行政院國家科學委員會專題研究計劃成果報告

The Investigation on the Equivalence between the Rigid-Body Rotational Dynamics and Particle Dynamics and Its Applications

剛體旋轉動力學與質點動力學的等價原理與應用之研究

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1.ABSTRACT

We present, in this project, a novel method by which the problem of rigid-body rotational dynamics is converted to the one of particle dynamics.. Then, the problem of maneuvering the rotation of a rigid body can be handled more easily by the theory of equivalent particle dynamics than by the theory of rigid-body rotational dynamics. An example of a rigid body tracking the desired time-varying orientation by supplying it with an appropriate control torque is given.

Keywords: equivalent principle, attitude control

中文摘要

在本計劃我們提出一個等價理論，即將剛體旋轉動力學轉換成質點動力學。此等價質點動力學的方程式比原來剛體旋轉動力方程式簡單的多較容易分析。並應用到一個剛體姿態追蹤控制的問題上。

關鍵誌字： 等價原理，姿態控制。

2. INTRODUCTION

Both Euler's angles [1-3] and Euler's parameters (or quaternion)[4-6] are often used to describe the orientation of the rigid body in most physical or engineering problems. While Euler's parameters have been extensively used because they require less computational efforts and have no geometrical singularities as do Euler's angles[3]. The motivation of converting three-dimensional rigid-body rotational dynamics into four-dimensional particle dynamics comes from the following factors: (1) The inertial terms in the equations of motion are nonlinear for large-angle rigid-body rotation but are linear for particle translation. Intuition tells us that the position control of a particle may be simpler than the attitude control of a rigid body. (2) The four parameters of quaternion with unit length constraint, besides presents the attitude of a rigid body, can be viewed as the components of position vector of a particle moving on a three-dimensional unit hyper-sphere embedded in a four-dimensional Euclidean space($\mathbf{R}^4$). This new idea facilitates the conversion of rigid-body rotational dynamics to particle's dynamics. (3) Some things that cannot be done in lower dimensional space, may be done in a higher dimensional one. For example, two-dimensional insect is assumed to move only on a two-dimensional surface. If this surface is cut into two subsurfaces, the insert cannot cross from one subsurface to the other. However, if this insect is equipped with wings so that it can move in the third dimension, it can fly over.

The proposed theory can be used as a general tool for designing the control torque needed to drive a rigid body to rotate following the desired stationary (or moving) orientation. Practical applications may include the case where a levitated rigid body is rotated by a magnetic torque to an expected orientation or the case where a satellite is maneuvered by a gas jet to reach the command attitude.

3. ROTATIONAL EQUATIONS OF MOTION

The rotational motion of a rigid body about its center of mass which is fixed in space is considered. The Euler's parameters that describe the orientation of the rigid body are denoted by $\mathbf{p}=(e_0,e_1,e_2,e_3)^{\mathrm{T}}$. Let $\mathbf{s}$ denote the position vector of an arbitrary point P of the rigid body in a fixed inertial reference frame with origin coinciding with the center of mass of the rigid body, and $\mathbf{s}'$ the position vector of the same point P in the body-fixed reference frame that has the same origin as the inertial frame. The rotational transformation matrix $\mathbf{A}$ that relates $\mathbf{s}$ to $\mathbf{s}'$ by $\mathbf{s}=\mathbf{A}\mathbf{s}'$ can be decomposed into the product of two 3×4 matrices as [4,5]

$$\mathbf{A}=\mathbf{G}\mathbf{L}^{\mathrm{T}} \tag{1}$$

Euler's parameters are not independent of each other, and they are subject to the constraint

$$\mathbf{p}^{\mathrm{T}}\mathbf{p}=1 \tag{2}$$

Let $\boldsymbol{\omega}$ denote the angular velocity of the rigid body relative to the inertial frame with components resolved along the axes of the body-fixed frame. Then it satisfies

$$\boldsymbol{\omega}=2\mathbf{L}\dot{\mathbf{p}}, \quad \text{or} \quad \dot{\mathbf{p}}=\frac{1}{2}\mathbf{L}^{\mathrm{T}}\boldsymbol{\omega} \tag{3}$$

For generality, it is assumed that the rigid body is maneuvered to track thedesired moving orientation denoted by $\mathbf{P}_d$; if the desired orientation is stationary, $\mathbf{P}_d$ is a constant vector . Let $\mathbf{p}_r(t)$ denote the error (or relative) quaternion that describes the orientation of the rigid body relative to that of the desired one. Then the relation among $\mathbf{p}_r$, $\mathbf{p}$, and $\mathbf{p}_d$ is given by [4]

$$\mathbf{p}_r=\left[\mathbf{p},-\mathbf{G}^{\mathrm{T}}(\mathbf{p})\right]_{4\times 4}\mathbf{p}_d=\left[\mathbf{p}_d,\mathbf{L}^{\mathrm{T}}(\mathbf{p}_d)\right]_{4\times 4}^{\mathrm{T}}\mathbf{p}=\mathbf{M}^{\mathrm{T}}(\mathbf{p}_d)\mathbf{p} \tag{4}$$

It is easy to prove that $\mathbf{M}^{-1}=\mathbf{M}^{\mathrm{T}}$.

Let $\mathbf{J}$ denote the inertial matrix of the rigid body with respect to the body-fixed frame, $\mathbf{h}$ the absolute angular momentum of the rigid body, then $\mathbf{h}=\mathbf{J}\boldsymbol{\omega}$. The governing equations are

$$\begin{cases}\dot{\mathbf{p}}_r=\dfrac{1}{2}\mathbf{L}^{\mathrm{T}}(\mathbf{p}_r)\boldsymbol{\omega}_r\\ \dot{\boldsymbol{\omega}}_r=\mathbf{J}^{-1}\left(\boldsymbol{\tau}-\boldsymbol{\omega}_r\times\mathbf{h}-\boldsymbol{\omega}_d\times\mathbf{h}\right)-\dot{\boldsymbol{\omega}}_d\end{cases} \tag{5a,b}$$

It is obvious that both equations (5a,b) are nonlinear in the state variables $\mathbf{p}_r$ and $\boldsymbol{\omega}_r$ Furthermore, equation (7b) is nonautonomous if $\boldsymbol{\omega}_d$ is time-varying.

4. EQUIVALENT PARTICLE DYNAMICS

The f relative Euler's parameters $\mathbf{p}_r=(e_{0r},e_{1r},e_{2r},e_{3r})^{\mathrm{T}}$ are first proposed to be viewed as the components of the position vector of a particle of mass m in a four-dimensional Euclidean space, $\mathbf{R}^4$. Due to the constraint $\Phi(\mathbf{p}_r)=\mathbf{p}_r^{\mathrm{T}}\mathbf{p}_r-1=0$, the particle is confined to move on the surface of a three-dimensional unit hyper-sphere The kinetic energy of the particle is $T=m\dot{\mathbf{p}}_r^{\mathrm{T}}\dot{\mathbf{p}}_r/2$. Let $\mathbf{v}'$ denote the external forces exerted on the particle. Then, from the Lagrange-multiplier equations of motion [4,7]

$$\left[\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\mathbf{p}}_r}\right)-\frac{\partial T}{\partial\mathbf{p}_r}\right]^{\mathrm{T}}=\mathbf{v}'+\Phi_{\mathbf{p}_r}^{\mathrm{T}}\lambda' \tag{6}$$

where $\lambda'(t)$ is the Lagrange multiplier associated with the quaternion constraint, we can derive the equations of motion of a particle as

$$m\ddot{\mathbf{p}}_r=\mathbf{v}'+\lambda'\mathbf{p}_r \tag{7}$$

Dividing both sides of Eq. (7) by m, and denoting $\mathbf{v}=\mathbf{v}'/m$ and $\lambda=\lambda'/m$, the governing equations become

$$\begin{cases}\ddot{\mathbf{p}}_r=\mathbf{v}+\lambda\mathbf{p}_r\\ \mathbf{p}_r^{\mathrm{T}}\mathbf{p}_r-1=0.\end{cases} \tag{8a,b}$$

System (8) is said to be equivalent to the system (5), if the solution for the position vector $\mathbf{p}_r(t)$ of system (8) is equal to the solution for the attitude $\mathbf{p}_r(t)$ of system (5). This means that the applied force $\mathbf{v}$ and the constraint force $\lambda\mathbf{p}_r$ must be furnished appropriately such that the resulting position vector $\mathbf{p}_r(t)$ of the corresponding particle happens to be the exact relative attitude of the rigid body. It should be noted that the constraint force $\lambda\mathbf{p}_r$ cannot affect particle motion along the spherical surface, since it is in the direction normal to the hyper-sphere. It is clear from this geometrical point of view that only the tangential component of the applied force $\mathbf{v}$ affects the motion of the particle. Before we derive the equation that relates the torque $\boldsymbol{\tau}$ applied to the rigid body to the tangential component of $\mathbf{v}$ exerted on the particle, the following preliminary background must be given a priori.

Proposition 1 .The three column vectors $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$ of the 3×4 matrix $\mathbf{L}^T$ constitute the basis of the tangent plane of the unit hyper-sphere, and they are orthogonal mutually.

The following theorem provides the necessary and sufficient condition for system (8) to be equivalent to system (5).

Proposition 2. Particle dynamics (8) is equivalent to the rigid-body rotational dynamics (5) if and only if the condition

$$\mathbf{J}^{-1}\left(\boldsymbol{\tau}-\boldsymbol{\omega}\times\mathbf{h}-\boldsymbol{\omega}_d\times\mathbf{h}\right)-\dot{\boldsymbol{\omega}}_d=2\mathbf{L}(\mathbf{p}_r)\mathbf{v} \qquad (9)$$

is satisfied.

The mapping $\mathbf{L}:\ \mathbf{v}\in\mathbf{R}^4\rightarrow\boldsymbol{\tau}\in\mathbf{R}^3$ is not homeomorphic. When $\mathbf{v}$ is given, Eq. (9) determines the unique $\boldsymbol{\tau}$, although the inverse is not true.

Proposition 3 Given the control torque $\boldsymbol{\tau}$ for system (5), the control force $\mathbf{v}$ for system (8) is

$$\mathbf{v}=\frac{1}{2}\mathbf{L}^T\left[\mathbf{J}^{-1}\left(\boldsymbol{\tau}-\boldsymbol{\omega}_r\times\mathbf{h}-\boldsymbol{\omega}_d\times\mathbf{h}\right)-\dot{\boldsymbol{\omega}}_d\right]+\mathbf{p}_r\mathbf{p}_r^T\mathbf{g} \quad (10)$$

where $\mathbf{g}$ is an arbitrary 4 by 1 vector and is function of time.

The results of Eqs. (9) and (10) reveal that there are many choices in the design of $\mathbf{v}$, all of which correspond to the same $\boldsymbol{\tau}$. Once $\mathbf{v}$ is chosen, the control torque $\boldsymbol{\tau}$ is uniquely determined.

5. APPLICATION OF EQUIVALENT PARTICLE DYNAMICS

If the rigid-body attitude $\mathbf{p}(t)$ is made to approach the desired one $\mathbf{p}_d(t)$, that is, $\mathbf{p}_r$ approaches $(\pm1,0,0,0)^T$, the applied (or control) torque $\boldsymbol{\tau}$ in Eq. (5b) must be designed properly. The control force $\mathbf{v}$ is designed in the form

$$\mathbf{v}=-c\dot{\mathbf{p}}_r-k\left(\mathbf{p}_r-\mathbf{p}_e\right), \qquad (11)$$

where $\mathbf{p}_e=(1,0,0,0)^T$. The substitution of (11) into (8) yields

$$\begin{cases}\ddot{\mathbf{p}}_r=-c\dot{\mathbf{p}}_r-k\left(\mathbf{p}_r-\mathbf{p}_e\right)+\lambda\mathbf{p}_r\\ \mathbf{p}_r^T\mathbf{p}_r-1=0\end{cases} \qquad (12)$$

The physical model of system (12) is that. the particle and hyper-sphere are immersed together in a viscous fluid. The term $-c\dot{\mathbf{p}}_r$ is the viscous damping force acting on the particle by the fluid in the direction opposite to the particle velocity. The constant c denotes the damping coefficient. The term $-k(\mathbf{p}_r-\mathbf{p}_e)$ is the restoring force acting on the particle by the spring with one end attached to the north pole of the sphere. This spring is of zero free length and of spring coefficient k. From the above interpretation, we know that the control force is so designed that Eqs. (12) mimic a spring-mass-dashpot system with a massive point moving on the spherical surface. It can be shown that system (12) with constants c and k being any positive real number has a solution approaching the state $\mathbf{p}_r=(1,0,0,0)^T$ and $\dot{\mathbf{p}}_r=\mathbf{0}$, which is a stable equilibrium state.

Using the control force $\mathbf{v}$ in Eq. (10), the control torque input to the rigid body is obtained from Eq. (3)

$$\boldsymbol{\tau}=\boldsymbol{\omega}\times\mathbf{h}+\mathbf{J}\left(-c\boldsymbol{\omega}_r+2k\mathbf{L}(\mathbf{p}_r)\mathbf{p}_e+\dot{\boldsymbol{\omega}}_d\right). \qquad (13)$$

The substitution of (13) into Eq. (5b) gives the closed-loop system

$$\begin{cases}\dot{\boldsymbol{\omega}}_d = -c\boldsymbol{\omega}_r + 2k\mathbf{L}(\mathbf{p}_r)\mathbf{p}_e \\ \dot{\mathbf{p}}_r = \frac{1}{2}\mathbf{L}^T(\mathbf{p}_r)\boldsymbol{\omega}_r\end{cases} \qquad (14)$$

By Proposition 2, system (12) is equivalent to system (14). System (12) is asymptotically stable about the equilibrium point $(\mathbf{p}_r, \dot{\mathbf{p}}_r) = (\mathbf{p}_e, \mathbf{0})$. Therefore, system (14) should be asymptotically stable about the point $(\mathbf{p}_r, \boldsymbol{\omega}_r) = (\mathbf{p}_e, 2\mathbf{L}(\mathbf{p}_r)\dot{\mathbf{p}}_r) = (\mathbf{p}_e, \mathbf{0})$. A rest-to-spin case where a rigid body is initially at rest and is driven by an applied torque to reach the purely spinning state about an axis which could be any fixed direction in space, is given below as an example.

Rest-to spin case The inertial matrix of the rigid body is assumed to be $\mathbf{J} = diag[20,20,50]kg - m^2$ Let $OXYZ$ frame be a fixed inertial one and $Oxyz$ frame be the body-fixed one with the origin O at the center of mass of the rigid body. The rigid body is initially at rest in space with the inertially symmetric x-axis coinciding with the Z-axis. The desired state is that the rigid body spins at a constant angular speed $2rad/s$ about the body-fixed $z-$ axis that coincides with the inertial $X-$axis.. The numerical integration of Eq. (14) gives the relative quaternion and angular velocity of the rigid body with respect to the desired target. The use of Eq. (4) yields the current absolute quaternion of the rigid body as shown in Fig. 1. Figure 2 reveals that when the rigid body reaches the purely rotational state at constant angular rate, the control torque is zero as expected.

6. CONCLUSIONS

A novel method that converts rigid-body rotational motion to particle motion in a high-dimensional space is presented. The problem of designing the control torque applied to the rigid body for rotational maneuvers can be handled more easily by changing it to the problem of designing the force applied to the particle for orbital control.

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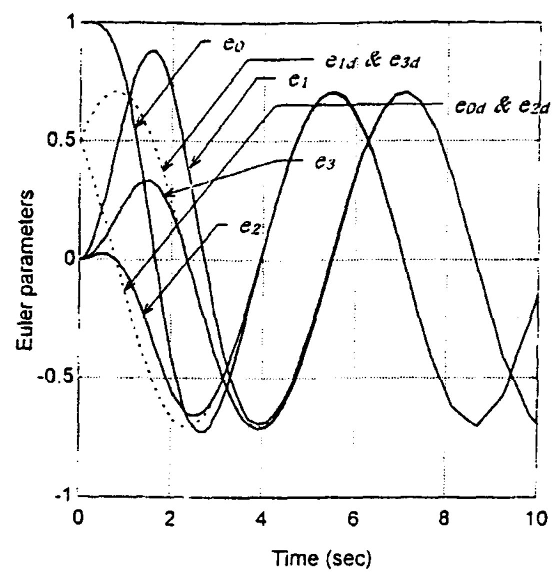

Fig.1 Rtime history of quaternion, $c = 1.6$, $k = 1.2$

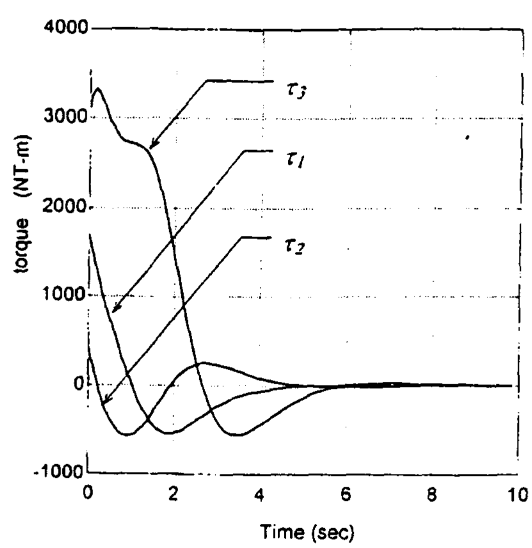

Fig.2 Torque applied to the rigid body, $c = 1.6$, $k = 1.2$.