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The stress intensity factors of slightly undulating interface cracks of bimaterials

CHAO-HSUN CHEN and JINHON HSU
Institute of Applied Mechanics, National Taiwan University, Taipei 106, Taiwan, Republic of China

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Abstract. A modified interface crack with slightly undulating profile, which has a good agreement with reality and retains the simplicity of a mathematical model, is presented in this paper. This model is utilized to reveal some of the properties of uneven cracks, especially the stress intensity factors. As we know, many failures occurring in the interface are induced by crucial lateral stresses which are parallel to the interface. Hence, when the lateral stresses are much stronger than others, the corresponding solution is also derived for understanding how the lateral stresses affect the stress intensity factors as the crack is uneven. In the present paper, the Hilbert's problem enables different perturbed-interface cracks to be solved in an unified manner. Muskhelishvili's potential formulation is used to derive, by means of a perturbation analysis technique, an homogeneous and general Hilbert's problem.

1. Introduction

Most of the major failure modes of fiber-reinforced composites, thin films and adhesive joints are the propagation of interfacial cracks. As a result, a great deal of effort has been devoted to the study of interfacial fracture of bimaterials. The basic solution to this problem was formulated by Williams [1]. There have been many discussions on its unusual local characteristics throughout the years. The complex stress intensity factor K associated with an elastic interface crack, for which contact is ignored, is discussed by Rice [2], and especially its validity as a crack tip characterizing parameter is noted for cases of small scale nonlinear material behavior and/or small scale contact zones at the crack tip. That is, similar values of K for two cracked bodies then imply similar states at the crack tip, so that conditions for crack growth can be phrased in terms of K reaching a critical failure locus in a complex plane. There have been many definitions of stress intensity factor proposed for the purpose of characterizing the near-tip field of an interfacial crack (e.g., Sih and Rice [3]; Rice and Sih [4]; Cherepanov [5]; Hutchinson, Mear, and Rice [6]; Sih and Asaro [7]; Rice [2]. Most of these definitions differ from one another only by the phase factor.

It is known that the balance between elastic energy and surface energy in a stressed solid may lead to morphological change via diffusional mass transport. Srolvitz [8] considered the stability of a flat surface of an isotropic solid and found that the surface becomes unstable beyond a critical wave length. Stress concentration effects caused by slightly undulating surfaces have been recently studied by Gao [9, 10]. Using the aspect ratio, ε, of a shallow dimple as the small parameter, he has derived a procedure that is capable of generating uniformly valid asymptotic solutions accurate to the order of ε. While not explicitly stipulated by Gao, the slightly uneven surface profile must have continuous derivative in order for the first order perturbation solution to be uniformly valid throughout the region of consideration. Wu [11] has studied the problem of a depression/protrusion profile having corners, and proposed the method of matched asymptotic expansions to obtain the desired analytic solution.

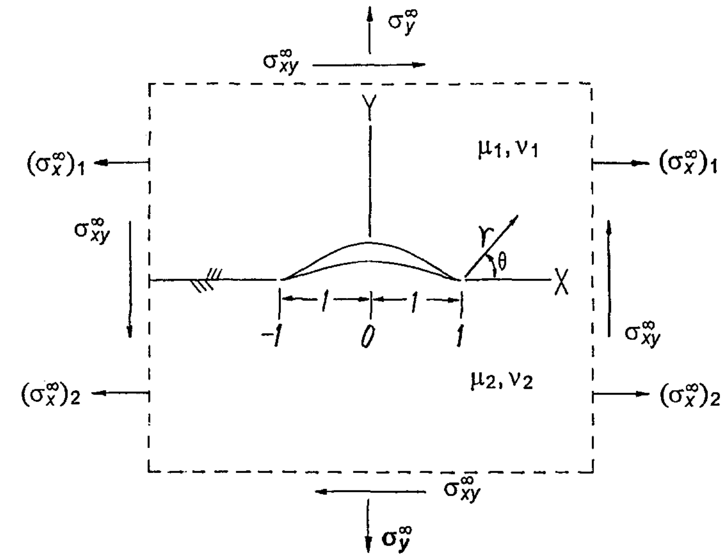

Figure 1. Infinite bimaterial-plate with a perturbed-interface hole subjected to stresses at infinity.

Similarly, the interface of stressed bimaterials via the diffusional mass transport or general loading conditions, also presents the undulating behavior. In reviewing these papers we note that none of them studied the undulating interfacial crack problem. Therefore, this paper is aimed at examining the role of the undulating interfacial surface of interfacial fracture phenomena or issues such as undulating interface-crack extensions for crack propagation under general loading conditions. For investigating the unevenness of cracks in which the problem of the non-straight crack existing in a monolithic material is studied, Cotterell and Rice [12] and Wu [13] provide good guides to solving the undulating interface crack problem. Based on the works mentioned above, in this paper a perturbation analysis, Cole [14], valid to the first order accuracy in the deviation of the interfacial crack surfaces from a straight line is developed for an undulating interfacial crack of bimaterials. Perturbation solutions in remarkably concise forms are given for the stress intensity factors at the tip of an undulating crack.

2. Statement of problem

Let a material with elastic properties μ_1 and ν_1 occupy the upper half-plane and a material with elastic properties μ_2 and ν_2 occupy the lower half-plane. The two materials are bonded along straight-line segments of the x-axis. A slightly undulating crack exists on the interface, and all quantities in length are normalized with a half of horizontal projection length along the x-axis of the slightly undulating crack such that the two tips locate at $(-1, 0)$ and $(1, 0)$, respectively. This configuration is shown in Figure 1.

Muskhelishvili [15] and others have shown that the solution to an individual problem in the plane theory of elasticity can be reduced to finding two independent complex functions, which satisfy the boundary conditions of the problem.

In the case of two different materials, however, the elastic properties are discontinuous across the bonded line, and a complete solution to the problem requires knowledge of the four complex functions, $W_j(z)$ and $w_j(z) j = 1, 2$, of the complex variable $z = x + iy$. The basic equations for the two-dimensional isotropic elasticity in the forms used by Kolosov–Muskhelishvili are

$$(\sigma_x)_j + (\sigma_y)_j = 4\,\mathrm{Re}[W_j'(z)], \tag{1}$$

$$(\sigma_y)_j - \mathbf{i}(\sigma_{xy})_j = W_j'(z) + \overline{W_j'(z)} + \overline{w_j'(z)}, \tag{2}$$

$$2\mu_j(u_1 + \mathbf{i}u_2)_j = \kappa W_j(z) - z\overline{W_j'(z)} - \overline{w_j(z)}, \tag{3}$$

and

$$\mathbf{i}R_j = \mathbf{i}\int [(\sigma_{\beta 1})_j n_\beta + \mathbf{i}(\sigma_{\beta 2})_j n_\beta]\,\mathrm{d}s = W_j(z) + z\overline{W_j'(z)} + \overline{w_j(z)}, \tag{4}$$

where $(u_1, u_2)_j$ are components of displacement, $(\sigma_x)_j, (\sigma_y)_j, (\sigma_{xy})_j$ are components of stress, μ_j is the shear modulus, $\kappa_j = 3 - 4\nu_j$ for plane strain and $\kappa_j = (3-\nu_j)/(1+\nu_j)$ for generalized plane stress, and ν_j is the Poisson ratio. Also $R_j = (R_1 + \mathbf{i}R_2)_j$ is the resultant force. $W_j'(z), w_j'(z)$ represent the derivatives of $W_j(z), w_j(z)$ with respect to z, respectively.

For the class of problems to be discussed in this paper it is more convenient to employ a third holomorphic function $f(z)$, which may be expressed in terms of $W(z)$ and $w(z)$ as shown

$$f_j(z) = W_j(z) - z\overline{W_j'(\bar{z})} - \overline{w_j(\bar{z})}. \tag{5}$$

Using the above to eliminate $w(z)$ from (2–4), we get (England [16])

$$(\sigma_y)_j - \mathbf{i}(\sigma_{xy})_j = W_j'(z) + W_j'(\bar{z}) + (z - \bar{z})\overline{W_j''(z)} - f_j'(\bar{z}), \tag{6}$$

$$2\mu_j(u_1 + \mathbf{i}u_2)_j = \kappa_j W_j(z) - W_j(\bar{z}) + (\bar{z} - z)\overline{W_j'(z)} + f_j(\bar{z}), \tag{7}$$

$$\mathbf{i}R_j = W_j(z) + W_j(\bar{z}) + (z - \bar{z})\overline{W_j'(z)} - f_j(\bar{z}). \tag{8}$$

The boundary conditions at infinity are subjected to an uniform stress field as shown in Figure 1. A notable condition is that the stress σ_x^∞ is discontinuous across the interface, i.e., $(\sigma_x^\infty)_1 \neq (\sigma_x^\infty)_2$. With this condition the continuity conditions along the bonded lines then can be satisfied such that this elasticity problem will be solved without further problems.

3. Formulation of problem

The interface between two media can be divided into two parts, one is the bonded part including $x < -1$ and $x > 1$ to be named C, the other is the slightly undulating crack in $-1 \leqslant x \leqslant 1$ to be named C'. It is assumed that C is perfectly bonded so that tractions and displacements along it are continuous. In particular, the traction continuity condition may be integrated once to become a continuity condition in R (c.f. (8)). Hence, we obtain two continuity equations as shown in (9) and (10),

$$\begin{aligned} &W_1(z_c) + W_1(\overline{z_c}) + (z_c - \overline{z_c})\overline{W_1'(z_c)} - f_1(\overline{z_c}) \\ &\quad = W_2(z_c) + W_2(\overline{z_c}) + (z_c - \overline{z_c})W_2'(z_c) - f_2(\overline{z_c}), \end{aligned} \tag{9}$$

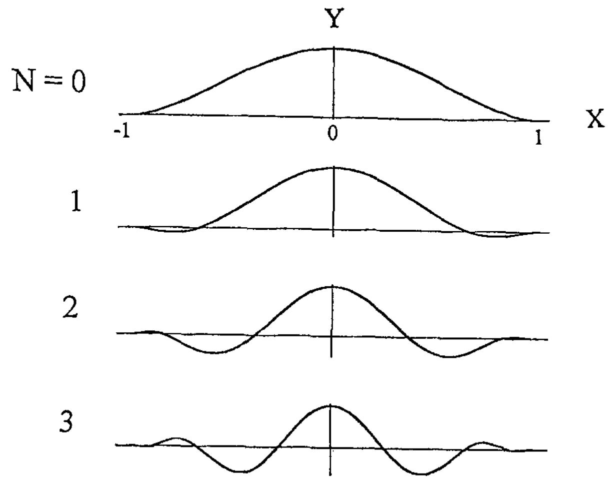

Figure 2. Schematics of smoothly undulating-interface cracks.

$$W_1(z_c) = \gamma W_2(z_c) + \gamma^*[W_2(z_c) + W_2(\overline{z_c}) + (z_c - \overline{z_c})\overline{W_2'(z_c)} - f_2(\overline{z_c})], \tag{10}$$

where (9) is the integrated form which represents the continuity in R, and (10) shows the displacement continuity condition after being rearranged by using (9). Two constants γ and γ^* appearing in (10) are composed of four material constants as shown below

$$\gamma = \frac{(1+\kappa_2)\mu_1}{(1+\kappa_1)\mu_2}, \qquad \gamma^* = \frac{1}{1+\kappa_1}\left(1 - \frac{\mu_1}{\mu_2}\right). \tag{11}$$

The constants γ and γ^* can be expressed as Dundur's constants α and β, via the following relations.

$$\alpha = \frac{\gamma - 1}{\gamma + 1}, \quad \beta = \frac{2\gamma^* + \gamma - 1}{\gamma + 1} \quad \text{or} \quad \gamma = \frac{1+\alpha}{1-\alpha}, \quad \gamma^* = \frac{\beta - \alpha}{1 - \alpha}. \tag{12}$$

The slightly undulating crack is assumed to be

$$C'\colon z_{c'} = x + \mathbf{i}y, \quad y = \varepsilon(1 - x^2)^2 \cos\left[N \ln\left(\frac{1+x}{1-x}\right)\right]; \quad x \in (-1, 1), \tag{13}$$

where N can be a number of $O(1)$ to make the crack smooth and the perturbation theory valid for the following solving processes, the cosine function generates the undulating behavior, $\ln(1 + x^*/1 - x^*)$ is so designed to let triangular functions converge to single values as z goes to infinity, the oscillating frequency can be adjusted by changing the value of N, and the expression of crack shape is assumed to be in the form such that its values and derivatives exit and tend to be zero at $x = \pm 1$. Some configurations of such cracks are depicted in Figure 2.

According to the perturbation analysis, the four complex functions $W_j(z)$, $f_j(z)$ can be rewritten as $W_j(z;\varepsilon)$, $f_j(z;\varepsilon)$, $j = 1, 2$ after involving the perturbation variable ε. And they could be in powers of ε as follows

$$W_j(z;\varepsilon) \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} W_{jn}(z). \tag{14.1}$$

$$f_j(z;\varepsilon) \sim \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} f_{jn}(z), \tag{14.2}$$

where $W_{jn}(z)$ and $f_{jn}(z)n = 1, 2, 3\ldots$ can be expanded as $z = z_{c'}$.

$$W_{jn}(z_{c'}) \sim W_{jn}^{+}(x) + \mathbf{i}\varepsilon(1-x^2)^2 \cos\left[N \ln\left(\frac{1+x}{1-x}\right)\right] W_{jn}'^{+}(x)$$
$$+\frac{-\varepsilon^2(1-x^2)^4 \cos^2\left[N \ln\left(\frac{1+x}{1-x}\right)\right]}{2} W_{jn}''^{+}(x) + O(\varepsilon^3), \tag{15.1}$$

$$f_{jn}(z_{c'}) \sim f_{jn}^{+}(x) + \mathbf{i}\varepsilon(1-x^2)^2 \cos\left[N \ln\left(\frac{1+x}{1-x}\right)\right] f'^{+}_{jn}(x)$$
$$+\frac{-\varepsilon^2(1-x^2)^4 \cos^2\left[N \ln\left(\frac{1+x}{1-x}\right)\right]}{2} f_{jn}''^{+}(x) + O(\varepsilon^3). \tag{15.2}$$

Similarly, $W_{jn}(\overline{z_{c'}})$ and $f_{jn}(\overline{z_{c'}})$ also have their expanded forms.

$$W_{jn}(\overline{z_{c'}}) \sim W_{jn}^{-}(x) - \mathbf{i}\varepsilon(1-x^2)^2 \cos\left[N \ln\left(\frac{1+x}{1-x}\right)\right] W_{jn}'^{-}(x)$$
$$+\frac{-\varepsilon^2(1-x^2)^4 \cos^2\left[N \ln\left(\frac{1+x}{1-x}\right)\right]}{2} W_{jn}''^{-}(x) + O(\varepsilon^3), \tag{15.3}$$

$$f_{jn}(\overline{z_{c'}}) \sim f_{jn}^{-}(x) - \mathbf{i}\varepsilon(1-x^2)^2 \cos\left[N \ln\left(\frac{1+x}{1-x}\right)\right] f_{jn}'^{-}(x)$$
$$+\frac{-\varepsilon^2(1-x^2)^4 \cos^2\left[N \ln\left(\frac{1+x}{1-x}\right)\right]}{2} f_{jn}''^{-}(x) + O(\varepsilon^3). \tag{15.4}$$

In deriving the above equations, the interpretations of $W_{jn}^{+}(x)$ and $W_{jn}^{-}(x)$ need to be clarified.

$$W_{1n}(x+\mathbf{i}0) = W_{1n}^{+}(x), \qquad W_{1n}(x-\mathbf{i}0) = W_{1n}^{-}(x),$$
$$W_{2n}(x-\mathbf{i}0) = W_{2n}^{+}(x) \quad \text{and} \quad W_{2n}(x+\mathbf{i}0) = W_{2n}^{-}(x).$$

The same interpretation is also valid for $f_{jn}^{+}(x)$ and $f_{jn}^{-}(x)$.

The boundary condition along the crack face is traction free. Hence, (8) leads to

$$W_1(z_{c'}) + W_1(\overline{z_{c'}}) + (z_{c'} - \overline{z_{c'}})\overline{W_1'(z_{c'})} - f_1(\overline{z_{c'}}) = 0, \tag{16}$$

$$W_2(z_{c'}) + W_2(\overline{z_{c'}}) + (z_{c'} - \overline{z_{c'}})\overline{W_2'(z_{c'})} - f_2(\overline{z_{c'}}) = 0, \tag{17}$$

where

$$z_{c'} - \overline{z_{c'}} = \mathbf{i}2\varepsilon(1-x^2)^2 \cos\left[N \ln\left(\frac{1+x}{1-x}\right)\right], \quad x \in (-1, 1).$$

Equations (9), (10), (16) and (17) are the formulation of this type of problem. Substitute (14) into (9) and (10), and substitute (15) into (16) and (17) to find two groups of equations for solving the solutions to the orders of ε^0 and ε^1.

$$\varepsilon^0\colon W_{10}^+(x_c) + W_{10}^-(x_c) - f_{10}^-(x_c) = W_{20}^+(x_c) + W_{20}^-(x_c) - f_{20}^-(x_c), \tag{18}$$

$$W_{10}^+(x_c) = \gamma W_{20}^+(x_c) + \gamma^*[W_{20}^+(x_c) + W_{20}^-(x_c) - f_{20}^-(x_c)], \tag{19}$$

$$W_{10}^+(x_{c'}) + W_{10}^-(x_{c'}) - f_{10}^-(x_{c'}) = 0, \tag{20}$$

$$W_{20}^+(x_{c'}) + W_{20}^-(x_{c'}) - f_{20}^-(x_{c'}) = 0, \tag{21}$$

$$\varepsilon^1\colon W_{11}^+(x_c) + W_{11}^-(x_c) - f_{11}^-(x_c) = W_{21}^+(x_c) + W_{21}^-(x_c) - f_{21}^-(x_c), \tag{22}$$

$$W_{11}^+(x_c) = \gamma W_{21}^+(x_c) + \gamma^*[W_{21}^+(x_c) + W_{21}^-(x_c) - f_{21}^-(x_c)], \tag{23}$$

$$\begin{aligned}&[W_{11}^+(x_{c'}) + W_{11}^-(x_{c'}) - f_{11}^-(x_{c'})] + \mathbf{i}(1 - x_{c'}^2)^2 \cos\left[N \ln\left(\frac{1 + x_{c'}}{1 - x_{c'}}\right)\right]\\ &\quad \times \{W_{10}'^+(x_{c'}) - W_{10}'^-(x_{c'}) + 2W_{10}'^+(x_{c'}) + f_{10}'^-(x_{c'})\} = 0,\end{aligned} \tag{24}$$

$$\begin{aligned}&[W_{21}^+(x_{c'}) + W_{21}^-(x_{c'}) - f_{21}^-(x_{c'})] + \mathbf{i}(1 - x_{c'}^2)^2 \cos\left[N \ln\left(\frac{1 + x_{c'}}{1 - x_{c'}}\right)\right]\\ &\quad \times \{W_{20}'^+(x_{c'}) - W_{20}'^-(x_{c'}) + 2W_{20}'^+(x_{c'}) + f_{20}'^-(x_{c'})\} = 0,.\end{aligned} \tag{25}$$

where $x_c \in (-\infty, -1] \cup [1, \infty)$, $x_{c'} \in (-1, 1)$.

4. Solution to the order of ε^0

By some manipulations, (18) and (19) can be rewritten as

$$W_{10}^+(x_c) - W_{20}^-(x_c) + f_{20}^-(x_c) = W_{20}^+(x_c) - W_{10}^-(x_c) + f_{10}^-(x_c), \tag{26}$$

$$W_{10}^+(x_c) - \gamma^*[W_{20}^-(x_c) - f_{20}^-(x_c)] = (\gamma + \gamma^*)W_{20}^+(x_c). \tag{27}$$

Based on (26) and (27), two new complex functions $\Phi_0(z)$ and $\Psi_0(z)$ can be assigned as follows.

$$\begin{aligned}\Phi_0(z) &= W_{10}(z) - W_{20}(z) + f_{20}(z)\\ &= W_{20}(z) - W_{10}(z) + f_{10}(z),\end{aligned} \tag{28}$$

$$\begin{aligned}\Psi_0(z) &= W_{10}(z) - \gamma^*[W_{20}(z) - f_{20}(z)]\\ &= (\gamma + \gamma^*)W_{20}(z),\end{aligned} \tag{29}$$

where $\Phi_0(z)$ and $\Psi_0(z)$ will satisfy $\Phi_0^+(x_c) = \Phi_0^-(x_c)$ and $\Psi_0^+(x_c) = \Psi_0^-(x_c)$. Then we can conclude that $\Phi_0(z)$ and $\Psi_0(z)$ are holomorphic functions in the whole plane cut along -1 to 1.

Manipulations between (20) and (21) will induce two expressions which describe a relation between Φ_0^+ and Φ_0^-, and a relation between Ψ_0^+ and Ψ_0^-.

Equation (20) subtracted by (21) yields

$$[W_{10}^+(x_{c'}) - W_{20}^-(x_{c'}) + f_{20}^-(x_{c'})] - [W_{20}^+(x_{c'}) - W_{10}^-(x_{c'}) + f_{10}^-(x_{c'})] = 0. \tag{30}$$

And adding the two equations (20) and (21) yields

$$[W_{10}^+(x_{c'}) + W_{20}^-(x_{c'}) - f_{20}^-(x_{c'})] + [W_{20}^+(x_{c'}) + W_{10}^-(x_{c'}) - f_{10}^-(x_{c'})] = 0. \tag{31}$$

According to the definition in (28), Eqn. (30) gives the first expression

$$\Phi_0^+(x_{c'}) - \Phi_0^-(x_{c'}) = 0. \tag{32}$$

With (28) and (29), the four complex functions $W_{10}(z), W_{20}(z), f_{10}(z)$ and $f_{20}(z)$ can be expressed as functions of $\Phi_0(z)$ and $\Psi_0(z)$.

$$W_{10}(z) = \frac{1}{1-\gamma^*}[\Psi_0(z) - \gamma^*\Phi_0(z)], \tag{33}$$

$$W_{20}(z) = \frac{1}{\gamma+\gamma^*}\Psi_0(z), \tag{34}$$

$$f_{10}(z) = \left[\frac{1}{1-\gamma^*} - \frac{1}{\gamma+\gamma^*}\right]\Psi_0(z) + \frac{1-2\gamma^*}{1-\gamma^*}\Phi_0(z), \tag{35}$$

$$f_{20}(z) = \left[\frac{1}{\gamma+\gamma^*} - \frac{1}{1-\gamma^*}\right]\Psi_0(z) + \frac{1}{1-\gamma^*}\Phi_0(z). \tag{36}$$

With (32–36) and the definition in (29), Eqn. (31) gives the second expression

$$\Psi_0^+(x_{c'}) + \lambda\Psi_0^-(x_{c'}) = \Phi_0^+(x_{c'}), \tag{37}$$

where $\lambda = (1-\gamma^*)/(\gamma+\gamma^*)$.

Equations (32) and (37) are the homogeneous and general Hilbert's problems, respectively. Referring to the solutions of Hilbert's problems in [16], the two cases of problems can be solved without further difficulties.

The solution of (32) would be

$$\Phi_0(z) = Bz, \tag{38}$$

where B is a complex constant.

The solution of (37) would be

$$\Psi_0(z) = \frac{1}{1+\lambda}Bz + P\chi(z), \tag{39}$$

where P is also a complex constant

$$\chi(z) = (z^2-1)^{1/2}\left(\frac{z+1}{z-1}\right)^{i\alpha}, \tag{40}$$

$$\alpha = \frac{1}{2\pi}\ln(\lambda). \tag{41}$$

Let the boundary condition at infinity be satisfied, the complex constants B and P will be evaluated, as well as a notable relation of $(\sigma_x^\infty)_1$, $(\sigma_x^\infty)_2$ and σ_y^∞ can be obtained.

$$P = \frac{1-\gamma^*}{1+\lambda}[\sigma_y^\infty - \mathbf{i}\sigma_{xy}^\infty], \tag{42}$$

$$B = \frac{1+\gamma}{4}[(\sigma_x^\infty)_2 + \sigma_y^\infty] - (1-\gamma^*)\sigma_y^\infty \tag{43}$$

and

$$(\sigma_x^\infty)_2 = \frac{1}{\gamma}[(\sigma_x^\infty)_1 + (1-\gamma-4\gamma^*)\sigma_y^\infty], \tag{44}$$

where the imaginary part of B is neglected, because this part will not influence the stress field. Equation (44) is the same as (19) in [4], and it reveals the relation among the stresses at infinity under the satisfaction of the continuity conditions along the bonded lines.

And now the four complex functions $W_{10}(z)$, $W_{20}(z)$, $f_{10}(z)$ and $f_{20}(z)$ are found.

$$W_{10}(z) = A_{10}z + B_{10}\chi(z), \tag{45}$$

$$W_{20}(z) = A_{20}z + B_{20}\chi(z), \tag{46}$$

$$f_{10}(z) = C_{10}z + D_{10}\chi(z), \tag{47}$$

$$f_{20}(z) = C_{20}z + D_{20}\chi(z), \tag{48}$$

where

$$A_{10} = \frac{\gamma}{1+\gamma}B, \qquad B_{10} = \frac{1}{1-\gamma^*}P, \qquad A_{20} = \frac{1}{1+\gamma}B, \qquad B_{20} = \frac{1}{\gamma+\gamma^*}P,$$

$$C_{10} = \frac{2\gamma}{1+\gamma}B, \qquad D_{10} = \frac{\gamma+2\gamma^*-1}{(1-\gamma^*)(\gamma+\gamma^*)}P,$$

$$C_{20} = \frac{2}{1+\gamma}B \quad \text{and} \quad D_{20} = -D_{10}.$$

5. Solution to the order of ε^1

The solution to the order of ε^1 indicates how the slightly oscillatory crack will influence the physical quantities. The solving process here would be almost the same as the one to the order ε^0. In (22), move the last two terms in the right-hand side to the left and the last two terms in the left-hand side to the right, then (22) goes to (49). Also move the bracketed term in the right-hand side of (23) to the left, (23) then goes to (50).

$$W_{11}^+(x_c) - W_{21}^-(x_c) + f_{21}^-(x_c) = W_{21}^+(x_c) - W_{11}^-(x_c) + f_{11}^-(x_c), \tag{49}$$

$$W_{11}^{+}(x_c) - \gamma^*[W_{21}^{-}(x_c) - f_{21}^{-}(x_c)] = (\gamma + \gamma^*)W_{21}^{+}(x_c). \tag{50}$$

Based on (49) and (50), two new complex functions $\Phi_1(z)$ and $\Psi_1(z)$ can be assigned as follows

$$\Phi_1(z) = W_{11}(z) - W_{21}(z) + f_{21}(z) = W_{21}(z) - W_{11}(z) + f_{11}(z), \tag{51}$$

$$\Psi_1(z) = W_{11}(z) - \gamma^*[W_{21}(z) - f_{21}(z)] = (\gamma + \gamma^*)W_{21}(z), \tag{52}$$

where $\Phi_1(z)$ and $\Psi_1(z)$ will satisfy $\Phi_1^+(x_c) = \Phi_1^-(x_c)$ and $\Psi_1^+(x_c) = \Psi_1^-(x_c)$. Then we can conclude that $\Phi_1(z)$ and $\Psi_1(z)$ are holomorphic functions in the whole plane cut along -1 to 1.

Manipulations between (24) and (25) will induce two expressions which describe a relation between Φ_1^+ and Φ_1^-, and a relation between Ψ_1^+ and Ψ_1^-.

Equation (24) subtracted by (25) yields

$$\begin{aligned}\Phi_1^+(x_{c'}) - \Phi_1^-(x_{c'}) &= \mathbf{i}(1 - x_{c'}^2)^2 \cos\left[N \ln\left(\frac{1 + x_{c'}}{1 - x_{c'}}\right)\right] \\ &\quad \times \{A_{\phi_1} + B_{\Phi_1}[\chi'^+(x_{c'}) + \chi'^-(x_{c'})] + 2\overline{B_{\Phi_1}\chi'^+(x_{c'})}\}.\end{aligned} \tag{53}$$

Adding the two equations (20) and (21) yields

$$\begin{aligned}&2[\Psi_1^+(x_{c'}) + \lambda\Psi_1^-(x_{c'})] \\ &\quad = [\Phi_1^+(x_{c'}) + \Phi_1^-(x_{c'})] + \mathbf{i}(1 - x_{c'}^2)^2 \cos\left[N \ln\left(\frac{1 + x_{c'}}{1 - x_{c'}}\right)\right] \\ &\quad\quad \times \{A_{\Psi_1} + B_{\Psi_1}\chi'^+(x_{c'}) + C_{\Psi_1}\chi'^-(x_{c'}) + 2\overline{B_{\Psi_1}\chi'^+(x_{c'})}\},\end{aligned} \tag{54}$$

where

$$A_{\Phi_1} = \frac{4(1-\gamma)}{1+\gamma}B, \quad B_{\Phi_1} = \frac{1 - \gamma - 2\gamma^*}{(1-\gamma^*)(\gamma+\gamma^*)}P, \quad A_{\Psi_1} = \frac{4(2\gamma^* - \gamma - 1)}{1+\gamma}B,$$

$$B_{\Psi_1} = \frac{2\gamma^*(1-\gamma^*) - (1+\gamma)}{(1-\gamma^*)(\gamma+\gamma^*)}P \quad \text{and} \quad C_{\Psi_1} = \frac{(1+\gamma) - 2\gamma^*(\gamma+\gamma^*)}{(1-\gamma^*)(\gamma+\gamma^*)}P.$$

The four complex functions $W_{11}(z)$, $W_{21}(z)$, $f_{11}(z)$ and $f_{21}(z)$ can be expressed in terms of $\Phi_1(z)$ and $\Psi_1(z)$ as follows.

$$W_{11}(z) = \frac{1}{1-\gamma^*}[\Psi_1(z) - \gamma^*\Phi_1(z)], \tag{55}$$

$$W_{21}(z) = \frac{1}{\gamma+\gamma^*}\Psi_1(z), \tag{56}$$

$$f_{11}(z) = \left[\frac{1}{1-\gamma^*} - \frac{1}{\gamma+\gamma^*}\right]\Psi_1(z) + \frac{1-2\gamma^*}{1-\gamma^*}\Phi_1(z), \tag{57}$$

$$f_{21}(z) = \left[\frac{1}{\gamma+\gamma^*} - \frac{1}{1-\gamma^*}\right]\Psi_1(z) + \frac{1}{1-\gamma^*}\Phi_1(z). \tag{58}$$

The following functions, which must be considered before solving (53) and (54), will give us a guide to finding the solutions.

$$K_{1,2}(z) = \left(\frac{z-1}{z+1}\right)^{1/2}\left(\frac{z+1}{z-1}\right)^{\pm \mathrm{i}\alpha},$$

$$K_{3,4}(z) = \left(\frac{z-1}{z+1}\right)^{-(1/2)}\left(\frac{z+1}{z-1}\right)^{\pm \mathrm{i}\alpha}, \tag{59a–d}$$

$$L_1(z) = \cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right], \quad L_2(z) = \sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right], \tag{60ab}$$

$$\begin{aligned} M(z) = \frac{1}{\Delta}\bigg\{&\left(1+\frac{1}{\lambda}\right)\cosh(N\pi)\cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right] \\ &-\mathbf{i}\left(1-\frac{1}{\lambda}\right)\sinh(N\pi)\sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\bigg\}, \end{aligned} \tag{61}$$

$$\begin{aligned} N(z) = \frac{1}{\nabla}\bigg\{&(1-\lambda^2)\cosh(N\pi)\sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right] \\ &+\mathbf{i}(1+\lambda^2)\sinh(N\pi)\cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\bigg\}, \end{aligned} \tag{62}$$

where

$$\Delta = (1+\lambda^{-1})^2\cosh^2(N\pi) - (1-\lambda^{-1})^2\sinh^2(N\pi),$$

$$\nabla = (1-\lambda^2)^2\cosh^2(N\pi) - (1+\lambda^2)^2\sinh^2(N\pi).$$

The functions (59–62) satisfy the following conditions

$$K_1^+(x_{c'}) = \mathbf{i}\lambda^{1/2}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{1/2}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{\mathrm{i}\alpha},$$

$$K_1^-(x_{c'}) = -\mathrm{i}\lambda^{-(1/2)}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{1/2}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{\mathrm{i}\alpha}, \tag{63}$$

$$K_2^+(x_{c'}) = \mathbf{i}\lambda^{-(1/2)}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{1/2}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{-\mathrm{i}\alpha},$$

$$K_2^-(x_{c'}) = -\mathbf{i}\lambda^{1/2}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{1/2}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{-\mathrm{i}\alpha}. \tag{64}$$

$$K_3^+(x_{c'}) = -\mathbf{i}\lambda^{1/2}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{-(1/2)}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{\mathrm{i}\alpha},$$

$$K_3^-(x_{c'}) = \mathbf{i}\lambda^{-(1/2)}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{-(1/2)}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{\mathrm{i}\alpha}, \tag{65}$$

$$K_4^+(x_{c'}) = -\mathbf{i}\lambda^{-(1/2)}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{-(1/2)}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{-\mathrm{i}\alpha},$$

$$K_4^-(x_{c'}) = \mathbf{i}\lambda^{1/2}\left(\frac{1-x_{c'}}{1+x_{c'}}\right)^{-(1/2)}\left(\frac{1+x_{c'}}{1-x_{c'}}\right)^{-\mathrm{i}\alpha}, \tag{66}$$

$$L_1^+(x_{c'}) = \cosh(N\pi)\cos\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right] + \mathbf{i}\sinh(N\pi)\sin\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right];$$

$$L_1^-(x_{c'}) = \cosh(N\pi)\cos\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right] - \mathbf{i}\sinh(N\pi)\sin\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right], \tag{67}$$

$$L_2^+(x_{c'}) = \cosh(N\pi)\sin\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right] - \mathbf{i}\sinh(N\pi)\cos\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right];$$

$$L_2^-(x_{c'}) = \cosh(N\pi)\sin\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right] + \mathbf{i}\sinh(N\pi)\cos\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right], \tag{68}$$

$$M^+(x_{c'}) + \lambda^{-1}M^-(x_{c'}) = \cos\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right] \tag{69}$$

and

$$N^+(x_{c'}) - \lambda^2 N^-(x_{c'}) = \sin\left[N\ln\left(\frac{1+x_{c'}}{1-x_{c'}}\right)\right]. \tag{70}$$

Using the above functions, (53) and (54), we can find the solutions $\Phi_1(z)$ and $\Psi_1(z)$. The solution to (53) would be

$$\begin{aligned}\Phi_1(z) &= \frac{-A_{\Phi_1}}{2\sinh(N\pi)}(1-z^2)^2\sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\\ &\quad + h(z)[C_{\Phi_1}g_1(z) + D_{\Phi_1}g_2(z) + E_{\Phi_1}g_3(z) + F_{\Phi_1}g_4(z)]\\ &\quad + \Phi_{14}z^4 + \Phi_{13}z^3 + \Phi_{12}z^2 + \Phi_{11}z,\end{aligned} \tag{71}$$

where

$$\begin{aligned}h(z) &= \left(1+\frac{1}{\lambda}\right)\cosh(N\pi)\cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\\ &\quad -\mathbf{i}\left(1-\frac{1}{\lambda}\right)\sinh(N\pi)\sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right],\end{aligned}$$

$$g_1(z) = (1-z^2)^2\left(\frac{z+1}{z-1}\right)^{\mathrm{i}\alpha}\left(\frac{z-1}{z+1}\right)^{1/2},$$

$$g_2(z) = (1-z^2)^2\left(\frac{z+1}{z-1}\right)^{\mathrm{i}\alpha}\left(\frac{z+1}{z-1}\right)^{1/2},$$

$$g_3(z) = (1-z^2)^2\left(\frac{z+1}{z-1}\right)^{-\mathbf{i}\alpha}\left(\frac{z-1}{z+1}\right)^{1/2},$$

$$g_4(z) = (1-z^2)^2\left(\frac{z+1}{z-1}\right)^{-\mathbf{i}\alpha}\left(\frac{z+1}{z-1}\right)^{1/2},$$

$$C_{\Phi_1} = \frac{1}{\Delta}\left[-B_{\Phi_1}\left(1-\frac{1}{\lambda}\right)\left(\alpha-\mathbf{i}\frac{1}{2}\right)\right],\quad D_{\Phi_1} = \frac{1}{\Delta}\left[B_{\Phi_1}\left(1-\frac{1}{\lambda}\right)\left(\alpha+\mathbf{i}\frac{1}{2}\right)\right];$$

$$E_{\Phi_1} = \frac{1}{\Delta}\left[-2\overline{B_{\Phi_1}}\lambda\left(\alpha+\mathbf{i}\frac{1}{2}\right)\right],\quad F_{\Phi_1} = \frac{1}{\Delta}\left[2\overline{B_{\Phi_1}}\lambda\left(\alpha-\mathbf{i}\frac{1}{2}\right)\right]$$

and

$$\Delta = \left(1+\frac{1}{\lambda}\right)^2\cosh^2(N\pi) - \left(1-\frac{1}{\lambda}\right)^2\sinh^2(N\pi).$$

The constants $\Phi_{14}, \Phi_{13}, \Phi_{12}$ and Φ_{11} are so chosen that $\Phi_1'(z)$ tends to $O(1/z^2)$ as z tends to infinity, hence they are

$$\Phi_{14} = -\left(1+\frac{1}{\lambda}\right)\cosh(N\pi)[C_{\Phi_1}+D_{\Phi_1}+E_{\Phi_1}+F_{\Phi_1}],$$

$$\begin{aligned}\Phi_{13} =\ & \frac{NA_{\Phi_1}}{\sinh(N\pi)} + \left(1+\frac{1}{\lambda}\right)\cosh(N\pi)[(1-\mathbf{i}2\alpha)(C_{\Phi_1}-F_{\Phi_1})\\ & +(1+\mathbf{i}2\alpha)(E_{\Phi_1}-D_{\Phi_1})] + \mathbf{i}2N\left(1-\frac{1}{\lambda}\right)\sinh(N\pi)\\ & \times[C_{\Phi_1}+D_{\Phi_1}+E_{\Phi_1}+F_{\Phi_1}],\end{aligned}$$

$$\begin{aligned}\Phi_{12} =\ & \left(1+\frac{1}{\lambda}\right)\cosh(N\pi)[(\tfrac{3}{2}+\mathbf{i}2\alpha+2\alpha^2+2N^2)(C_{\Phi_1}+F_{\Phi_1})\\ & +(\tfrac{3}{2}-\mathbf{i}2\alpha+2\alpha^2+2N^2)(D_{\Phi_1}+E_{\Phi_1})]\\ & +\mathbf{i}2N\left(1-\frac{1}{\lambda}\right)\sinh(N\pi)[(1-\mathbf{i}2\alpha)(F_{\Phi_1}-C_{\Phi_1})+(1+\mathbf{i}2\alpha)(D_{\Phi_1}-E_{\Phi_1})]\end{aligned}$$

and

$$\begin{aligned}\Phi_{11} =\ & \frac{-NA_{\Phi_1}}{3\sinh(N\pi)}(5+2N^2) + \left(1+\frac{1}{\lambda}\right)\cosh(N\pi)[(\tfrac{3}{2}+\mathbf{i}\tfrac{7}{3}\alpha+2\alpha^2\\ & +\mathbf{i}\tfrac{4}{3}\alpha^3+2N^2+\mathbf{i}4\alpha N^2)(D_{\Phi_1}-E_{\Phi_1})\\ & +(\tfrac{3}{2}-\mathbf{i}\tfrac{7}{3}\alpha+2\alpha^2-\mathbf{i}\tfrac{4}{3}\alpha^3+2N^2-\mathbf{i}4\alpha N^2)(F_{\Phi_1}-C_{\Phi_1})]\\ & -\mathbf{i}\left(1-\frac{1}{\lambda}\right)\sinh(N\pi)[(\tfrac{7}{3}N+\mathbf{i}4\alpha N+4\alpha^2N+\tfrac{4}{3}N^3)(C_{\Phi_1}+F_{\Phi_1})\\ & +(\tfrac{7}{3}N-\mathbf{i}4\alpha N+4\alpha^2N+\tfrac{4}{3}N^3)(D_{\Phi_1}+E_{\Phi_1})].\end{aligned}$$

The solution to (54) would be

$$
\begin{aligned}
2\Psi_1(z) = {} & (1-z^2)^2\left[T_1 \cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right] + T_2 \sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\right] \\
& + g_1(z)\left[S_1 \cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right] + S_2 \sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\right] \\
& + g_2(z)\left[S_3 \cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right] + S_4 \sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\right] \\
& + g_3(z)\left[S_5 \cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right] + S_6 \sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\right] \\
& + g_4(z)\left[S_7 \cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right] + S_8 \sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\right] \\
& + \frac{2}{1+\lambda}[\Phi_{14}z^4 + \Phi_{13}z^3 + \Phi_{12}z^2 + \Phi_{11}z] \\
& + (z^2-1)^{1/2}\left(\frac{z+1}{z-1}\right)^{\mathrm{i}\alpha}[q_3z^3 + q_2z^2 + q_1z + q_0], \qquad (72)
\end{aligned}
$$

where

$$T_1 = \frac{\mathbf{i}\cosh(N\pi)[(1+\lambda)A_{\Psi_1} - (1-\lambda)A_{\Phi_1}]}{(1+\lambda)^2\cosh^2(N\pi) - (1-\lambda)^2\sinh^2(N\pi)},$$

$$T_2 = \frac{\sinh(N\pi)[(1-\lambda)A_{\Psi_1} - (1+\lambda)A_{\Phi_1}\coth^2(N\pi)]}{(1+\lambda)^2\cosh^2(N\pi) - (1-\lambda)^2\sinh^2(N\pi)},$$

$$S_1 = \frac{-G_2}{2\lambda^{1/2}\sinh(N\pi)}, \quad S_2 = \frac{G_1}{2\lambda^{1/2}\sinh(N\pi)}, \quad S_3 = \frac{G_4}{2\lambda^{1/2}\sinh(N\pi)},$$

$$S_4 = \frac{-G_3}{2\lambda^{1/2}\sinh(N\pi)},$$

$$S_5 = \frac{1}{\nabla}[-\mathbf{i}\lambda^{1/2}(1-\lambda^2)\cosh(N\pi)G_5 + \lambda^{1/2}(1+\lambda^2)\sinh(N\pi)G_6],$$

$$S_6 = \frac{1}{\nabla}[-\mathbf{i}\lambda^{1/2}(1-\lambda^2)\cosh(N\pi)G_6 - \lambda^{1/2}(1+\lambda^2)\sinh(N\pi)G_5],$$

$$S_7 = \frac{1}{\nabla}[\mathbf{i}\lambda^{1/2}(1-\lambda^2)\cosh(N\pi)G_7 - \lambda^{1/2}(1+\lambda^2)\sinh(N\pi)G_8],$$

$$S_8 = \frac{1}{\nabla}[\mathbf{i}\lambda^{1/2}(1-\lambda^2)\cosh(N\pi)G_8 + \lambda^{1/2}(1+\lambda^2)\sinh(N\pi)G_7],$$

$$G_1 = LC_{\Phi_1} + \lambda^{-1/2}(\tfrac{1}{2} + \mathbf{i}\alpha)[C_{\Psi_1} - \lambda B_{\Psi_1}], \qquad G_2 = -MC_{\Phi_1},$$

$$G_3 = -LD_{\Phi_1} + \lambda^{-1/2}(-\tfrac{1}{2} + \mathbf{i}\alpha)[C_{\Psi_1} - \lambda B_{\Psi_1}], \qquad G_4 = MD_{\Phi_1},$$

$$G_5 = -L^* E_{\Phi_1} + 2\lambda^{1/2}(\tfrac{1}{2} - \mathrm{i}\alpha)\overline{B_{\Psi_1}}, \qquad G_6 = -M^* E_{\Phi_1},$$

$$G_7 = L^* F_{\Phi_1} - 2\lambda^{1/2}(\tfrac{1}{2} + \mathrm{i}\alpha)\overline{B_{\Psi_1}}, \qquad G_8 = M^* F_{\Phi_1},$$

$$L = \mathrm{i}\lambda^{1/2}(1 - \lambda^{-2}), \qquad M = 2\lambda^{-1/2}\sinh(2N\pi),$$

$$L^* = \mathrm{i}\lambda^{1/2}(1 - \lambda^{-2})[\cosh^2(N\pi) + \sinh^2(N\pi)]$$

and

$$M^* = \lambda^{1/2}(1 + \lambda^{-2})\sinh(2N\pi).$$

$q(z) = q_3 z^3 + q_2 z^2 + q_1 z + q_0$ is a polynomial so chosen that $\Psi_1'(z)$ tends to $O(1/z^2)$ as z tends to infinity, and its coefficients are

$$q_3 = -\left\{T_1 + S_1 + S_3 + S_5 + S_7 + \frac{2}{1+\lambda}\Phi_{14}\right\},$$

$$\begin{aligned} q_2 = {} & \mathrm{i}2\alpha q_3 - \Big\{2N[T_2 + S_2 + S_4 + S_6 + S_8] - (1 - \mathrm{i}2\alpha)[S_1 - S_7] \\ & +(1 + \mathrm{i}2\alpha)[S_3 - S_5] + \frac{2}{1+\lambda}\Phi_{13}\Big\}, \end{aligned}$$

$$\begin{aligned} q_1 = {} & (\tfrac{1}{2} + 2\alpha^2)q_3 - \mathrm{i}2\alpha q_2 + \Big\{2(1 + N^2)T_1 + (\tfrac{3}{2} + \mathrm{i}2\alpha + 2\alpha^2 + 2N^2)(S_1 + S_7) \\ & +(\tfrac{3}{2} - \mathrm{i}2\alpha + 2\alpha^2 + 2N^2)(S_3 + S_5) + 2N(1 - \mathrm{i}2\alpha)(S_2 - S_8) \\ & -2N(1 + \mathrm{i}2\alpha)(S_4 - S_6) - \frac{2}{1+\lambda}\Phi_{12}\Big\} \end{aligned}$$

and

$$\begin{aligned} q_0 = {} & \frac{\mathrm{i}\alpha}{3}(1 + 4\alpha^2)q_3 + \tfrac{1}{2}(1 + 4\alpha^2)q_2 - \mathrm{i}2\alpha q_1 + \frac{2N}{3}(5 + 2N^2)T_2 \\ & +(\tfrac{3}{2} - \mathrm{i}\tfrac{7}{3}\alpha + 2\alpha^2 - \mathrm{i}\tfrac{4}{3}\alpha^3 + 2N^2 - \mathrm{i}4\alpha N^2)(S_7 - S_1) \\ & +(\tfrac{3}{2} + \mathrm{i}\tfrac{7}{3}\alpha + 2\alpha^2 + \mathrm{i}\tfrac{4}{3}\alpha^3 + 2N^2 + \mathrm{i}4\alpha N^2)(S_3 - S_5) \\ & +(\tfrac{7}{3}N + \mathrm{i}4\alpha N + 4\alpha^2 N + \tfrac{4}{3}N^3)(S_2 + S_8) \\ & +(\tfrac{7}{3}N - \mathrm{i}4\alpha N + 4\alpha^2 N + \tfrac{4}{3}N^3)(S_4 + S_6) - \frac{2}{1+\lambda}\Phi_{11}. \end{aligned}$$

6. Stress intensity factor

This singular stress field right ahead of the interface crack tip has been described with a complex stress intensity factor K in [4]. And the complex expression of K is $K_{\mathrm{I}} + iK_{\mathrm{II}}$. K_{I}

and $K_{\rm II}$ represent the stress intensity factors of the mode I and mode II in classical fracture mechanics, respectively. Then the stresses at the immediate vicinity of the crack tip can be expressed as

$$\sigma_y + \mathbf{i}\sigma_{xy} = \mathbf{K}(2\pi\mathbf{r})^{-1/2}\mathbf{r}^{\mathbf{i}\alpha}. \tag{73}$$

For the sake of using (6), take the complex conjugate of (61), then

$$K_{\rm I} - \mathbf{i}K_{\rm II} = \sqrt{2\pi}\,\mathbf{r}^{1/2+\mathbf{i}\alpha}[\sigma_y - \mathbf{i}\sigma_{xy}]. \tag{74}$$

Finally, based on the solutions obtained in previous sections, the complex conjugate of K is

$$\begin{aligned} K_{\rm I} - \mathbf{i}K_{\rm II} &= \sqrt{\pi}[\cos(\alpha\ln 2) + \mathbf{i}\,\sin(\alpha\ln 2)](1 - \mathbf{i}2\alpha)\left\{[\sigma_y^\infty - \mathbf{i}\sigma_{xy}^\infty]\right. \\ &\quad \left. +\varepsilon\left[\frac{1+\lambda}{1-\gamma^*}(q_3+q_2+q_1+q_0)\right] + O(\varepsilon^2)\right\}. \end{aligned} \tag{75}$$

In view of (75), there are only $q_3+q_2+q_1+q_0$, which are the coefficients of the polynomial multiplied by $\chi(z)$ in (72), and material constants appearing in the ε^1 term. Hence, the stress singularity at the tip to the order of ε is provided by $\chi'(z)$.

For the special fracture mechanisms in which the lateral stresses dominate, the results derived in this undulating case can be reduced to some simple forms for revealing the effects of the lateral stresses. The far-field stresses σ_y^∞ and σ_{xy}^∞ are neglected in the present determinations, the complex functions $\Phi_1(z)$ and $\Psi_1(z)$ are given below.

$$\Phi_1(z) = -\frac{1}{2\,\sinh(N\pi)}A_\phi(z^2-1)^2\,\sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right] + \Phi_{13}z^3 + \Phi_{11}z, \tag{76}$$

$$\begin{aligned} 2\Psi_1(z) &= (1-z^2)^2\left[T_1\,\cos\left[N\ln\left(\frac{z+1}{z-1}\right)\right] + T_2\,\sin\left[N\ln\left(\frac{z+1}{z-1}\right)\right]\right] \\ &\quad + \frac{2}{1+\lambda}[\Phi_{13}z^3 + \Phi_{11}z] + (z^2-1)^{1/2} \\ &\quad \times\left(\frac{z+1}{z-1}\right)^{\mathbf{i}\alpha}[q_3z^3 + q_2z^2 + q_1z + q_0], \end{aligned} \tag{77}$$

where

$$A_{\Phi_1} = \frac{1-\gamma}{\gamma}(\sigma_x^\infty)_1, \quad \Phi_{13} = \frac{N(1-\gamma)}{\gamma\,\sinh(N\pi)}(\sigma_x^\infty)_1,$$

$$\Phi_{11} = -\frac{N(5+2N^2)(1-\gamma)}{3\gamma\,\sinh(N\pi)}(\sigma_x^\infty)_1, \quad A_{\Psi_1} = \frac{2\gamma^*-\gamma-1}{\gamma}(\sigma_x^\infty)_1,$$

$$T_1 = \frac{\mathbf{i}}{\Delta\lambda^2}\,\cosh(N\pi)[(1+\lambda)A_{\Psi_1} - (1-\lambda)A_{\Phi_1}],$$

$$T_2 = \frac{1}{\Delta\lambda^2}\sinh(N\pi)[(1-\lambda)A_{\Psi_1} - (1+\lambda)A_{\Phi_1}\coth^2(N\pi)],$$

$$q_3 = -T_1, \quad q_2 = \mathrm{i}2\alpha q_3 - 2\left[NT_2 + \frac{1}{1+\lambda}\Phi_{13}\right],$$

$$q_1 = (\tfrac{1}{2} + 2\alpha^2)q_3 - \mathrm{i}2\alpha q_2 + 2(1+N^2)T_1,$$

$$q_0 = \frac{\mathrm{i}\alpha}{3}(1+4\alpha^2)q_3 + \tfrac{1}{2}(1+4\alpha^2)q_2 - \mathrm{i}2\alpha q_1 + \frac{2N(5+2N^2)}{3}T_2 - \frac{2}{1+\lambda}\Phi_{11}.$$

With the results shown above, we can then have $q_3 + q_2 + q_1 + q_0$ determine the stress intensity factors presented in (75) for the fracture mechanism dominated by the crucial lateral stress.

$$\begin{aligned} q_3 + q_2 + q_1 + q_0 = {} & T_1[2(1-\mathrm{i}2\alpha)(1+N^2) - \tfrac{1}{6}(\mathrm{i}14\alpha - \mathrm{i}40\alpha^3 + 9 + 36\alpha^2)] \\ & + T_2[\tfrac{2}{3}N(5+2N^2) - N(3-4\alpha^2 - \mathrm{i}4\alpha)] \\ & + \frac{1}{1+\lambda}\Phi_{13}[\tfrac{2}{3}(5+2N^2) - (3-4\alpha^2-\mathrm{i}4\alpha)]. \end{aligned} \tag{78}$$

7. Results and discussion

In this paper, a slightly undulating interfacial crack has been used to simulate the brittle fracture phenomena of bimaterials. Some basic notions of interface fracture mechanics are summarized in [2], but they are always used to analyze a flat thin-cut of semi-infinite or finite length between the interface. Here we tried to extend the theory to include the slightly undulating interface cracks. The associated stress intensity factors are determined, via a regular perturbation procedure, in terms of a small aspect ratio ε that measures the slight unevenness of the interface. The dominating effect in the first order term of the expression of complex stress intensity factor is the summation of coefficients of the homogeneous polynomial solution, multiplied by $\chi(z)$, to the complex function $\Psi_1(z)$. This is because the slightly undulating case cited in the present paper has the same local geometry around the tip as the conventional thin-cut model. Hence, it can be called a *regularly* perturbed interface crack. There will be regularly and singularly perturbed interface cracks studied in a forthcoming paper by the same authors.

It is known that the associated stress intensity factors of straight-line interface cracks of bimaterials are functions of σ_{yy}^{∞} and σ_{xy}^{∞} only. But for the case of the undulating interfacial cracks of bimaterials their associated stress intensity factors are not only functions of σ_{yy}^{∞} and σ_{xy}^{∞} but also functions of σ_{xx}^{∞}. When the lateral stress is much stronger than the others, the zeroth order solution including only σ_{yy}^{∞} and σ_{xy}^{∞} will be possible to ignore and the first order effect is then notable because it still retains the crucial stress. Obviously, this kind of fracture mechanism cannot be solved under the convenient thin-cut assumption. In this paper, there are several explicit results derived for this mechanism by considering the unevenness of the interface crack, those solutions present the effects induced by the lateral stress in composites.

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