily simplified as

$$y_3 = k_p\left\{[1-2+2] - a_1 e^{-\frac{t}{\tau_1}}\left[e^{-\frac{D+\frac{pu}{2}}{\tau_1}} - 2e^{-\frac{pu}{2\tau_1}} + 2\right] + b_1 e^{-\frac{t}{\tau_2}}\left[e^{-\frac{D+\frac{pu}{2}}{\tau_2}} - 2e^{-\frac{pu}{2\tau_2}} + 2\right]\right\} \tag{3}$$

It can be seen that the terms in the right hand side (RHS) of the above equation are slowly forming a series.

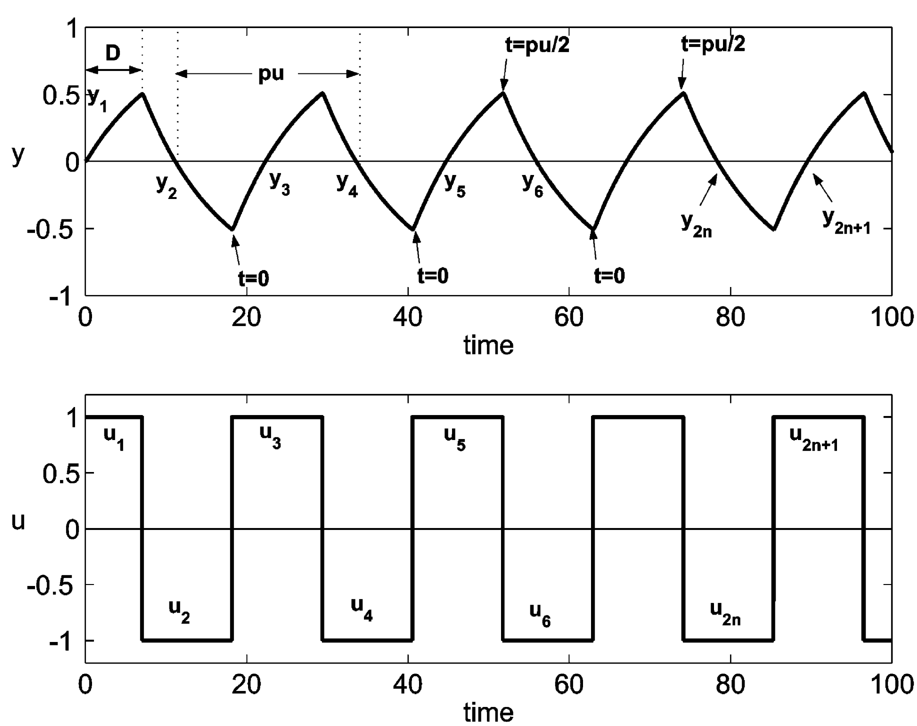


Fig. 2. Shifted version of relay output response of a typical SOPDT system.

With the progress of time, response becomes stabilized and the general expression for the nth term can be described as

$$\begin{aligned} y_n = k_p\Big\{&[1-2+2-\ldots] - a_1 e^{-\frac{t}{\tau_1}}\Big[e^{-D+(n-2)pu/2/\tau_1} \\ &-2e^{-\frac{(n-2)pu}{2\tau_1}} + 2e^{-\frac{(n-1)pu}{2\tau_1}} - \ldots + 2e^{-\frac{pu}{2\tau_1}} - 2\Big] \\ &+b_1 e^{-\frac{t}{\tau_1}}\Big[e^{-D+(n-2)pu/2/\tau_2} - 2e^{-\frac{(n-2)pu}{2\tau_2}} + 2e^{-\frac{(n-1)pu}{2\tau_2}} \\ &-\ldots + 2e^{-\frac{pu}{2\tau_2}} - 2\Big]\Big\} \end{aligned} \tag{4}$$

The RHS of the above Eq. (4) has three parts and each part consists of an infinite series, F_1, F_2 and F_3.

$$y_n = k_p\Big\{F_1 - a_1 e^{-\frac{t}{\tau_1}} F_2 + b_1 e^{-\frac{t}{\tau_2}} F_3\Big\}$$

Let $v_1 = \dfrac{pu}{2\tau_1}$ and $r = e^{-v_1}$. The first series, F_1 is simply

$$F_1 = [1-2+2-2+\ldots] = (1)^{2n-1} = 1$$

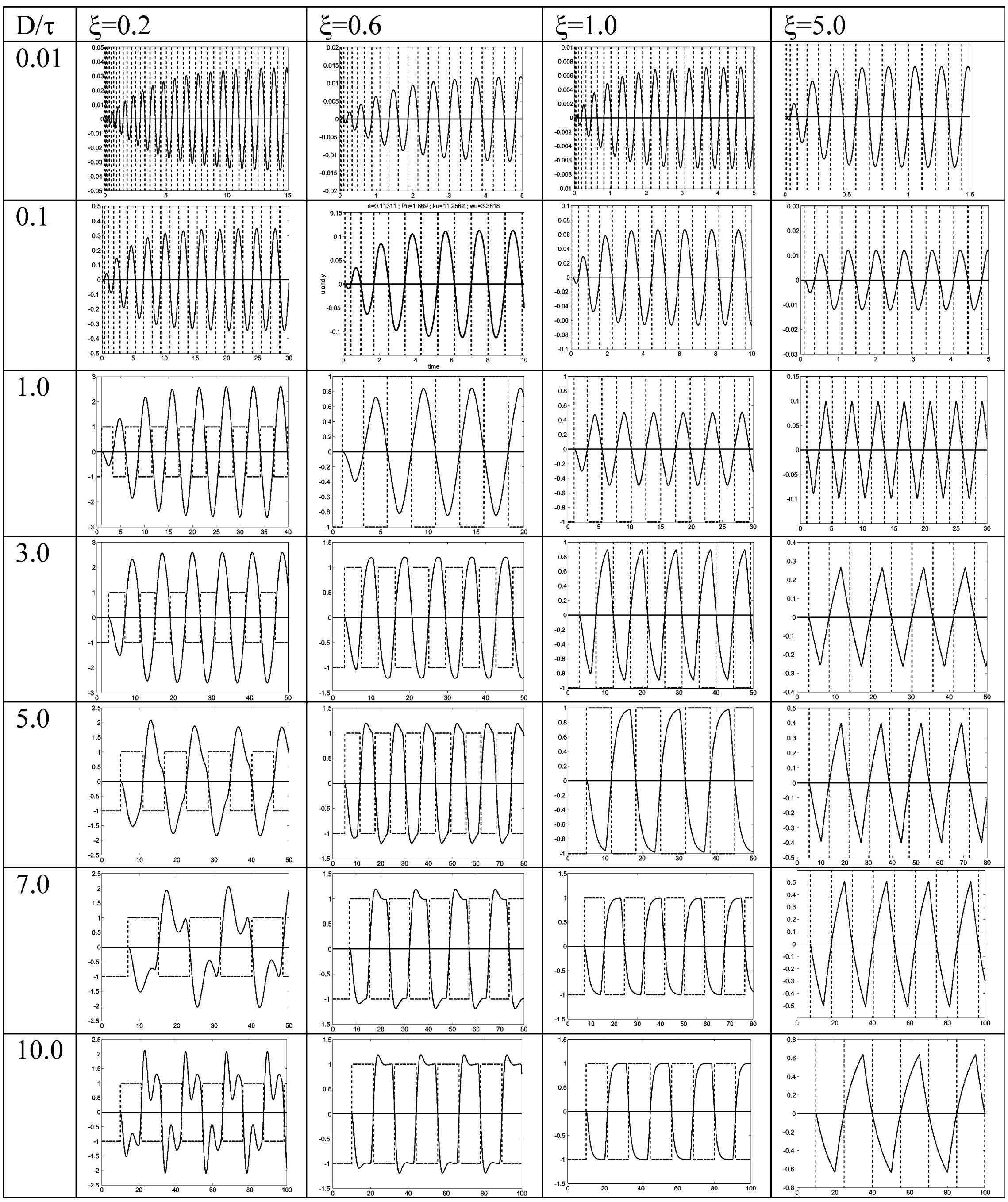


Fig. 3. Relay feedback responses for some SOPDT processes with different D/τ and ξ values.

If n is odd, the second series becomes,

$$F_2 = \left[e^{-\frac{D}{\tau_1}}r^{n-2} - 2r^{n-2} + 2r^{n-3} - 2r^{n-4} + \cdots - 2r + 2\right]$$

This above series is convergent and can be put into the following form (note that terms are rearranged from the back side of above expression)

$$F_2 = \lim_{n\to\infty}\left(e^{-D/\tau}r^{n-2}\right) + 2\left(1 - r + r^2 - r^3 + \ldots\right)$$

hence

$$F_2 = 2\left[1 - r + r^2 - r^3 + \cdots\right] = \frac{2}{1+r} = \frac{2}{1+e^{-\frac{pu}{2\tau_1}}}$$

In a similar way, the F_3 of RHS of Eq. (4) can also be simplified. Ultimately, the response can be given by :

$$y_n = k_p\left\{1 - a_1 e^{-\frac{t}{\tau_1}}\left(\frac{2}{1+e^{-Pu/2/\tau_1}}\right) + b_1 e^{-\frac{t}{\tau_2}}\left(\frac{2}{1+e^{-Pu/2/\tau_2}}\right)\right\} \tag{5}$$

This represents the ascending response (n is odd). Since this response is dissymmetric, the general form can be employed as:

$$y_n = k_p\left\{-1 + a_1 e^{-\frac{t}{\tau_1}}\left(\frac{2}{1+e^{-Pu/2/\tau_1}}\right) - b_1 e^{-\frac{t}{\tau_2}}\left(\frac{2}{1+e^{-Pu/2/\tau_2}}\right)\right\}(-1)^n \tag{6}$$

2.2. Critically damped systems

The transfer function of a criticallydamped SOPDT system with damping coefficient greater than one can be expressed as: $G(s) = \frac{k_p e^{-Ds}}{(\tau s+1)^2}$ where k_p is the steady state gain, τ is process time constant, and D is the time delay. The original step response of a criticallydamped SOPDT process can be given by:

$$y = k_p\left[1 - \left(1 + \frac{t-D}{\tau}\right)e^{-\frac{t-D}{\tau}}\right]$$

Similar to the procedure followed in Section 2.1, the first segment of the relay response (y_1) is simply the step response without time delay in the time index.

$$y_1 = k_p\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-\frac{t}{\tau}}\right] \tag{7}$$

For the second segment, the time is reset to zero at the initial point and, the relay response can be given by [i.e., introducing a time shift of D in Eq. (7)]

$$y_2 = k_P\left[1 - \left(1 + \frac{t+D}{\tau}\right)e^{-\frac{t+D}{\tau}}\right] - 2k_p\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-\frac{t}{\tau}}\right]$$

The above equation can be simplified as

$$y_2 = k_p\left\{[1-2] - e^{-\frac{t}{\tau}}\left[e^{-\frac{D}{\tau}} - 2\right] - e^{-\frac{t}{\tau}}\left[\frac{t+D}{\tau}e^{-\frac{D}{\tau}} - 2\frac{t}{\tau}\right]\right\} \tag{8}$$

The relay output at the third interval is the result of three step changes and [i.e., introducing a time shift $D + pu/2$ in Eq. (7) can be expressed as

$$y_3 = k_p\left\{[1-2+2] - e^{-\frac{t}{\tau}}\left[e^{-\frac{D+\frac{pu}{2}}{\tau}} - 2e^{-pu/2/\tau} + 2\right] - e^{-t/\tau}\left[\frac{t+D+\frac{pu}{2}}{\tau}e^{-\frac{D+\frac{pu}{2}}{\tau}} - 2\frac{t+\frac{pu}{2}}{\tau}e^{-pu/2/\tau} + 2\frac{t}{\tau}\right]\right\} \tag{9}$$

Proceeding similarly as described in Section 2.1 above, the stabilized response can be expressed as

$$\begin{aligned} y_n = k_p\Big\{&[1-2+2-2+\cdots] \\ &- e^{-\frac{t}{\tau}}\left[e^{-D+(n-2)pu/2/\tau} - 2e^{-(n-2)pu/2/\tau} + \cdots + 2\right] \\ &- e^{-t/\tau}\Big[\frac{t+D+(n-2)pu/2}{\tau}e^{-D+(n-2)pu/2/\tau} \\ &- 2\frac{t+(n-2)pu/2}{\tau}e^{-(n-2)pu/2/\tau} \\ &+ 2\frac{t+(n-1)pu/2}{\tau}e^{-(n-1)pu/2/\tau} - \cdots + 2\frac{t}{\tau}\Big]\Big\} \end{aligned} \tag{10}$$

It can be seen that, in the above Eq. (10), the RHS has three parts of which each part is an infinite series (F_1, F_2 and F_3) and they are convergent.

$$y_n = k_P\{F_1 - F_2 - F_3\}$$

Let $r = -e^{-\frac{pu}{2\tau}}$ and $q = \frac{t}{\tau}$ and $v = \frac{pu}{2\tau}$.

then $F_1 = [1-2+2-2\cdots] = (1)^{2n-1} = 1$

$$F_2 = e^{-\frac{t}{\tau}}\cdot 2\left[1 - r + r^2 - r^3 + \cdots\right] \quad = e^{-\frac{t}{\tau}}\left[\frac{2}{1+e^{-\frac{Pu}{2\tau}}}\right]$$

and $F_3 = e^{-\frac{t}{\tau}}\left[q + (q+v)r + (q+2v)r^2 + \cdots\right]$ or

$$\text{then } F_3 = 2e^{-\frac{t}{\tau}}\left[\frac{\frac{t}{\tau}}{1+e^{-\frac{pu}{2\tau}}} + \frac{-\frac{pu}{2\tau}e^{-\frac{Pu}{2\tau}}}{\left(1+e^{-\frac{pu}{2\tau}}\right)^2}\right]$$

So, ultimately, the response can be expressed analytically as

$$y_n = k_p\left\{1 - e^{-\frac{t}{\tau}}\left[\frac{2}{1+e^{-\frac{pu}{2\tau}}}\right] - 2e^{-\frac{t}{\tau}}\left[\frac{\frac{t}{\tau}}{1+e^{-\frac{pu}{2\tau}}} + \frac{-\frac{pu}{2\tau}e^{-\frac{pu}{2\tau}}}{\left(1+e^{-\frac{pu}{2\tau}}\right)^2}\right]\right\}$$

This represents the ascending response (n is odd). The general form can be employed as (n = odd for ascending trends or even for descending trends),

$$y_n = k_p\left\{-1 + e^{-\frac{t}{\tau}}\left[\frac{2}{1+e^{-\frac{pu}{2\tau}}}\right] + 2e^{-\frac{t}{\tau}}\left[\frac{\frac{t}{\tau}}{1+e^{-\frac{pu}{2\tau}}} + \frac{-\frac{pu}{2\tau}e^{-\frac{pu}{2\tau}}}{\left(1+e^{-\frac{pu}{2\tau}}\right)^2}\right]\right\}(-1)^n \quad (11)$$

2.3. Underdamped system

The second order under damped process can be described as

$$\frac{y(s)}{u(s)} = \frac{k_p e^{-Ds}}{\tau^2 s^2 + 2\xi\tau s + 1}$$

or

$$\frac{y(s)}{u(s)} = \frac{e^{-Ds}}{(s+p_1)(s+p_2)}$$

where $p_1 = \frac{-\xi - i\beta}{\tau}$ and $p_2 = \frac{-\xi + i\beta}{\tau}$ with $\beta = \sqrt{1-\xi^2}$.

Here k_p is the steady-state gain, τ is process time constant, ξ is damping coefficient, and D is the time delay. The original step response for unit step change can be expressed as

$$y = k_p\left[1 - a_1 e^{p_2(t-D)} + b_1 e^{p_1(t-D)}\right]$$

where $a_1 = \dfrac{p_1}{p_1 - p_2}$ and $b_1 = \dfrac{p_2}{p_1 - p_2}$

Similar to the procedure followed in Sections 2.1 and 2.2 above, the relay response for the first segment can be given, by shifting the output by D (as in Fig. 2), as

$$y_1 = k_p\left[1 - a_1 e^{p_2 t} + b_1 e^{p_1 t}\right] \quad (12)$$

At second instant, the relay response is the result of two step changes, and can be given by [i.e., introducing a time shift by D amount in the Eq. (12)]

$$y_2 = k_p\left[1 - a_1 e^{p_2(t-D)} + b_1 e^{p_1(t-D)}\right] - 2k_p\left[1 - a_1 e^{p_2 t} + b_1 e^{p_1 t}\right]$$

The above equation can be rearranged as

$$y_2 = k_p\left\{[1-2] - a_1 e^{p_2 t}\left(e^{-Dp_2} - 2\right) + b_1 e^{p_1 t}\left(e^{-Dp_1} - 2\right)\right\} \quad (13)$$

Similarly, the relay response for the third interval is the result of three step changes. After introducing a time shift of $D + pu/2$ in Eq. (12), the net effect becomes

$$y_3 = k_p\left\{[1-2+2] - a_1 e^{p_2 t}\left[e^{-(D+\frac{pu}{2})p_2} - 2e^{-\frac{pu}{2}\cdot p_2} + 2\right] + b_1 e^{p_1 t}\left[e^{-(D+\frac{pu}{2})p_1} - 2e^{-\frac{pu}{2}\cdot p_1} + 2\right]\right\} \quad (14)$$

By proceeding this way, the general expression for the relay response can be given as

$$y_n = k_p(-1)^n\{[1-2+2-2+\ldots] - a_1 e^{p_2 t}\left[e^{-(D+(n-2)pu/2)p_2} - 2e^{-((n-2)pu/2)p_2} + 2e^{-((n-1)pu/2)p_2} - \cdots + 2\right] + b_1 e^{p_1 t}\left[e^{-(D+(n-2)pu/2)p_1} - 2e^{-((n-2)pu/2)p_1} + 2e^{-((n-1)pu/2)p_1} - \cdots + 2\right]\} \quad (15)$$

It can be seen that, in the above Eq. (15), the RHS has three parts of which each part is an infinite series (convergent). These series can be simplified and Eq. (15) can be rewritten as

$$y_n = k_p(-1)^n\left\{-1 + a_1\left[\frac{2}{1+e^{\frac{pu}{2p_2}}}\right]e^{p_2 t} - b_1\left[\frac{2}{1+e^{\frac{pu}{2p_1}}}\right]e^{p_1 t}\right\} \quad (16)$$

The above equation is the general form of relay output (n = odd for ascending trends or even for descending trends).

Eq. (16) is a function of complex poles. It can be further simplified, as it contains conjugate poles, as follows: e^{ix} can be written as

$$e^{ix} = \cos(x) + i\cdot\sin(x)$$

hence,

$$y_n = k_p(-1)^n\left\{-1 + e^{-\frac{\xi t}{\tau}}\left[(c_1 - c_2)\cos\left(\frac{\beta t}{\tau}\right) + i(c_1 + c_2)\sin\left(\frac{\beta t}{\tau}\right)\right]\right\} \quad (17)$$

where

$$c_1 = \frac{p_1}{p_1 - p_2}\cdot\frac{2}{1+e^{\frac{Pu}{2p_2}}} \quad \text{and} \quad c_2 = \frac{p_2}{p_1 - p_2}\cdot\frac{2}{1+e^{\frac{Pu}{2p_1}}}$$

Now mathematically, it can be further simplified astaking $r = e^{-\frac{pu.\xi.\tau}{2}}$ and $\theta = \frac{pu.\beta.\tau}{2}$

$$c_1 - c_2 = \frac{2}{\beta}\left\{\frac{\beta + \beta r\cos(\theta) - \xi r\sin(\theta)}{1 + 2r\cos(\theta) + r^2}\right\}$$

and,

$$c_1 + c_2 = \frac{2}{i\beta}\left\{\frac{\xi + \xi r\cos(\theta) + \beta r\sin(\theta)}{1 + 2r\cos(\theta) + r^2}\right\}$$

After simplification, the complex conjugate parts cancel each other and the equation can be rewritten as

$$y_n = k_p\left\{-1 + 2\frac{e^{-\frac{\xi t}{\tau}}}{\beta}\sin\left(\frac{\beta t}{\tau} + \alpha\right)\right\}(-1)^n \tag{18}$$

where

$$\alpha = \tan^{-1}\left(\frac{\beta + \beta r\cos(\theta) - \xi r\sin(\theta)}{\xi + \xi r\cos(\theta) + \beta r\sin(\theta)}\right)$$

Eq. (18) represents the ascending response (n is odd).

Responses obtained after simulating the above models are compared with original relay responses from similar systems.

3. Results

Different types of transfer functions are considered and the analytical expressions for their relay feedback output response are developed following the above procedure (as discussed in Section 2 above). Table 1 gives a list of first order plus dead time and second order plus dead time processes and their corresponding mathematical expressions for the stabilized relay feedback output responses. These equations (y_n) denote the upward or ascending trend (or sometimes, curves in the lower part of midline for higher order systems) of relay feedback output (while time, t, changes from 0 to $pu/2$). The downward or descending trend can be obtained by reversing the sign of the output ($-y_n$).

In this Table 1, individual expression, for relay feedback responses of 1st, 2nd and 3rd order systems, contains terms similar to that of corresponding equation for step responses; except, they differ only in weighing factor ($\frac{2}{1+e^{-pu/2\tau}}$). (i.e., If we compare the terms of expressions of relay feedback response with those of step response of a process, we see that they differ by weighing factor of $\frac{2}{1+e^{-pu/2\tau}}$). For a FOPDT system, the response starts ($t=0$) from the minimal point, at $y=-a$, and ends ($t=pu/2$) at maximal point, at $y=a$. Also note that for an unstable FOPDT system, stable limit cycles can occur only if $D/\tau < \ln(2)$. For the lead/lag second order system (No.6 in Table 1), the expression is applicable to systems with left-half plane ($\tau_3 > 0$) or right half plane ($\tau_3 < 0$) zero.

Analytical expressions of relay feed back output responses for higher order systems are presented in Table 2. They are of much interest because when we see, for example, the expression for fifth order process, the equation contains mainly 5 terms (except '1'). Each of these terms represents corresponding lower order processes. The first term inside the third bracket of first line/row appears to be for a FOPDT. The second term

Table 1
Time response (y_n) of relay feedback for FOPDT, SOPDT and third order processes

No.	Process transfer functions	Time response (y_n) of relay feedback (y_n is for ascending part of response: $-y_n$ is for descending part)	Remarks
1	$\frac{K_P e^{-Ds}}{\tau s+1}$	$K_p\left(1 - e^{-t/\tau}\left[\frac{2}{1+e^{-pu/2\tau}}\right]\right)$	In a particular cycle, response starts from min point and reaches to a max. point
2	$\frac{K_P e^{-Ds}}{\tau s-1}$	$K_p\left(1 - e^{t/\tau}\left[\frac{2}{1+e^{pu/2\tau}}\right]\right)$	D/τ <ln (2)
3	$\frac{K_P e^{-Ds}}{(\tau s+1)^2}$	$K_p\left\{1 - e^{-\frac{t}{\tau}}\left[\frac{2}{1+e^{-pu/2\tau}}\right] - 2e^{-\frac{t}{\tau}}\left[\frac{t/\tau}{1+e^{-pu/2\tau}} + -(pu/2\tau)e^{-pu/2\tau}/\left(1+e^{-pu/2\tau}\right)^2\right]\right\}$	
4	$\frac{K_P e^{-Ds}}{(\tau_1 s+1)(\tau_2 s+1)}$	$k_p\left\{1 - a_1 e^{-\frac{t}{\tau_1}}\left(\frac{2}{1+e^{-pu/2\tau_1}}\right) + b_1 e^{-\frac{t}{\tau_2}}\left(\frac{2}{1+e^{-pu/2\tau_2}}\right)\right\}$ where $a_1 = \frac{\tau_1}{\tau_1-\tau_2}$ and $b_1 = \frac{\tau_2}{\tau_1-\tau_2} : \tau_1 > \tau_2$	
5	$\frac{K_P e^{-Ds}}{\tau^2 s^2 + 2\xi\tau s+1}$	$k_p\left\{1 - 2\frac{e^{-\frac{\xi t}{\tau}}}{\beta}\sin\left(\frac{\beta t}{\tau}+\alpha\right)\right\}$ where $\alpha = \tan^{-1}\left(\frac{\beta+\beta r\cos(\theta)-\xi r\sin(\theta)}{\xi+\xi r\cos(\theta)+\beta r\sin(\theta)}\right)$, $\beta = \sqrt{1-\xi^2}$, $r = e^{-\frac{pu.\xi.\tau}{2}}$ and $\theta = pu.\beta.\tau/2$	
6	$\frac{K_P(\tau_3 s+1)e^{-Ds}}{(\tau_1 s+1)(\tau_2 s+1)}$	$k_p\left\{1 - a_1 e^{-\frac{t}{\tau_1}}\left(\frac{2}{1+e^{-pu/2\tau_1}}\right) + b_1 e^{-\frac{t}{\tau_2}}\left(\frac{2}{1+e^{-pu/2\tau_2}}\right)\right\}$ where $a_1 = \frac{\tau_1-\tau_3}{\tau_1-\tau_2}$ and $b_1 = \frac{\tau_2-\tau_3}{\tau_1-\tau_2} : \tau_1 > \tau_2$ $\tau_3 >$ or < 0	
7	$\frac{K_P}{(\tau s+1)^3}$	$K_p\left\{1 - 2e^{-t/\tau}\left[\left(\frac{1}{1-r}\right) + \left(\frac{q}{1-r} + \frac{rv}{(1-r)^2}\right) + \left(\frac{1}{2!}\right)\left(\frac{q^2}{1-r} + \frac{2qvr}{(1-r)^2} + \frac{v^2 r(1+r)}{(1-r)^3}\right)\right]\right\}$ where $q = t/\tau$, $v = pu/2\tau$ and $r = -e^{-pu/2\tau}$	
8	$\frac{K_P}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$	$k_p\left\{1 + a_1 e^{-\frac{t}{\tau_1}}\left(\frac{2}{1+e^{-pu/2\tau_1}}\right) + b_1 e^{-\frac{t}{\tau_2}}\left(\frac{2}{1+e^{-pu/2\tau_2}}\right) + c_1 e^{-\frac{t}{\tau_3}}\left(\frac{2}{1+e^{-pu/2\tau_3}}\right)\right\}$ where $a_1 = \frac{\tau_1^2(\tau_1-\tau_2)}{(\tau_1-\tau_3)}$, $b_1 = \frac{\tau_2^2(\tau_3-\tau_2)}{(\tau_1-\tau_2)}$ and $c_1 = \frac{\tau_3^2}{(\tau_1\tau_2 - \tau_2\tau_3 - \tau_3\tau_1 + \tau_3^2)} : \tau_1 > \tau_2 > \tau_3$	

Table 2
Time response (y_n) of relay feedback for fourth and higher order processes (see Appendix for derivation)

No.	Process transfer functions	Time response (y_n) of relay feedback (y_n is for ascending part of response: $-y_n$ is for descending part)
9	$\frac{K_P}{(\tau s+1)^4}$	$K_p\left\{1-2e^{-t/\tau}\left[\frac{1}{1-r}+\frac{q}{1-r}+\frac{rv}{(1-r)^2}+\frac{1}{2!}\left(\frac{q^2}{1-r}+\frac{2qvr}{(1-r)^2}+\frac{v^2r(1+r)}{(1-r)^3}\right)\right] -\frac{2}{3!}e^{-t/\tau}\left[\frac{q^3}{1-r}+\frac{3q^2dr}{(1-r)^2}+\frac{3qv^2r(1+r)}{(1-r)^3}+\frac{v^3r(r^2+4r+1)}{(1-r)^4}\right]\right\}$ where $q=t/\tau$, $v=pu/2\tau$ and $r=-e^{-pu/2\tau}$
10	$\frac{K_P}{(\tau s+1)^5}$	$K_p\left\{1-2e^{-t/\tau}\left[\frac{1}{1-r}+\left(\frac{q}{1-r}+\frac{rv}{(1-r)^2}\right)+\frac{1}{2!}\left(\frac{q^2}{1-r}+\frac{2qvr}{(1-r)^2}+\frac{v^2r(1+r)}{(1-r)^3}\right)\right] -\frac{2}{3!}e^{-t/\tau}\left[\frac{q^3}{1-r}+\frac{3q^2vr}{(1-r)^2}+\frac{3a_1v^2r(1+r)}{(1-r)^3}+\frac{v^3r(r^2+4r+1)}{(1-r)^4}\right] -\frac{2}{4!}e^{-t/\tau}\left[\frac{q^4}{1-r}+\frac{4q^3vr}{(1-r)^2}+\frac{6q^2v^2r(1+r)}{(1-r)^3}+\frac{4qv^3r(r^2+4r+1)}{(1-r)^4}+\frac{v^4r(r^3+11r^2+11r+1)}{(1-r)^5}\right]\right\}$ where $q=t/\tau$, $v=pu/2\tau$ and $r=-e^{-pu/2\tau}$
11	$\frac{K_P}{(\tau s+1)^n}$	$K_p\left\{1-2e^{-q}\left[\frac{1}{0!}\left(\frac{1}{1-r}\right)+\frac{1}{1!}\left(\frac{q}{1-r}+\frac{rv}{(1-r)^2}\right)+\frac{1}{2!}\left(\frac{q^2}{1-r}+\frac{2qvr}{(1-r)^2}+\frac{v^2r(1+r)}{(1-r)^3}\right)\right] -\frac{2}{3!}e^{-q}\left[\frac{q^3}{1-r}+\frac{3q^2vr}{(1-r)^2}+\frac{3qv^2r(1+r)}{(1-r)^3}+\frac{v^3r(r^2+4r+1)}{(1-r)^4}\right]+\ldots -\frac{2}{(n-1)!}e^{-q}\left[\frac{{}^{n-1}C_0q^{n-1}}{1-r}+\frac{{}^{n-1}C_1q^{n-2}vr}{(1-r)^2}+\frac{{}^{n-1}C_2q^{n-3}v^2r(1+r)}{(1-r)^3}+\frac{{}^{n-1}C_3q\ v^3r(r^2+4r+1)}{(1-r)^4}+\ldots+{}^{n-1}C_{n-1}v^{n-1}\sum_{d=0}^{\infty}d^{n-2}r^d\right]\right\}$ where $q=t/\tau$, $v=pu/2\tau$ and $r=-e^{-pu/2\tau}$

(having two terms inside first bracket) is for a SOPDT (critically damped). The third term (having three terms inside first bracket) is for a third order process. The terms in the second row/line (having 4 terms inside) are for a fourth order process. In the third or last row/line there are five terms for a fifth order process. Hence, the number of terms (size of the series) for a particular order of process is rhythmic. These tables are similar to the tables of Inverse Laplace transform and will help to find out equation for relay feedback responses. Derivation of higher order systems is shown in Appendix.

4. Discussion

4.1. Shapes

Luyben [9] and Thyagarajan and Yu [10] discussed different kinds of shapes evolved in relay feedback response curves. Depending on its D/τ ratio, a system takes time to reach to stabilized oscillations. These swings contain process information like oscillation frequency, amplitude etc. It can be noted that the term 'D' does not appear in the analytical expressions (Tables 1 and 2) of relay feedback. D is inherent in pu. When D/τ or D is small, pu becomes smaller and in effect ω_u becomes larger. This gives rise to oscillations with smooth and rounded peaks. Fig. 4 explains the effect of dead time on pu and ω_u for any system. The values of corresponding pu and ω_u are shown in Table 3. So, processes with small D/τ produce oscillations with low pu but high ω_u values and the shapes of relay response become almost triangular (Fig. 4a). On the other hand, processes with large D/τ yield oscillations with higher pu but lower ω_u values and the shapes of relay response become almost rectangular (Fig. 4c). If we see the analytical equation for FOPDT systems, for a very low D/τ, the expression reduces almost to $y=1-e^{-t/\tau}$. Moreover, small pu implies only the initial response is utilized to shape the relay response and therefore, response becomes triangular.

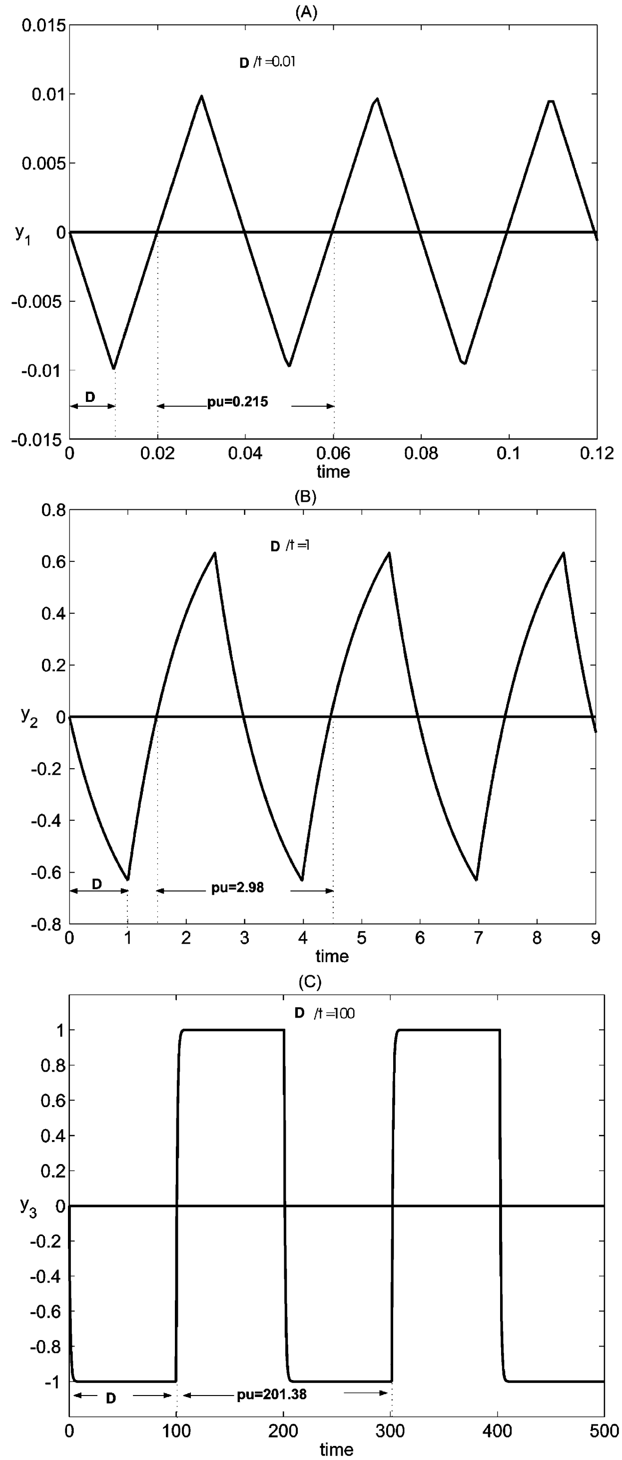


Fig. 4. Effect of D on pu and ω_u for $y_1 = \frac{1.0e^{-1S}}{S+1}$, $y_2 = \frac{1.0e^{-10S}}{S+1}$ and $y_3 = \frac{1.0e^{-100S}}{S+1}$.

Similarly, for a very large D/τ, the analytical expression for FOPDT becomes $y = 1 - 2e^{-t/\tau}$. Because of large pu, the response is fully developed and the shape becomes rectangular. Hence pu has an direct effect on time delay as well as shape of the response.

Table 3
Effect of D on pu and ω_u

Transfer function	D/τ	pu	ω_u
$y_1 = \frac{1.0e^{-0.01S}}{S+1}$	0.01	0.215	29.2241
$y_2 = \frac{1.0e^{-10S}}{S+1}$	1.0	2.98	2.1085
$y_3 = \frac{1.0e^{-100S}}{S+1}$	100	201.38	0.0312

4.2. Types of equations

Two kinds of responses can be observed in the analytical expressions in Tables 1 and 2. These responses are tabulated in Fig. 5. Systems with serial numbers 1 and 2 in Table 1, produce monotonic response where, at $t = 0$, response from model starts at lower most (or upper most) point (A or B) and, at $t = pu/2$, it ends at the other extreme point (B or C). Whereas processes with serial numbers 3, 4, 5 and 6 in Table 1, give non-monotonic response with time (t) where $0 < t < pu/2$. At $t = 0$, the position of starting point (A or B) of response is in between $y = 0$ and $y = \pm a$ along the relay output curve (lower or upper meniscus). At $t = pu/2$, the response ends at point (B or C) which is in between $y = 0$ and $y = \pm a$ along the relay output curve (upper or lower meniscus).

As the D/τ value increases, pu increases and the point A, (the starting point of response) shifts towards one pinnacle position along the relay output curve. The time (Δt) between point A and lower most point (at which $y = -a$) decreases with increase in the D/τ value. Based on the values of Δt, class of non-monotonic responses can be subdivided into three types. The first type (with large D/τ) has Δt almost equal to zero and they may be considered as practically monotonic systems. The second type has moderate Δt values. The third type (nos. 7 and 8 in Table 1 and 9–11 in Table 2) is higher order systems without time delay (i.e., $n \geqslant 3$). For this type of systems, this value occurs at the mid-point of the half period (i.e. $\Delta t = pu/4$). This is important because this simplify the parameter identification procedure.

Fig. 5 shows trueness of validation of the derived mathematical models. If the relay height is other than unity, then the model for the relay output response will be just multiplied by actual value of relay height (h).

4.3. Application

It can be observed from Fig. 5 that the responses of generalized analytical expressions start at point A and end at point B for a time period from 0 to $pu/2$. The position of point A lies either in the peak (minimum or maximum point) or in-between in these ascending or descending curves. In identification of parameters, the landmark points and their values can be observed from

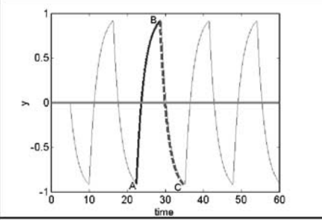
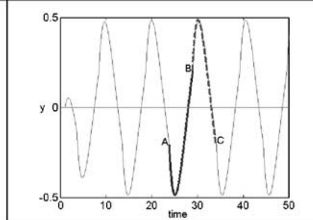
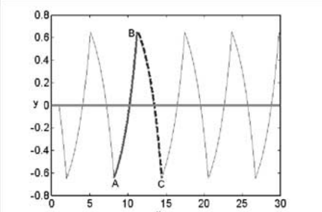
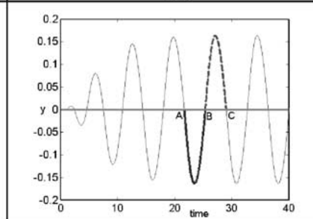
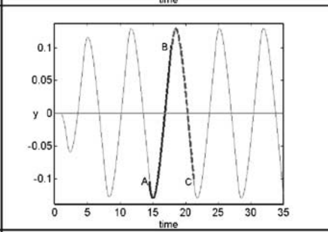
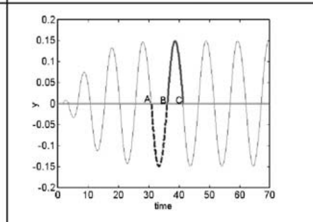
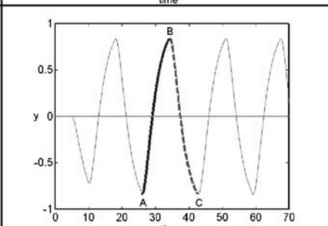
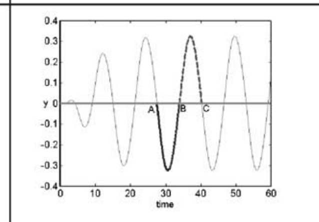
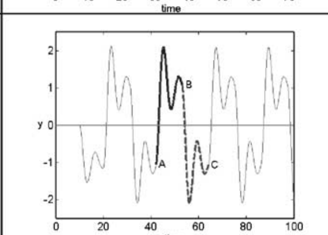
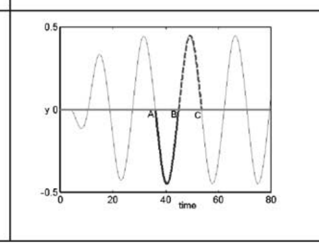
Transfer function	Responses	Transfer function	Responses
$\dfrac{1.0e^{-5s}}{2s+1}$		$\dfrac{1.0(1\text{-}s)se^{-1s}}{(3s+1)(2s+1)}$	
$\dfrac{1.0e^{-1s}}{-1+2s}$		$\dfrac{1.0}{(2s+1)^{3}}$	
$\dfrac{1.0e^{-1s}}{(8s+1)(s+1)}$		$\dfrac{1.0}{(4s+1)(3s+1)(2s+1)}$	
$\dfrac{1.0e^{-5s}}{(2s+1)^{2}}$		$\dfrac{1.0}{(2s+1)^{4}}$	
$\dfrac{1.0e^{-10S}}{s^{2}+0.4s+1}$		$\dfrac{1.0}{(2s+1)^{5}}$	

Fig. 5. Validation of analytical expressions for relay output of different systems: solid line is relay output and dashed line is model output. (A denotes starting of one cycle that ends at B. Again from B next cycle starts and ends at C.)

system. For monotonic systems, one can apply following boundary conditions

$$y(t = pu/2 - D) = 0,$$

$$y(t = 0) = -a \qquad \text{and}$$

$$y(t = pu/2) = a$$

where y is response, a is amplitude, D is dead time and pu is ultimate period. For a FOPDT process, the first condition yields

$$2e^{D/\tau - pu/2\tau} - e^{-pu/2\tau} - 1 = 0$$

and the second condition gives

$$k_p = -a/\left(h\left(1 - \left[2/\left(1 + e^{-pu/2\tau}\right)\right]\right)\right)$$

These equations can be solved for τ and k_p. Time delay, D, can be found by calculating time to reach peak of response in any one cycle in stabilized response.

For type 2 of non-monotonic class the following boundary conditions are valid

$$y(\text{at } t = \Delta t) \pm a,$$

$$y(\text{at } t = D^* + \Delta t)0$$

where D* is the time taken by the response to reach its peak, often called as apparent death time

$$y(\text{at } t = pu/2 - D) = 0$$

$$\frac{dy}{dt}(\text{at } t = \Delta t) = 0$$

The above conditions can be used to identify unknown process parameters.

For type 3 of non-monotonic systems, the following boundary conditions can be used to estimate its unknown parameters.

(i) at time $t = 0$, response, $y = 0$
(ii) at time $t = pu/2$, response, $y = 0$
(iii) at time $t = pu/4$, response, $y = \pm a$

Once model structure and parameters become available, we proceed with the controller tuning. The users can use their favorite tuning rules for PID controller design.

5. Conclusion

A systematic approach is proposed to derive exact expressions for relay feedback responses. In this work, time domain model equations are derived for first, second, third and higher order processes. These results are tabulated in a closed form solution for the first time. These model equations are useful to back calculate the exact process model parameters. This method of modeling can be extended to different types of transfer functions.

Acknowledgements

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Appendix

Derivation of analytical equation of relay feedback response for 5th order process: process transfer function is $G(s) = \frac{K_p}{(\tau S+1)^5}$

Similar to the procedure followed in Section 3.1 for over damped process, the initial response in the first period is given by y_1

$$y_1 = K_P\left(1 - \left(1 + \frac{t}{1!\tau} + \frac{t^2}{2!\tau^2} + \frac{t^3}{3!\tau^3} + \frac{t^4}{4!\tau^4}\right)e^{-t/\tau}\right)$$

where $t = t^*$ (A1)

In the second interval, the response will be

$$y_2 = K_P\left(1 - \left(1 + \frac{t+D}{1!\tau} + \frac{(t+D)^2}{2!\tau^2} + \frac{(t+D)^3}{3!\tau^3} + \frac{(t+D)^4}{4!\tau^4}\right)e^{-(t+D)/\tau}\right) - 2K_P\left(1 - \left(1 + \frac{t}{1!\tau} + \frac{t^2}{2!\tau^2} + \frac{t^3}{3!\tau^3} + \frac{t^4}{4!\tau^4}\right)e^{-t/\tau}\right)$$

where $t = t^* - D$ (A2)

For the response in the third interval, we can write

$$y_3 = K_P\left(1 - \left(1 + \frac{t+D+pu/2}{1!\tau} + \frac{(t+D+pu/2)^2}{2!\tau^2} + \frac{(t+D+pu/2)^3}{3!\tau^3} + \frac{(t+D+pu/2)^4}{4!\tau^4}\right)e^{-(t+D+pu/2)/\tau}\right) - 2K_P\left(1 - \left(1 + \frac{t+D}{1!\tau} + \frac{(t+D)^2}{2!\tau^2} + \frac{(t+D)^3}{3!\tau^3} + \frac{(t+D)^4}{4!\tau^4}\right)e^{-(t+D)/\tau}\right) + 2K_P\left(1 - \left(1 + \frac{t}{1!\tau} + \frac{t^2}{2!\tau^2} + \frac{t^3}{3!\tau^3} + \frac{t^4}{4!\tau^4}\right)e^{-t/\tau}\right)$$

where $t = t^* - D - pu/2$ (A3)

In a similar way, the generalized expression for the response in the nth interval can be given as:

$$\begin{aligned} y_n = K_p\Bigg\{&[1 - 2 + 2 - \ldots] - 2e^{-t/\tau}\left[\frac{t}{\tau} - \left(\frac{t}{\tau} + \frac{pu}{2\tau}\right)e^{-pu/2\tau} + \left(\frac{t}{\tau} + \frac{2pu}{2\tau}\right)e^{-2pu/2\tau} - \ldots\right] - \frac{2}{2!}e^{-t/\tau}\Bigg[\left(\frac{t}{\tau}\right)^2 \\ &- \left(\frac{t}{\tau} + \frac{pu}{2\tau}\right)^2 e^{-pu/2\tau} + \left(\frac{t}{\tau} + \frac{2pu}{2\tau}\right)^2 e^{-2pu/2\tau} - \left(\frac{t}{\tau} + \frac{3pu}{2\tau}\right)^2 e^{-3pu/2\tau} + \ldots\Bigg] - \frac{2}{3!}e^{-t/\tau}\Bigg[\left(\frac{t}{\tau}\right)^3 \\ &- \left(\frac{t}{\tau} + \frac{pu}{2\tau}\right)^3 e^{-pu/2\tau} + \left(\frac{t}{\tau} + \frac{2pu}{2\tau}\right)^3 e^{-2pu/2\tau} - \left(\frac{t}{\tau} + \frac{3pu}{2\tau}\right)^3 e^{-3pu/2\tau} + \ldots\Bigg] - \frac{2}{4!}e^{-t/\tau}\Bigg[\left(\frac{t}{\tau}\right)^4 \\ &- \left(\frac{t}{\tau} + \frac{pu}{2\tau}\right)^4 e^{-pu/2\tau} + \left(\frac{t}{\tau} + \frac{2pu}{2\tau}\right)^4 e^{-2pu/2\tau} - \left(\frac{t}{\tau} + \frac{3pu}{2\tau}\right)^4 e^{-3pu/2\tau} + \ldots\Bigg]\Bigg\} \end{aligned} \quad \text{(A4)}$$

The RHS of above equation has 5 parts and can be described as

$$y_n = K_P\{F_1 - F_2 - F_3 - F_4 - F_5\} \quad \text{(A5)}$$

in which, F_1, F_2, F_3, F_4 and F_5 can be simplified as follows:

$$\begin{aligned} F_1 &= [1] \\ F_2 &= 2e^{-t/\tau}\left[\frac{1}{1+e^{-pu/2\tau}}\right] \\ &\quad + 2e^{-t/\tau}\left[\frac{t/\tau}{1+e^{-pu/2\tau}} + -(pu/2\tau)e^{-pu/2\tau}/\left(1+e^{-pu/2\tau}\right)^2\right] \end{aligned}$$

Let us take $q = t/\tau$, $v = pu/2\tau$ and $r = -e^{-pu/2\tau}$ then F_3 becomes

$$\begin{aligned} F_3 &= \frac{2}{2!}e^{-q}\left[q^2 + \left(q^2+2qv+v^2\right)r + \left(q^2+2q(2v)+(2v)^2\right)r^2 \right. \\ &\quad \left. + \left(q^2+2q(3v)+(3v)^2\right)r^3 + \ldots\right] \\ &= \frac{2}{2!}e^{-q}\left[\left(q^2+q^2r+q^2r^2+q^2r^3+\ldots\right) \right. \\ &\quad \left. +2qv\left(r+2r^2+3r^3+\ldots\right)+v^2\left(r+2^2r^2+3^2r^3+\ldots\right)\right] \\ &= \frac{2}{2!}e^{-q}\left[\frac{q^2}{1-r}+\frac{2qvr}{(1-r)^2}+\frac{v^2r(1+r)}{(1-r)^3}\right] \end{aligned}$$

F_4 becomes

$$\begin{aligned} F_4 &= \frac{2}{3!}e^{-q}\left[q^3 + \left(q^3+3q^2v+3qv^2+v^3\right)r \right. \\ &\quad \left. + \left(q^3+3q^2(2v)+3q(2v)^2+(2v)^3\right)r^2 + \ldots\right] \\ &= \frac{2}{3!}e^{-q}\left[\left(q^3+q^3r+q^3r^2+q^3r^3+\ldots\right) \right. \\ &\quad +3q^2v\left(r+2r^2+3r^3+\ldots\right) \\ &\quad +3qv^2\left(r+2^2r^2+3^2r^3+\ldots\right) \\ &\quad \left. +v^3\left(r+2^3r^2+3^3r^3+\ldots\right)\right] \\ &= \frac{2}{3!}e^{-q}\left[\frac{q^3}{1-r}+\frac{3q^2vr}{(1-r)^2}+\frac{3qv^2r(1+r)}{(1-r)^3}+\frac{v^3r\left(r^2+4r+1\right)}{(1-r)^4}\right] \end{aligned}$$

Similarly, F_5 can be simplified as

$$\begin{aligned} F_5 &= \frac{2}{4!}e^{-q}\left[q^4 + \left(q^4+4q^3v+6q^2v^2+4qv^3+v^4\right)r \right. \\ &\quad \left. + \left(q^4+4q^3(2v)+6q^2(2v)^2+4q(2v)^3+(2v)^4\right)r^2 + \ldots\right] \\ &= \frac{2}{4!}e^{-q}\left[\left(q^4+q^4r+q^4r^2+q^4r^3+\ldots\right) \right. \\ &\quad +4q^3v\left(r+2r^2+3r^3+\ldots\right) \\ &\quad +6q^2v^2\left(r+2^2r^2+3^2r^3+\ldots\right) \\ &\quad +4qv^3\left(r+2^3r^2+3^3r^3+\ldots\right) \\ &\quad \left. +v^4\left(r+2^4r^2+3^4r^3+\ldots\right)\right] \\ &= \frac{2}{4!}e^{-q}\left[\begin{array}{l} \frac{q^4}{1-r}+\frac{4q^3vr}{(1-r)^2}+\frac{6q^2v^2r(1+r)}{(1-r)^3} \\ +\frac{4qv^3r\left(r^2+4r+1\right)}{(1-r)^4}+\frac{v^4r\left(r^3+11r^2+11r+1\right)}{(1-r)^5} \end{array}\right] \end{aligned}$$

Thus Eq. (A5) can be rewritten as :

$$\begin{aligned} y_n &= K_P\left\{1 - 2e^{-q}\left[\left(\frac{1}{1-r}\right)\right] \right. \\ &\quad -\frac{2}{1!}e^{-q}\left[\left(\frac{q}{1-r}+\frac{vr}{(1-r)^2}\right)\right] \\ &\quad -\frac{2}{2!}e^{-q}\left[\frac{q^2}{1-r}+\frac{2qvr}{(1-r)^2}+\frac{v^2r(1+r)}{(1-r)^3}\right] \\ &\quad -\frac{2}{3!}e^{-q}\left[\frac{q^3}{1-r}+\frac{3q^2vr}{(1-r)^2}+\frac{3qv^2r(1+r)}{(1-r)^3}\right. \\ &\quad \left.+\frac{v^3r\left(r^2+4r+1\right)}{(1-r)^4}\right]\frac{2}{4!}e^{-q}\left[\frac{q^4}{1-r}+\frac{4q^3vr}{(1-r)^2}\right. \\ &\quad +\frac{6q^2v^2r(1+r)}{(1-r)^3}+\frac{4qv^3r\left(r^2+4r+1\right)}{(1-r)^4} \\ &\quad \left.\left.+\frac{v^4r\left(r^3+11r^2+11r+1\right)}{(1-r)^5}\right]\right\} \end{aligned} \quad \text{(A6)}$$

This is the equation given in Table 2.

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