QUALITATIVE PROPERTIES OF FREQUENCIES AND MODES OF BEAMS MODELED BY DISCRETE SYSTEMS

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ABSTRACT

In this paper, a discrete system model and its equation of motion for beams with arbitrary supports at two ends are established. These supports include elastic, rigid and free supports in translation and rotation directions. Based on theory of oscillatory matrices, a series of qualitative properties of frequencies and modes of this system are derived. The basic properties include: non-zero frequencies are distinct; the ith displacement mode has $i-1$ nodes; nodes of ith mode and $(i+1)$th mode interlace.

Some additional important qualitative properties owned by rotation modes and strain modes are given as well.

Keywords : Frequency, Mode, Beam, Qualitative property.

1. INTRODUCTION

The quantitative analysis of frequencies and modes of engineering structures is usually needed and high precision is required. In these cases, the experimental or computational approaches are employed. While, in some cases, only qualitative properties of frequencies and modes interest us, in other words, the knowledge of law, but not quantities is paid to attention. For example: (1) What general properties does the frequency spectrum of the structure have? Are there any repeated frequencies? What general properties does their mode shape have? How many nodes does a certain mode have? (2) For some special structures (e.g., mirror symmetric structures, axisymmetric structures and cyclic periodic structures), what characteristics do the modes and frequency spectrum have? (3) With structure parameters (stiffness, masses, constrains and shape) changing, how will frequencies and modes change consequently? To investigate these qualitative properties, mathematical analyzing approach is needed to be employed but not experimental or numeral approach. Therefore we call approach such as this, investigation of qualitative properties of frequencies and modes by applying analytical approach, theoretical mode analysis.

What do the qualitative properties of frequencies and modes mean to theory of vibration and its application? At least, they mean: (1) They are important criteria to check the results of experiments and calculations. For example, consider a rocket regarded as a slender beam, if two bending modes, one with two nodes, another one with three nodes that do not strictly interlace with those two nodes of the first mode, are calculated out, according to the qualitative properties, the result must be incorrect. For another example, only two longitude modes of a certain pile, one has two nodes, another has four nodes is measured out, it can be said that, according to the qualitative properties, a mode with three nodes must be lost whose frequency is located between those of these two modes. (2) They are helpful to simplify experimental and calculation schemes. For example, consider a symmetrical structure with a mirror symmetrical plane, all the frequencies and modes can be obtained by testing or computing in the half structure. (3) It is necessary to guarantee the reasonableness of the data given in dynamic design, structure modification and the inverse problem in vibration. The given data should meet the qualitative

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properties. For example, when we design a beam, it is impossible that the node line of a certain mode lie along the beam or repeated frequencies exist. (4) In some problems, only the knowledge of the trend is desired while the quantity is not necessary. For example, with stiffness or constraints changing, the frequencies will increase or decrease, that can be analyzed by using qualitative properties and that complex experimental or calculation approaches are unnecessary. (5) A continuous system is often modeled by a discrete system, is the discrete system reasonable? One important judgement is that the qualitative properties of two systems are parallel and not contrary to each other. It can be seen from the above that the knowledge of qualitative properties of frequencies and modes of vibrating system is very helpful to analysis, numerical calculations, and experiments in theory of vibration and its application including vibration control.

Because it is quite difficult to investigate the qualitative properties of vibrating system, the investigation develops slowly and few references can be found. Basic analysis was done by Gantmakher, Krein [1] and Gladwell [2,3]. In Ref. [1], the particular theory of oscillatory matrix and oscillatory kernel is established, that is the base of finding the qualitative properties of frequencies and modes of discrete and continuous vibrating system, and furthermore, some basic results were obtained. The work of Ref. [1] was developed in Ref. [2], especially some important results concerning the properties of beam was obtained; e.g., the frequencies interlacing property of beam with different supports, etc. Gladwell, *et al.*, [4] and Wang, *et al.*, [5] proved the necessary and sufficient conditions which should be satisfied by a single mode of discrete beam system was proved. Gladwell, *et al.*, [4] and Wang, *et al.*, [6] discussed the necessary and sufficient conditions which should be satisfied for two modes in same system.

In this paper, the researches mentioned above are developed systematically, some important results are obtained: (1) The discrete model of beam with arbitrary supports is established, and the qualitative properties of frequencies and modes of beam with arbitrary supports including free-free support are obtained through the concept of conjugate beam. (2) By use of geometry and physical relations, the qualitative properties of rotation modes, strain modes, and shearing force modes of beams are obtained.

2. THE MODEL AND EQUATIONS OF DISCRETE SYSTEM OF A BEAM

An Euler-Bernoulli beam with arbitrary supports may be modeled by a physical model (as shown in Fig. 1), rigid and massless rods are hinged at the mass points, m_i; m_n, k_i, …, k' , k'' are spring constants of rotational spring, h' , h'' are spring constants of translation spring, l_i is length of rigid rod, all of them are positive.

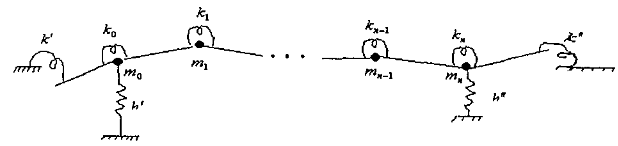

Fig. 1 Mass point-spring-rigid rod system

We label the lateral displacement of the ith point as u_i, rotational angle of the ith rod as θ_i, rotational angle of left and right rod as θ_0, θ_{n+1} respectively, relative rotational angle of the ith and the $(i + 1)$th rods at the ith hinged point as w_i and spring moment in rotational spring at this point as τ_i, shearing force at both ends of the ith rod as Q_i, shearing force at the left and right end as Q_0, Q_{n+1}, respectively. We introduce the following matrices and vectors,

$$\mathbf{K} = \text{diag}\,(k_0, k_1, \cdots, k_n)$$
$$\mathbf{M} = \text{diag}\,(m_0, m_1, \cdots, m_n)$$
$$\mathbf{L} = \text{diag}\,(l_1, \cdots, l_n)$$

$$\mathbf{E} = \begin{bmatrix} 1 & -1 & & & 0 \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & -1 \end{bmatrix}_{n\times(n+1)}, \quad \mathbf{e}_1 = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}_{n+1}, \quad \mathbf{e}_{n+1} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{Bmatrix}_{n+1}$$

The relations between displacement, rotation, relative rotation (corresponding to strain of beam), rotational spring moment (corresponding to stress of beam) and shearing force are

$$\theta_i = (u_i - u_{i-1})\, l_i^{-1}, \qquad i = 1, \cdots, n$$

$$w_i = \theta_{i+1} - \theta_i \qquad i = 0, \cdots, n$$

$$\tau_i = k_i\, w_i \qquad i = 0, \cdots, n$$

$$\theta_i = (\tau_{i-1} - \tau_i)\, l_i^{-1} \qquad i = 1, \cdots, n$$

the vectors formed by them are

$$\mathbf{u} = (u_0, \cdots, u_n)^T$$
$$\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)^T = -\mathbf{L}^{-1}\,\mathbf{E}\,\mathbf{u} \tag{1}$$

$$\mathbf{w} = (w_0, \cdots, w_n)^T = \mathbf{E}^T\boldsymbol{\theta} - \theta_0\,\mathbf{e}_1 + \theta_{n+1}\,\mathbf{e}_{n+1}$$
$$= -\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{u} - \theta_0\,\mathbf{e}_1 + \theta_{n+1}\,\mathbf{e}_{n+1} \tag{2}$$

$$\boldsymbol{\tau} = (\tau_0, \cdots, \tau_n)^T = K\,\mathbf{w} = -\mathbf{K}\,\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{u}$$
$$- \theta_0 k_0 \mathbf{e}_1 + \theta_{n+1} k_n \mathbf{e}_{n+1} \tag{3}$$

$$\mathbf{Q} = (Q_1, \cdots, Q_n)^T = \mathbf{L}^{-1}\mathbf{E}\boldsymbol{\tau} = -\mathbf{L}^{-1}\mathbf{E}\,\mathbf{K}\,\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{u}$$
$$- \theta_0 l_1^{-1} k_0 \mathbf{e}_1 + \theta_{n+1} l_n^{-1} k_n \mathbf{e}_{n+1} \tag{4}$$

From dynamic equations of every mass point and system

$$m_i \ddot{u}_i = Q_{i+1} - Q_i, \qquad i = 0,1,\cdots,n$$
$$\mathbf{M}\ddot{\mathbf{u}} = \mathbf{E}^T\mathbf{Q} - Q_0\,\mathbf{e}_1 + Q_{n+1}\,\mathbf{e}_{n+1} \tag{5}$$

and from force and moment balance equations of rods at left and right ends

$$Q_0 = -h' u_0 \qquad Q_{n+1} = h'' u_n \tag{6}$$

$$\theta_0 = \frac{k_0\theta_1}{(k' + k_0)} \qquad \theta_{n+1} = \frac{k_n\theta_n}{(k'' + k_n)} \tag{7}$$

We can derive the equation of motion of the system

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{A}\mathbf{u} + k_0\theta_0 l_1^{-1}(\mathbf{e}_1 - \mathbf{e}_2) + k_n\theta_{n+1} l_n^{-1}(\mathbf{e}_n - \mathbf{e}_{n+1}) + h' u_0 \mathbf{e}_1 + h'' u_n \mathbf{e}_{n+1} = 0 \tag{8}$$

or

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{A}\mathbf{u} + k_0^2 l_1^{-2}(k' + k_0)^{-1}(u_1 - u_0)(\mathbf{e}_1 - \mathbf{e}_2) + k_n^2 l_n^{-2}(k'' + k_n)^{-1}(u_n - u_{n-1})(\mathbf{e}_n - \mathbf{e}_{n+1}) + h' u_0 \mathbf{e}_1 + h'' u_n \mathbf{e}_{n+1} = 0 \tag{9}$$

where matrix

$$\mathbf{A} = \mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\,\mathbf{K}\,\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E} = (a_{ij})_1^{n+1} \tag{10}$$

is a symmetrical penta-diagonal matrix.

For beams with different supports, Eq. (8) or Eq. (9) should be handled properly. For example, for supports at left end, if displacement is fixed, let $u_0 = 0$, if without translation supports, let $h' = 0$; if rotation is fixed, let $\theta_0 = 0$; if without rotational support, let $k' = 0$. Note that the third term in equation (9) is $k_0^2 l_1^{-2}$ $(u_1 - u_0)$ $(\mathbf{e}_1 - \mathbf{e}_2)$ which counteract $k_0^2 l_1^{-2}$ $(u_0 - u_1)$ $(\mathbf{e}_1 - \mathbf{e}_2)$, the term with k_0 in the second term of $\mathbf{Au}$ in Eq. (9), so that the third term in Eq. (9) is deleted and let $k_0 = 0$ in $\mathbf{A}$. Thus for left end supports, if the end is fixed, let $u_0 = \theta_0 = 0$ in Eq. (8), delete the first equation, matrix $\mathbf{A}$ changes into $\mathbf{A}_1 = (a_{ij})_2^{n+1}$, which is minor of $\mathbf{A}$ with the first column and row deleted, matrix $\mathbf{M}$ changes into $\mathbf{M}_1$ = diag(m_1, ..., m_n); if the end is free, let $h' = k' = 0$, let $k_0 = 0$ in $\mathbf{A}$ and denote it as $\mathbf{A}_{k_0=0}$; if sliding, let $h' = \theta_0 = 0$; if pinned, let $u_0 = k' = 0$, $\mathbf{A}$ changes into $\mathbf{A}_{1,\,k_0=0}$, $\mathbf{M}$ changes into $\mathbf{M}_1$.

For different common non-elastic supports,

$$h' u_0 = 0, \quad h'' u_n = 0, \quad k'\theta_0 = 0, \quad k''\theta_{n+1} = 0 \tag{11}$$

by virtue of the handling for case of $k' = 0$, the general form of equations of motion of beams with different non-elastic supports are obtained as follows

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{A}\mathbf{u} = 0 \tag{12}$$

and modes equations are

$$\omega^2\mathbf{M}\mathbf{u} = \mathbf{A}\mathbf{u} = \mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{K}\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{u} \tag{13}$$

For different supports, mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{A}$ has different forms,

clamped-clamped: $\mathbf{M}_{cc} = \mathbf{M}_{1n} = \mathrm{diag}(m_1, \cdots, m_{n-1})$
$\mathbf{A}_{cc} = \mathbf{A}_{1n} = (a_{ij})_2^n$

clamped-pinned: $\mathbf{M}_{cp} = \mathbf{M}_{1n}$, $\mathbf{A}_{cp} = \mathbf{A}_{1n,k_n=0}$

clamped-sliding: $\mathbf{M}_{cs} = \mathbf{M}_1$, $\mathbf{A}_{cs} = \mathbf{A}_1$

clamped-free: $\mathbf{M}_{cf} = \mathbf{M}_1$, $\mathbf{A}_{cf} = \mathbf{A}_{1,k_n=0}$

pinned-pinned: $\mathbf{M}_{pp} = \mathbf{M}_{1n}$, $\mathbf{A}_{pp} = \mathbf{A}_{1n,k_0=k_n=0}$

pinned-sliding: $\mathbf{M}_{ps} = \mathbf{M}_1$, $\mathbf{A}_{ps} = \mathbf{A}_{1,k_0=0}$

pinned-free: $\mathbf{M}_{pf} = \mathbf{M}_1$, $\mathbf{A}_{pf} = \mathbf{A}_{1,k_0=k_n=0}$

sliding-sliding: $\mathbf{M}_{ss} = \mathbf{M}$, $\mathbf{A}_{ss} = \mathbf{A}$

sliding-free: $\mathbf{M}_{sf} = \mathbf{M}$, $\mathbf{A}_{sf} = \mathbf{A}_{k_n=0}$

free-free: $\mathbf{M}_{ff} = \mathbf{M}$, $\mathbf{A}_{ff} = \mathbf{A}_{k_0=k_n=0}$

where $A_{in} = (a_{ij})_2^n$ is minor of A with the first and the $(n + 1)$th columns and rows deleted.

3. QUALITATIVE PROPERTIES OF FREQUENCIES AND MODES OF STATICALLY DETERMINATE AND STATICALLY INDETERMINATE BEAMS

In this section, six kinds of beams, i.e., clamped-clamped, clamped-pinned, clamped-sliding, clamped-free, pinned-pinned and pinned-sliding beams are considered. At first, the singularities of the stiffness matrices of these systems are investigated.

Let $\overline{\mathbf{E}}$ be $n \times n$ matrix formed by deleting the 1st row of matrix $\mathbf{E}$. It can be verified easily that

$$\det(\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}) = 0, \qquad \det(\overline{\mathbf{E}}^T\mathbf{L}^{-1})(\mathbf{L}^{-1}\overline{\mathbf{E}}) = \prod_{i=1}^{n} l_i^{-2}$$

$$\det(\mathbf{E}\mathbf{K}\mathbf{E}^T) = \sum_{i=0}^{n} k_0 \cdots k_{i-1}k_{i+1} \cdots k_n$$

hence

$$\det\mathbf{A} = \det\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{K}\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E} = 0$$

$$\det\mathbf{A}_1 = \det\overline{\mathbf{E}}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{K}\mathbf{E}^T\mathbf{L}^{-1}\overline{\mathbf{E}}$$

$$= \prod_{i=1}^{n} l_i^{-2} \sum_{i=0}^{n} k_0 \cdots k_{i-1}k_{i+1}k_n \begin{cases} > 0, & k_0 + k_n > 0 \\ = 0, & k_0 = k_n = 0 \end{cases}$$

Because sign reverse matrix has the same determinant with that of original matrix, then $\mathbf{A}^*$, $\mathbf{A}_1^*$, the sign inverse matrix of $\mathbf{A}$, $\mathbf{A}_1$ respectively, satisfy:

$$\det\mathbf{A}^* = 0, \quad \det\mathbf{A}_1^* \begin{cases} > 0, & k_0 + k_n > 0 \\ = 0, & k_0 = k_n = 0 \end{cases}$$

Since $\mathbf{A}_1^*$ is non-singular when $k_0 + k_n > 0$, its main minor $\det \mathbf{A}_{1n}^* > 0$ when $k_0 + k_n > 0$. Although $\det \mathbf{A}_1^* = 0$ when $k_0 = k_n = 0$, A_{1n} can be expressed as

$$\mathbf{A}_{1n} = \mathbf{E}_1\mathbf{L}^{-1}\mathbf{E}_1^T\mathbf{K}_1\mathbf{E}_1\mathbf{L}^{-1}\mathbf{E}_1^T$$

where

$$\mathbf{E}_1 = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{n-1,n}, \quad \mathbf{K}_1 = \begin{bmatrix} k_1 & & \\ & \ddots & \\ & & k_{n-1} \end{bmatrix}$$

Meanwhile $\det \mathbf{E}_1\mathbf{L}^{-1}\mathbf{E}_1^T = \sum_{i=1}^{n-1} l_1^{-1}\cdots l_{i-1}^{-1}l_{i+1}^{-1}\cdots l_{n-1}^{-1} > 0$, $\det\mathbf{K} = \prod_{i=1}^{n-1} k_i > 0$, thus $\det \mathbf{A}_{1n} > 0$, i.e., matrix $\mathbf{A}_{1n}^*$ is non-singular when $k_0 = k_n = 0$.

Secondly, it will be verified that $\mathbf{A}^*$, $\mathbf{A}_{1n}^*$ and $\mathbf{A}_1^*$, when $k_0 + k_n > 0$, are completely non-negative matrices. We introduce matrices

$$\widetilde{\mathbf{L}}^{-1} = \mathrm{diag}(l_1^{-1},\cdots,l_n^{-1},0), \quad \widetilde{\mathbf{E}}^T = (E^T, e_{n+1}), \quad \widetilde{\mathbf{E}} = (\widetilde{\mathbf{E}}^T)^T$$

it can be identified that $\widetilde{\mathbf{E}}^T\widetilde{\mathbf{L}}^{-1}\widetilde{\mathbf{E}} = \mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}$. Consider

$$\begin{aligned}\mathbf{A}^* &= (\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{K}\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E})^* = (\widetilde{\mathbf{E}}^T\widetilde{\mathbf{L}}^{-1}\widetilde{\mathbf{E}}\mathbf{K}\widetilde{\mathbf{E}}^T\widetilde{\mathbf{L}}^{-1}\widetilde{\mathbf{E}})^* \\ &= (\widetilde{\mathbf{E}}^T)^*\widetilde{\mathbf{L}}^{-1}(\widetilde{\mathbf{E}})^*\mathbf{K}(\widetilde{\mathbf{E}}^T)^*\widetilde{\mathbf{L}}^{-1}(\widetilde{\mathbf{E}})^*\end{aligned}$$

Matrices $(\widetilde{\mathbf{E}})^*$ and $(\widetilde{\mathbf{E}}^T)^*$ are completely non- negative because all of their minors are non-negative. Therefore $\mathbf{A}^*$ is completely non-negative. Its truncation matrices $\mathbf{A}_1^*$, $\mathbf{A}_{1n}^*$ are completely non-negative also.

In addition, it can be verified that the quasi-diagonal elements of matrix $\mathbf{A}$ satisfy $a_{i,i+1} = a_{i+1,i} < 0$, $i = 1, \ldots, n$.

According to the definition of sign-oscillatory matrix, $\mathbf{A}_{1n}$, $\mathbf{A}_{1,}$, $\mathbf{A}_{1,k_0=0}$, $\mathbf{A}_{1,k_n=0}$ are sign-oscillatory matrices, it means, the stiffness matrices of six kinds of statically determinate and statically indeterminate beams are sign oscillatory matrices.

According to the theory of oscillatory matrix, the frequencies and modes of these beams have the following most important qualitative properties:

(1) Frequencies are distinct, that is

$$0 < \omega_1 < \omega_2 < \cdots < \omega_{n-1} \ (< \omega_n)$$

there are n frequencies for beams with sliding or free ends, $n-1$ frequencies for other three kinds of beams.

(2) The i th displacement mode $u^{(i)}$ corresponding to ω_i has $i-1$ interchanges of sign.

(3) The figure obtained by joining the points with coordinates (j, $u_j^{(i)}$) is called $u^{(i)}$-line. Then the nodes of u-line of two successive displacement modes, i.e., nodes of $u^{(i)}$-line and $u^{(i+1)}$-line, interlace.

4. QUALITATIVE PROPERTIES OF ROTATION MODES, STRAIN MODES, AND SHEARING FORCE MODES OF STATICALLY DETERMINATE AND STATICALLY INDETERMINATE BEAMS

From the viewpoint of theory and application, we are also concerned about the qualitative properties of rotation modes, strain modes and shearing force modes of beam.

In order to deduce the number of sign interchange, we give a property of vector first.

Set a vector

$$\begin{aligned} y = (y_1,\cdots,y_n)^T &= -\mathbf{E}\,x = -\mathbf{E}\,(x_0,x_1,\cdots,x_n)^T \\ &= (x_1-x_0, x_2-x_1,\cdots,x_n-x_{n-1})^T \end{aligned} \tag{14}$$

If vector $\mathbf{x}$ has a definite number, S_x, of sign interchanges, then the least value, S_y^-, of the number of sign interchanges in vector $\mathbf{y}$ satisfy following inequality

$$S_x - 1 + H(x_0) + H(x_n) \le S_y^- \tag{15}$$

where $H(x) = \begin{cases} 1, & x = 0 \\ 0, & x \ne 0 \end{cases}$

If vector $\mathbf{y}$ has a definite number of sign interchanges S_y, then the greatest value, S_x^+, of the number of sign interchanges in vector $\mathbf{x}$ satisfy following inequality

$$S_x^+ \le S_y + 1 - H(x_0) - H(x_n) \tag{16}$$

Firstly, according to the properties mentioned above, we investigate the least value of the number of sign interchanges of rotation modes, strain modes, and sheering force modes. Comparing Eq. (1) with (14), owing to inequality (15), we can obtain the relation of sign interchanges between rotation and displacement of beam

$$S_{\theta}^- \ge S_{\mathbf{u}} - 1 + H(u_0) + H(u_n) \tag{17}$$

By virtue of Eq. (2)

$$\begin{aligned}\mathbf{w} &= (\theta_1-\theta_0,\cdots,\theta_{n+1}-\theta_n)^T \\ &= -(\mathbf{E})_{(n+1)(n+1)}(\theta_0, \boldsymbol{\theta}^T, \theta_{n+1})^T = -\mathbf{E}\,\boldsymbol{\theta}^{*T}\end{aligned} \tag{18}$$

and inequality Eqs. (15) and (17), then

$$\begin{aligned} S_{\mathbf{w}}^- &\ge S_{\theta^*} - 1 + H(\theta_0) + H(\theta_{n+1}) \ge S_{\theta^*}^- - 1 + H(\theta_0) + H(\theta_{n+1}) \\ &\ge S_{\theta}^- - 1 + H(\theta_0) + H(\theta_{n+1}) \\ &\ge S_{\mathbf{u}} - 2 + H(u_0) + H(u_n) + H(\theta_0) + H(\theta_{n+1}) \end{aligned} \tag{19}$$

note that

$$\tau = \mathbf{K}\,\mathbf{w} = (k_0 w_0,\ \cdots,\ k_n, w_n)^T \tag{20}$$

$$\mathbf{Q} = (Q_1, L, Q_n)^T = \mathbf{L}^{-1}\,\mathbf{E}\,\tau$$

and by virtue of inequality (14), (15) and (19), we obtain

$$S_{\mathbf{Q}}^- \geq S_{\tau} - 1 + H(\tau_0) + H(\tau_n) \geq S_{\mathbf{w}}^- - 1 + H(\tau_0) + H(\tau_n)$$

$$\begin{aligned} S_{\mathbf{Q}}^- \geq S_{\mathbf{u}} - 3 + H(u_0) + H(u_n) + H(\theta_0) + H(\theta_{+1n}) \\ + H(\tau_0) + H(\tau_n) \end{aligned} \tag{21}$$

Secondly, the greatest value of the number of sign interchanges will be concerned in following. We introduce

$$\mathbf{Q}^* = (Q_1, Q^T, Q_{n+1})^T$$

From Eq. (5), it is known that

$$\omega^2 \mathbf{M}\mathbf{u} = (Q_1 - Q_0, \cdots, Q_{n+1} - Q_n)^T = (\mathbf{E})_{(n+1)(n+2)}\mathbf{Q}^*$$

By virtue of inequality (14) and (16), then

$$S_{\mathbf{Q}}^+ \leq S_{\mathbf{Q}^*}^+ \leq S_{\mathbf{u}} + 1 - H(Q_0) - H(Q_{n+1}) \tag{22}$$

From exp. (4) and inequality (14), (16) and (22), yield

$$\begin{aligned} S_{\tau}^+ &\leq S_{\mathbf{Q}} + 1 - H(\tau_0) - H(\tau_n) \leq S_{\mathbf{Q}}^+ + 1 - H(\tau_0) - H(\tau_n) \\ &\leq S_{\mathbf{u}} + 2 - H(Q_0) - H(Q_{n+1}) - H(\tau_0) - H(\tau_n) \end{aligned} \tag{23}$$

$$S_{\mathbf{w}}^+ \leq S_{\mathbf{u}} + 2 - H(Q_0) - H(Q_{n+1}) - H(\tau_0) - H(\tau_n)$$

Owing to exp. (18) and inequality (14), (16) and (23), obtain

$$\begin{aligned} S_{\theta^*}^+ &\leq S_{\mathbf{w}} + 1 - H(\theta_0) - H(\theta_{n+1}) \\ &\leq S_{\tau}^+ + 1 - H(\theta_0) - H(\theta_{n+1}) \\ &\leq S_{\mathbf{u}} + 3 - H(Q_0) - H(Q_{n+1}) - H(\theta_0) - H(\theta_{n+1}) \\ &\quad - H(\tau_0) - H(\tau_n) \end{aligned}$$

Furthermore

$$\begin{aligned} S_{\theta}^+ \leq S_{\mathbf{u}} + 3 - H(Q_0) - H(Q_{n+1}) - H(\theta_0) \\ - H(\theta_{n+1}) - H(\tau_0) - H(\tau_n) \end{aligned} \tag{24}$$

Lastly, it can be obtained from inequalities (17) and (24), (19) and (23'), (21) and (22) respectively that

$$\begin{aligned} S_{\mathbf{u}} - 1 + H(u_0) + H(u_n) \leq S_{\theta}^- \leq S_{\theta}^+ \leq S_{\mathbf{u}} + 3 - H(Q_0) \\ - H(Q_{n+1}) - H(\tau_0) - H(\tau_n) - H(\theta_0) - H(\theta_{n+1}) \end{aligned} \tag{25}$$

$$\begin{aligned} S_{\mathbf{u}} - 2 + H(u_0) + H(u_n) + H(\theta_0) + H(\theta_{n+1}) \leq S_w^- \leq S_w^+ \\ \leq S_{\mathbf{u}} + 2 - H(Q_0) - H(Q_{n+1}) - H(\tau_0) - H(\tau_n) \end{aligned} \tag{26}$$

$$\begin{aligned} S_{\mathbf{u}} - 3 + H(u_0) + H(u_n) + H(\theta_0) + H(\theta_{n+1}) + H(\tau_0) \\ + H(\tau_n) \leq S_{\mathbf{Q}}^- \leq S_{\mathbf{Q}}^+ \leq S_{\mathbf{u}} + 1 - H(Q_0) - H(Q_{n+1}) \end{aligned} \tag{27}$$

From above three equations, the number of sign interchanges of rotation θ modes, strain $\mathbf{w}$ modes, and shearing force $\mathbf{Q}$ modes of beams can be obtained.

For example, for a clamped-clamped beam, the number of sign interchanges of the ith displacement mode $u^{(i)}$ is $S_{u^{(i)}} = i - 1$, and meanwhile $u_0 = \theta_0 = u_n = \theta_{n+1} = 0$, τ_0, Q_0, τ_n and Q_{n+1} are non-zero, from equation (25) ~ (27) we can deduce that

$$i \leq S_{\theta^{(i)}}^- \leq S_{\theta^{(i)}}^+ \leq i$$

$$i + 1 \leq S_{\mathbf{w}^{(i)}}^- \leq S_{\mathbf{w}^{(i)}}^+ \leq i + 1$$

$$i \leq S_{\mathbf{Q}^{(i)}}^- \leq S_{\mathbf{Q}^{(i)}}^+ \leq i$$

Therefore

$$S_{\theta^{(i)}} = i, \quad S_{\mathbf{w}^{(i)}} = i + 1, \quad S_{\mathbf{Q}^{(i)}} = i$$

it is written in the third row in Table 1.

Table 1 Sign interchanges of u, θ, w and Q modes of beams

supports form	end conditions								sign interchanges			
	u_0	θ_0	τ_0	Q_0	u_n	θ_{n+1}	τ_n	Q_{n+1}	$S_{u^{(i)}}$	$S_{\theta^{(i)}}$	$S_{w^{(i)}}$	$S_{Q^{(i)}}$
clamped-clamped	0	0			0	0			$i-1$	i	$i+1$	i
clamped-pinned	0	0			0		0		$i-1$	i	i	i
clamped-sliding	0	0				0		0	$i-1$	$i-1$	i	$i-1$
clamped-free	0	0					0	0	$i-1$	$i-1$	$i-1$	$i-1$
pinned-pinned	0		0		0		0		$i-1$	i	$i-1$	i
pinned-sliding	0		0			0		0	$i-1$	$i-1$	$i-1$	$i-1$
pinned-free	0		0				0	0	$i-1$	$i-1$	$i-2$	$i-1$
sliding-sliding		0		0		0		0	$i-1$	$i-2$	$i-1$	$i-2$
sliding-free		0		0			0	0	$i-1$	$i-2$	$i-2$	$i-2$
free-free			0	0			0	0	$i-1$	$i-2$	$i-3$	$i-2$

Similarly, the sign interchanges of modes of the other five kinds of beams can be obtained, they are written in the 4th row to the 8th row in Table 1.

5. QUALITATIVE PROPERTIES OF FREQUENCIES AND MODES OF BEAMS WITH RIGID MOTIONS

The stiffness matrices of beams with rigid motions, i.e., beams with pinned-free, sliding-sliding, sliding-free, and free-free supports, are singular, we can not discuss their qualitative properties of frequencies and modes by employing oscillatory matrices directly. Therefore we introduce the conjugated system to change beams with rigid motions into beams without rigid motions, and then to deduce their qualitative properties.

For those four kinds of beams mentioned above, $\theta_0 k_0 = \theta_{n+1} k_n = 0$. From Eq. (13) and exp. (3), we get

$$\omega^2 \mathbf{K}^{-1}\boldsymbol{\tau} = \mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{M}^{-1}\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\boldsymbol{\tau} \tag{28}$$

it can be expressed as

$$\omega^2 \overline{\mathbf{M}}\,\overline{\mathbf{u}} = \mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\overline{\mathbf{K}}\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\overline{\mathbf{u}} \tag{29}$$

where $\overline{\mathbf{M}} = \mathbf{K}^{-1}$, $\overline{\mathbf{K}} = \mathbf{M}^{-1}$, $\overline{\mathbf{u}} = \boldsymbol{\tau}$.

Table 2 The corresponding relations between original beam and its conjugated beam

	modes				supports form			
original beam	Moment $\boldsymbol{\tau}$	Shearing force $\mathbf{Q}$	Displacement $\mathbf{u}$	Rotation $\boldsymbol{\theta}$	clamped	pinned	sliding	free
conjugated beam	Displace $\bar{\mathbf{u}}$	Rotation $\bar{\boldsymbol{\theta}}$	Moment $\bar{\boldsymbol{\tau}}$	Shearing force $\bar{\mathbf{Q}}$	free	pinned	sliding	clamped

Comparing Eq. (28) with (29), we can find that the spring moment $\underline{\tau}$ of original beam corresponds to the displacement of another beam, whose point mass is reciprocal of spring constant of original one, and whose spring constant is reciprocal of point mass of original one. We call that beam the conjugated beam of original one. The Eq. (28) together with (29) imply that the original beam and its conjugated one have the same frequencies, and the moment modes of original beam are the displacement modes of conjugated one. The relations of other variables and support types between original beam and its conjugated one are shown in Table 2.

For pinned-free beam, its conjugated beam is pinned-clamped beam. It is shown in section 3 that its frequencies are distinct, the ith displacement mode has $i-1$ sign interchange, that is $S_{\bar{\mathbf{u}}^{(i)}} = i-1$. It should be noted that the trivial solutions, $\omega = 0$, $\bar{\mathbf{u}} = \mathbf{0}$, to mode equations of the conjugated beam correspond to $\omega = 0$, $\boldsymbol{\tau} = \mathbf{0}$, of original beam, since the support is pinned-free, so original beam has a rigid rotation displacement mode $\mathbf{u}^{(1)}$, whose frequencies $\omega = 0$, moment mode $\boldsymbol{\tau}^{(1)} = \mathbf{0}$. Therefore the ith displacement mode of conjugated beam correspond to the $(i+1)$th moment mode of original beam, so $S_{\tau^{(i+1)}} = i-1$. It is shown in the 4th row in Table 1 that $S_{\bar{\tau}^{(i)}} = i$, it corresponds $S_{u^{(i+1)}} = i$, i.e., $S_{u^{(i)}} = i-1$.

For a sliding-free beam, whose conjugated beam is sliding-clamped beam; one rigid translation and one rigid rotation mode with zero-frequency exist. It can be seen in the 5th row in Table 1 that $S_{\bar{\tau}^{(i)}} = i$, so $S_{u^{(i)}} = i-1$.

For a free-free beam, whose conjugated beam is clamped-clamped beam; one rigid translation mode and one rigid rotation mode with zero-frequencies exist. It can be seen in the 3rd line in Table 1 that $S_{\bar{\tau}^{(i)}} = i+1$, so $S_{u^{(i)}} = i-1$.

By virtue of inequalities (25) to (27), we can deduce the number of sign interchanges of θ mode, τ mode, and Q mode of above three kinds beams, the results are provided in the 8th, 9th and 12th rows in Table 1.

The above approach cannot be applied for a sliding-sliding beam, since its conjugated beam is also a sliding-sliding beam. A new approach is needed, so Eq. (13) is rewritten as follows

$$\omega^2 \boldsymbol{\theta} = \mathbf{L}^{-1}\mathbf{E}\mathbf{M}^{-1}\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{K}\mathbf{E}^T\boldsymbol{\theta} \tag{30}$$

$$\omega^2 \mathbf{Q} = \mathbf{L}^{-1}\mathbf{E}\mathbf{M}^{-1}\mathbf{E}^T\mathbf{L}^{-1}\mathbf{E}\mathbf{K}\mathbf{E}^T\mathbf{Q} \tag{31}$$

It is simple to identify that $(\mathbf{EKE}^T)^*$ and $(\mathbf{EM}^{-1}\mathbf{E}^T)^*$ are non-singular and completely non-negative matrices, whose quasi-diagonal elements are positive, then Eqs. (30) and (31) are eigenvalue problems of sign-oscillatory matrix. The theory of oscillatory matrix implies that non-zero frequencies are distinct. For a sliding-sliding beam, there exist a rigid translation, the corresponding θ mode in Eq. (30) and Q mode in Eq. (31) are zero. Thus

$$S_{\theta^{(i)}} = S_{Q^{(i)}} = i-2$$

The relation between $\mathbf{u}$, $\boldsymbol{\theta}$, $\boldsymbol{\tau}$ and $\mathbf{Q}$ together with the property of vector mentioned in Section 3, lead to the properties of displacement modes and moment modes, which is shown in the 10th row in Table 1.

So far, we can obtain that for all beams with non-elastic supports, non-zero frequencies are distinct; and the regulation of the number of sign interchanges of $\mathbf{u}$, $\boldsymbol{\theta}$, $\mathbf{w}$, $\mathbf{Q}$ modes, i.e., the number of nodes, is as shown in Table 1.

Since displacement mode Eq. (13) can be rewritten as rotation mode Eq. (30), moment mode Eq. (28), and shearing force mode Eq. (31), when the matrices of these equations is sign-oscillatory matrices, the corresponding modes have the property that the nodes of two neighboring modes interlace.

6. THE INTERLACING PROPERTY BETWEEN NODES OF THE iTH MODE $u^{(i)}$, $\theta^{(i)}$, $w^{(i)}$ AND $Q^{(i)}$

By virtue of the regulation of the number of nodes of $\mathbf{u}^{(i)}$, $\boldsymbol{\theta}^{(i)}$, $\mathbf{w}^{(i)}$ and $\mathbf{Q}^{(i)}$ in Table 1, and the property that there is at least one node of y-line of vector $\mathbf{y} = -\mathbf{Ex} = (x_1 - x_0, \ldots, x_n - x_{n-1})^T$ existing between two neighboring nodes of x-line of vector $\mathbf{x} = (x_0, x_1, \ldots, x_n)^T$, we can prove the following significant and interesting properties.

For the ith displacement modes $\mathbf{u}^{(i)}$ and the ith rotation modes $\boldsymbol{\theta}^{(i)}$ of the ten kinds of beams listed in Table 1, their nodes interlace. Similarly, for the ith rotation modes $\boldsymbol{\theta}^{(i)}$ and the ith strain modes $\mathbf{w}^{(i)}$, the ith strain modes $\mathbf{w}^{(i)}$ and the ith shearing modes $\mathbf{Q}^{(i)}$, the ith shearing force modes $\mathbf{Q}^{(i)}$, and the ith displacement modes $\mathbf{u}^{(i)}$, their nodes interlace.

Here we give a proof to a clamped-clamped beam for example. The $\mathbf{u}^{(i)}$-line of the ith displacement mode $\mathbf{u}^{(i)}$

has $i-1$ nodes, plus two ends where displacements are zero, then $i+1$ zero-points divide the coordinates of $\mathbf{u}^{(i)}$-line into i domain. From expression (1), $\boldsymbol{\theta}^{(i)} = -\mathbf{L}^{-1}\mathbf{E}\mathbf{u}^{(i)}$, hence $\boldsymbol{\theta}^{(i)}$-line has at least one node in every domain. As shown in Table 1, the number of nodes of $\boldsymbol{\theta}^{(i)}$ is i exactly. Thus there is one and only one node in every domain, so that the nodes of $\mathbf{u}^{(i)}$ and that of $\boldsymbol{\theta}^{(i)}$ interlace.

The i nodes of $\boldsymbol{\theta}^{(i)}$ plus two ends where rotations are zero, $\theta_0 = \theta_{n+1} = 0$, vector $\overline{\boldsymbol{\theta}}^{(i)} = (\theta_0, (\boldsymbol{\theta}^{(i)})^T, \theta_{n+1})^T$ has $i+2$ zero points, that divide the coordinates of $\overline{\boldsymbol{\theta}}^{(i)}$-line into $i+1$ domain. From expression (2), $\mathbf{w}^{(i)} = (\theta_1^{(i)} - \theta_0, \cdots, \theta_{n+1} - \theta_n^{(i)})$, hence $\mathbf{w}^{(i)}$-line has at least one node in every domain. As shown in Table 1, the number of nodes of $\mathbf{w}^{(i)}$ is $i+2$ exactly, thus there is one and only one node in every domain, therefore the nodes of $\mathbf{w}^{(i)}$ and $\overline{\boldsymbol{\theta}}^{(i)}$ interlace. Apparently, the 1st [or $(i+1)$th] node of $w^{(i)}$ do not locate between θ_0 and θ_1(or θ_n and θ_{n+1}), therefore the nodes of $\mathbf{w}^{(i)}$ and $\boldsymbol{\theta}^{(i)}$ interlace.

$\mathbf{w}^{(i)}$-line has $i+1$ nodes, and $\mathbf{w}$ is non-zero at two ends, those $i+1$ nodes divide the coordinates of $w^{(i)}$-line into i domain. From expression (4), $\mathbf{Q}^{(i)} = \mathbf{L}^{-1}\mathbf{E}\mathbf{K}\mathbf{w}^{(i)}$, hence $\boldsymbol{\theta}^{(i)}$-line has at least one node in every domain. As shown in Table 1, the number of nodes is i exactly. Therefore the nodes of $\mathbf{w}^{(i)}$ and $\boldsymbol{\theta}^{(i)}$ interlace.

The i nodes of $Q^{(i)}$-line divide the coordinates of $Q^{(i)}$-line into $i-1$ domain, by expression (5), $\overline{\mathbf{u}}^{(i)} = (u_1, \ldots, u_{n+1})^T = -\omega^{-2}\mathrm{diag}(m_1^{-1}, \cdots, m_{n-1}^{-1})\,(Q_2 - Q_1, \ldots, Q_n - Q_{n-1})^T$, hence $\overline{u}^{(i)}$-line has at least one node in every domain, apparently, it is impossible that there are any nodes between $u_0 = 0$ and $u_1^{(i)}$, $u_n = 0$ and $u_{n-1}^{(i)}$. Therefore the nodes of $\underline{Q}^{(i)}$ and $\underline{u}^{(i)}$ interlace.

For the cases of other 9 kinds of beams, it can be proved similarly.

7. CONCLUSION

The discrete models of beams and corresponding equation of motion of Euler beam with general end supports are established in this paper. For the ten kinds of beams listed in Table 1, the following qualitative properties are proved:

(1) Non-zero frequencies are distinct.

(2) The ith displacement mode has $i-1$ nodes, and the number of nodes of corresponding rotation modes, strain modes, and shearing force modes are dependent on conditions of supports, that is shown in Table 1.

(3) The nodes of two successive modes interlace.

(4) The nodes of displacement modes and rotation modes of same order interlace each other. Same case occurs for nodes of rotation modes and strain modes, strain and shearing force, shearing force and displacement of same order.

For the beams with elastic supports, some basic qualitative properties of frequencies and modes are the same as those for the beams with non-elastic supports. The qualitative properties of beams modeled by continuous systems can be obtained by the use of limit-transition approach. These two issues will be demonstrated in detail in the follow-up paper.

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