行政院國家科學委員會專題研究計畫　成果報告

三維邊界積分法瑕積分計算之改良

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計畫主持人：黃維信

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行政院國家科學委員會專題研究計畫成果報告

三維邊界積分法瑕積分計算之改良

An Improvement on the Singular Integrals of 3D Boundary Integral Equations

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一、中文摘要

有鑒於一般三維的邊界元素法對於高階元素的瑕積分(singular integral)處理方式，多以適應式積分或極座標轉換為主。本計畫提出另一種方式將瑕積分轉換為一般積分加上部份沿著元素邊界的線積分，使得邊界元素法中的瑕積分計算能夠更精確、更簡潔。

關鍵詞：邊界元素法，邊界積分式，奇異積分

Abstract

In this report, a new approach is presented to handle singular integrals in the 3-D boundary integral integrals. By utilizing the subtracting and adding back technique, one integral keeps the singular kernel but the density is set to be the constant at the singular point. The other one becomes regular after the subtracting and adding back technique. The latter can be integrated by a common method. The former is transformed into a boundary line integral by fundamental mathematical operations.

Keywords: *Boundary Element Methods, Boundary Integral Equation, Singular Integral*

二、緣由與目的

There are many available methods in the literature, which can handle singular integrals, for example, analytical formulas, adaptive integration, etc. In boundary element methods, most textbooks introduce exact analytical formulas for singular integrals in constant and linear elements. However, for higher-order elements, all the textbooks cannot present any analytical formula. Some authors claimed it was too difficult to derive formulas to integrate singular integrals on space surface elements. Therefore, the polar transform with an adaptive quadrature is commonly used in the literature. In this project, a simple and direct method is proposed to replace the adaptive procedure for such singular integrals.

In the last report, we proposed a simple technique to handle the singular integral in a two-dimensional element. That idea used an analytical formula to subtract a singular integrand from a singular integral and then add it back. However, for a three-dimensional element, such an analytic formula is generally not available. In this report, instead of deriving an analytic form, a different approach is constructed. Two line integrals along the boundary of elements are derived to transform two types of singular integrals, source and doublet singularities, into line integrals. Then use such identities to reduce the order of singularity of singular which is similar to the last report.

三、.基本計算理論與方法

1. Boundary integral equation

The common expression of a boundary integral equation can be described as:

$$\varepsilon(P)\phi(P)=\int_S \phi(Q)\frac{\partial G}{\partial n_Q}dS_Q-\int_S G\frac{\partial \phi}{\partial n_Q}dS_Q\,, \tag{1}$$

where P represents a source point, and Q a point on the boundary S. The free space Green's function G, based on the three-dimensional Laplace equation, is given as

$$G=\frac{-1}{4\pi r(P,Q)} \tag{2}$$

where $r(P,Q)$ is the distance between points P and Q. In general, the free term coefficient ε equals 1 when P inside the domain, 1/2 when P on the smooth part of the boundary, and zero outside the domain. More precisely, the coefficient ε can be expressed as a flux such that

$$\varepsilon(P)=\int_S \frac{\partial G}{\partial n_Q}dS_Q \tag{3}$$

Substituting Eq.(3) into Eq. (1), and one has

$$0=\int_S \left[\phi(Q)-\phi(P)\right]\frac{\partial G}{\partial n_Q}dS_Q-\int_S G\frac{\partial \phi}{\partial n_Q}dS_Q \tag{4}$$

The singular term comes from the second integral in the right-hand side of Eq. (4), and it is only weakly singular. As for the first integral, it is regular one and can be computed easily.

2. Singular elements

The fundamental solution G becomes singular when P and Q are coincident. For an arbitrary smooth surface, $\varepsilon(P)$ in Eq. (3) can be interpreted as a solid angle, and it can be calculated as a line integral along its boundary curve. Its expression is simply shown in Eq. (5), as for the detailed derivation can be found in Refs. [1, 2].

$$\iint_S \frac{\partial G}{\partial n}dS_Q=\frac{1}{4\pi}\int_C \left(1-\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\frac{x\dot{y}-\dot{x}y}{x^2+y^2}dt \tag{5}$$

where (x(t),y(t),z(t)) represents the coordinates of the space boundary curve C with respect to its origin P. In Eq. (5), the surface S does not enclose a space region. It is different from the surface in Eq. (1). Similarly, the integral of source term can be rearranged as:

$$\int_S \frac{f(Q)}{r}dS_Q=\int_S \left[\frac{f(Q)}{r}-\frac{f(P)}{\rho}\left(\vec{n}_P\cdot\vec{n}_Q\right)\right]dS_Q$$
$$+\,f(P)\oint_C \frac{\left(\vec{n}_p\times\vec{r}\right)\cdot\vec{s}}{\left\|\vec{n}_p\times\vec{r}\right\|}ds \tag{6}$$

where $\vec{s}$ means the unit tangent vector along the boundary curve C, and ρ represents the project length of r on the tangent plane at P for surface S.

四、結果與討論

1. Examples

Figure 1 shows the numerical errors of the line integral in Eq. (6) for P at origin with a uniform source distribution on a unit square. Only a few integral points on each side can reach a highly accurate result.

The first test case is a sphere, which is chosen for its smooth boundary. The boundary condition in this case is a Dirichlet type with the velocity potential $\phi = x$. Figure 2 shows the error distribution at different position along x-axis with 15 Gaussian quadrature nodes. Figure 3 shows the root-mean-square errors for different computing nodes on x-axis. Double nodes are distributed on each section for simplicity. The slope of the root-mean-square line in Fig. 3 is about -2.85 which indicates this computation with a good convergent speed

The next example is a non-smooth boundary object, a cube. Again, the Dirichlet problem ϕ = x is tested. The root-mean-square errors by using the trapezoidal and the Sympson's methods are shown in Fig. 4.

2. Conclusions

This study provides a different approach to handle singular integrals for the three-dimensional boundary integral equations. According to the numerical examples, the present method gives a simple and efficient tool for solving such singular integrals.

五、參考文獻

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六、成果自評

本研究內容與原計畫相符、並達成預期目標、研究成果兼具學術及應用價值、適合在學術期刊發表。

附 圖

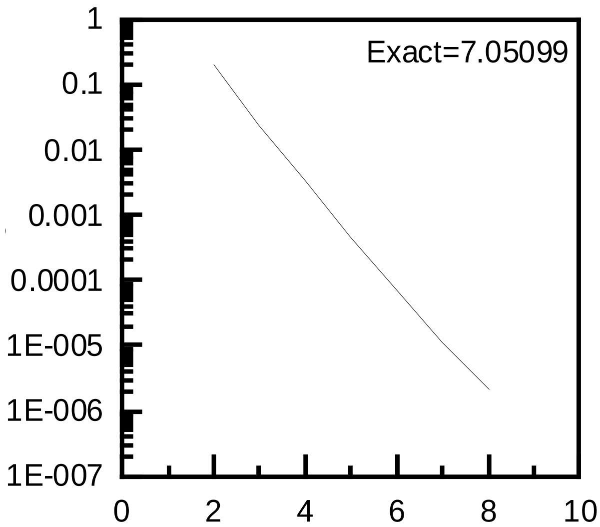

Fig. 1 Numerical errors of the line integral in Eq. (6) on a unit square for different numbers of nodes on each side

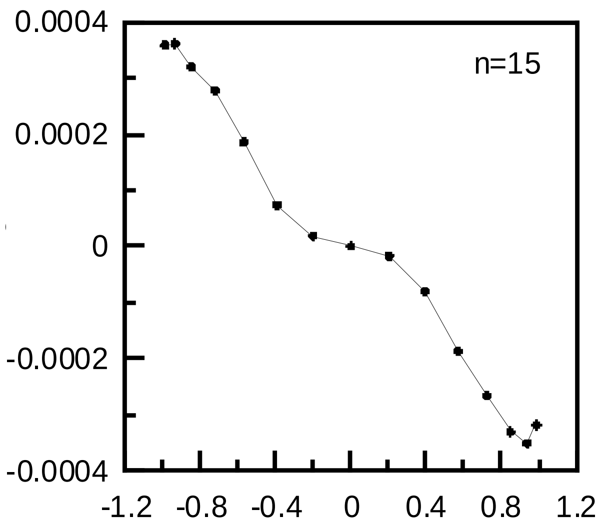

Fig. 2 Numerical error distribution of velocity potential along x- axis of a sphere surface

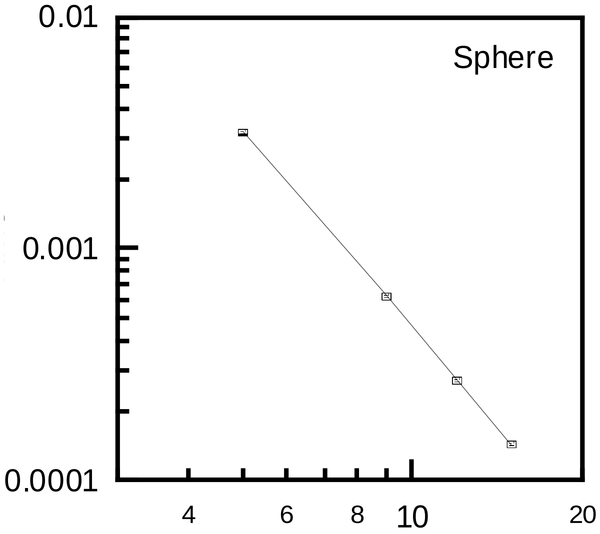

Fig. 3 Root-mean-square errors of velocity potentials for different numbers of computing nodes on the x-axis on a unit sphere surface

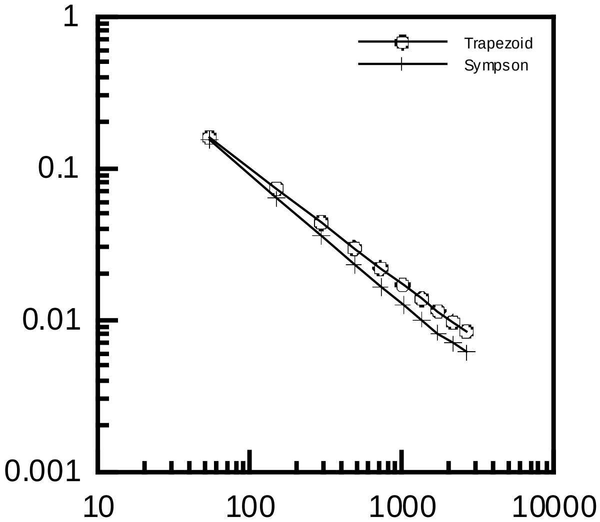

Fig. 4 Root-mean-square errors of velocity potentials for different numbers of computing nodes on a cube surface