行政院國家科學委員會專題研究計畫成果報告

離散渦漩法之進階研究(II)

計畫編號：NSC 88-2212-E-002-021
執行期限：87年8月1日至88年7月31日
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一、中文摘要

本研究旨在推展先前由作者開發的二維離散渦法，拓展至三維不可壓縮分離流場(separated flow)**的計算。此方法係結合網格計算與隨機渦漩法之方法，一般又稱為混成渦漩法**(hybrid vortex method)**。在此研究階段，我們提出了新的定義，在此新定義下，我們僅須將前期所提的數值算則作些微的修改，即可直接引用，保留此一數值算則於中高雷諾數流場計算原有高穩定性的優點。同時為從所解渦度場求得其速度場，我們定義了新的向量流線函數，此向量流線函數在固體邊界滿足** Dirichlet **條件，遠域則滿足** Neumann **條件，在此定義和邊界條件下，我們可證明其所得速度場在固體邊界直接滿足不可穿透條件**(non-penetrating condition)**。為驗證此方法，我們以環繞瞬間啟動之一球體流場為例，所得結果和文獻上實驗比較獲致相當一致的結果。**

關鍵詞：離散渦漩法、雷諾數、向量流線函數、不可穿透條件。

Abstract

Some time ago, the present authors proposed a hybrid vortex method for study of two-dimensional separated flows. It is hybrid in that a grid is required, and therefore is not fully Lagrangian. It is also deterministic that no random walk for diffusion is employed. The method is here extended to three-dimensional separated flows. Such an extension is not at all obvious but requires new definition and formulation of the previous method. The method may be briefly described as follows. At any instant, a collection of vortices forms a patch of the flow field. The methods consists of solving the viscous vorticity equation by evolving a total vorticity associated with each vortex, and then redistributing the evolved total vorticity back to the grid at the end of each time step. The total vorticity, when divided by the volume that it occupied, yields the mean vorticity associated with the vortex. The velocity field is recovered from the vorticity field by solving a Poisson equation for a vector stream fucntion. It is shown consistent to specify the gauge that a component of the vector stream function be identically zero; this facilitates imposing Dirichlet conditions for the other two components on the body surface to satisfy the non-penetrating condition. Vorticity is then updated on the body surface to fulfill the no-slip condition. The overall method here presented is a quite general setting for a finite body but with a particular application to flow about an impulsively started sphere. Preliminary results shows excellent comparisons with measured data in several detailed flow chatracteristics.

Keywords: separated flow, Lagrangian, viscous vorticity equation, Poisson equation, Dirichlet condition.

1. Introduction

Being a special technique as they might seem, the methods which use numerical vortices are highly attractive for computing vorticity-dominated incompressible flows. Earlier review articles on this subject include Leonard (1980, 1985). Recently, Chorin (1993) has an excellent survey of the methods that are fully Lagrangian, while Sarpkaya (1995) contains a remarkable account of almost all the existing methods that employ the usage of vortex elments. Though a large body of literature is devoted to the methods for two-dimensional flows, we have seen an increased interest toward the applicaitons of the methods to flows in three space dimensions. Referring to the review articles cited above, the present status toward this effort may be classified according to the individual interest on the subject. (1) One interest is in inviscid flow for fine structures of perturbed vortices. The flows considered are typically in free space and no existence of solid bodies is assumed. (2) Another interest is to simulate viscous flow by a fully Lagrangian vortex method. One is apt to employ Gaussian random walks for simulating viscous diffusion, and this approach is quite appealing for high-Reynolds number flows.

To complement the exisitng interest, the present work is to develop a hybrid vortex method for flow about a finite body in three space dimensions. The flow is assumed to be governed by the incompressible Navier-Stokes equations. The study is based on our earlier work on a deterministic vortex method for two-dimensional flows (Cf. Chang \& Chern 1991a,b). The methods is here modified and extended to be suitable for three-dimensional flows. In order to make an extension, three obvious obstacles are standing in front and the way to obviate these are briefly described as follows. (1) The diffusion-in-cell algorithm involves evolution of a circulation associated with each vortex (blob). Being not possibly to evolve three independent circulations, we introduce a total vorticity associated with each vortex (now in three dimensions). The time variation of a total vorticity can be related to the time variation of the circulations, defined for the six surfaces (in a Cartesian grid), enclosing the vortex. In this connection, the three-dimensional vortex method can actually exploit the usage of its two-dimensional counterpart. (2) As in the diffusion-in-cell algorithm, the evolved total vorticities are redistributed back to the grid, now by a volume-weighting scheme. (3) The velocity is recovered from the vorticity by solving a Poisson equation for a vector stream function. The three components of the equation are not independent, and moreover, the choice of the stream function is not unique, and usually requires specification of a gauge for it. In our formulation, the body surface is a coordinate surface; it is shown consistently to specify the third component of the vector function to be zero. Doing so enables the other two components to satisfy Dirichlet conditions on the body surface. Moreover, the multigrid technique can be incorporated to accelerate the convergence of numerical solution of the Poisson equation for the stream function. The vector stream function thus introduced enforces non-penetrating condition on the body surface; vorticity is then updated on the surface to fulfill the no-slip condition. In summary, the present vortex method has several advantages. (i) With the use of vortices, the viscous vortex-in-cell algorithm is highly stable. (ii) Since a grid is used, the mean values of vorticity lend themselves to high-order accurate reconstruction of the local vorticity field. (iii) The choice of the gauge facilitates solution of the Poisson equation for the stream function subject to Dirichlet conditions. The present hybrid vortex method is formulated for flow about a finite body in general coordinates, and is suitable for a body of rather general shape. In this preliminary study, it is applied to study laminar flow about an impulsively started sphere.

2. Three-Dimensional hybrid vortex method

In this section, we shall continue to develop the hybrid vortex method for three-dimensional flow. Two major issues have to be addressed toward the success of the method: (i) how can the velocity field be recovered from the vorticity field and (ii) how is the vorticity field (of the grid vortices) updated for each time step. The two points are examined respectively in the following two sub-sections with the full algorithm summarized at the end of the section.

2.1 Time evolution of vorticity

The idea of transporting vorticity via circulation plays a central role in the two-dimensional method. But this technique is hardly useful for the three-dimensional case; because here it is not possible to always define three independent circulations. Instead, we define, for a lump of fluid with volume *V*, a total vorticity to be

$$\Omega \equiv \int_V \vec{\omega}\, dV \tag{1}$$

which when divided by the occupied volume yields a mean vorticity for the lump of fluid:

$$\bar{\vec{\omega}} = \Omega / V \tag{2}$$

Since the velocity $\mathbf{u}$ is divergence-free, we have

$$\mathbf{u} = (\mathbf{u} \cdot \nabla)\mathbf{r} = \nabla \cdot (\mathbf{u}\,\mathbf{r}) \tag{3}$$

and therefore

$$\begin{aligned} \frac{d\Omega}{dt} &= \frac{d}{dt}\int_V \vec{\omega}\, dV = \frac{d}{dt}\int_V \nabla \cdot (\vec{\omega}\,\mathbf{r})\, dV \\ &= \frac{d}{dt}\oint_S \mathbf{r}\,\vec{\omega} \cdot \mathbf{n}\, dA \end{aligned} \tag{4}$$

Denoting $\vec{\omega} \cdot \mathbf{n}\, dA = d\Gamma$, we recast (4) into

$$\frac{d\Omega}{dt} = \underbrace{\oint_S \frac{d\mathbf{r}}{dt} d\Gamma}_{\mathrm{I}} + \underbrace{\oint_S \mathbf{r} \frac{d}{dt}(d\Gamma)}_{\mathrm{II}} \tag{5}$$

Because $d\mathbf{r}/dt = \mathbf{u}$, the first term becomes

$$\mathrm{I} = \oint_S \frac{d\mathbf{r}}{dr} d\Gamma = \oint_S \mathbf{u}\,\vec{\omega} \cdot \mathbf{n}\, dA \tag{6}$$

while in the second integral II, $d\Gamma$ is simply the differential version of the two-dimensional counterpart. In these connection, we can actually exploit the usage of two-dimensional algorithm for evolving the total vorticity.

2.2 Recover the velocity field

There are two strategies used to construct the velocity field from the vorticity field; one uses the vector stream function, and the other uses Helmholtz decomposition. Both approaches need one further constraint. In this study, we use the former strategy by introducing a vector stream function $\boldsymbol{\Psi}$ satisfying

$$\mathbf{u} = \nabla \times \boldsymbol{\Psi}. \tag{7}$$

This is possible because $\nabla \cdot \mathbf{u} = 0$. The velocity thus obtained satisfies the incompressibility condition automatically. Because $\vec{\omega} = \nabla \times \mathbf{u}$, we must have

$$\vec{\omega} = \nabla \times (\nabla \times \boldsymbol{\Psi}) = \nabla(\nabla \cdot \boldsymbol{\Psi}) - \nabla^2 \boldsymbol{\Psi}. \tag{8}$$

In (8), the three components are not independent because we must have $\nabla \cdot \breve{\mu} = 0$. One further constraint is needed here. It is found, instead of requiring Ψ to be divergence free, more convenient to specify that the component of Ψ normal to the surface be zero. Two in equations (8) are solved to obtain Ψ, and then the velocity field is obtained simply by using (7).

A gauge chosen for Ψ. Consider a uniform flow about a finite body with velocity U_∞. A general coordinate system $\left(\P^1, \P^2, \P^3\right)$ can be constructed with $\P^1 = 0$ to be the surface of the body and $\P^1 = \infty$ at infinity. The corresponding covariant basis vectors $\{\mathbf{e}_k\}$ are then defined by $\mathbf{e}_i = \partial \mathbf{x} / \partial \P^i$.

For the vector stream function, the approach adopted here is to solve (8) in $\mathbf{e}^2$ and $\mathbf{e}^3$-directions and specify the $\mathbf{e}^1$-component of Ψ to be identically zero. In other words, what to be solved are

$$\Psi_1 = \mathbf{e}^1 \cdot \Psi = 0 \qquad (9)$$

$$\mathbf{e}^k \cdot \nabla \times (\nabla \times \Psi) = \mathbf{e}^k \cdot \breve{\mu} \text{ for } k = 2,3 \qquad (10)$$

The boundary conditions on the surface of the body are constructed to satisfy the non-penetrating condition. The finite body considered may exhibit a motion that is a combination of translation and rotation. The velocity $\mathbf{u}^{(b)}$ at any point of the body can be written as $\mathbf{u}^{(b)} = \mathbf{U}_t + \Omega \times \mathbf{r}$, where $\mathbf{U}_t$ and Ù denote respectively the translation velocity and the angular speed of rotation. Notice the identity

$$\nabla \times \left(\frac{1}{2} \mathbf{U}_t \times \mathbf{r} + \frac{1}{3} (\Omega \times \mathbf{r}) \times \mathbf{r} \right) = \mathbf{U}_t + \Omega \times \mathbf{r} \qquad (11)$$

Define $\Psi^{(b)} = \frac{1}{2} \mathbf{U}_t \times \mathbf{r} + \frac{1}{3} (\Omega \times \mathbf{r}) \times \mathbf{r}$. To specify Ψ uniquely, we require

$$\mathbf{e}_k \cdot \Psi = \mathbf{e}_k \cdot \Psi^{(b)} \text{ at } \P^1 = 0 \text{ for } k = 2,3 \qquad (12)$$

while the far field condition is given by

$$\mathbf{e}_k \cdot (\nabla \times \Psi) = \mathbf{e}_k \cdot \mathbf{U}_\infty \text{ at } \P^1 = \infty \text{ for } k = 2,3 \qquad (13)$$

It can be shown that the stream function thus chosen have the following properties.

(i) $\nabla \times \Psi$ satisfies the non-penetrating boundary Condition.

(ii) The $\mathbf{e}^1$-component of the Poisson equation (8) for Ψ must also be satisfied.

No-slip condition. Once the velocity field that satisfies the non-penetrating condition is determined, a vorticity sheet has to be introduced on the body surface to fulfill the no-slip condition. But this is typically practiced numerically as follows. After Ø (thus **u**) has been obtained for its values on the grid, the velocity on the body surface is set to be $\mathbf{u}^{(b)}$. The vorticity of the body surface is then updated by applying a one-sided difference scheme to the equation of definition for $\breve{\mu} : \breve{\mu} = \nabla \times \mathbf{u}$. Another way is to extend Ø to the inside (`ghost points') of the solid body so that on the body surface the extended Ψ_e satisfies $\nabla \times \Psi_e = \mathbf{u}^{(b)}$, and then central differencing equation (8) to update vorticity on the body surface.

3. Numerical results and discussion

Consider a fluid of constant density ñ and kinematic viscosity í. Figure 1 is schematic of flow about a sphere considered in the present study. Notice that the reference length, velocity and time are taken respectively to be the radius of the sphere *a*, the velocity *U* of the incoming stream, and *a*/*U*. The pressure is non-dimensionalized by $°U^2$, while the Reynolds number is taken to be $\mathrm{Re} = Ua / ¬$.

Numerical calculations are all carried out up to $t = 50$ for $\mathrm{Re} = 100, 150, 200$ and 300. The number of grid points used here is (257×129) in terms of *r*- and *è*-directions. Figure 2(a) shows very close agreement between the computed drag coefficients at $t = 50$ and the drag curve taken from Schlichting (1979), while 2(b) shows also very close agreement of the computed separation angles with the measured data taken from Taneda (1956). Figure 3 shows the early-time development of the drag coefficients. For each case considered, the drag coefficient C_D exhibits a monotone behavior in time history, decreasing rapidly from the infinity to a steady value. Figure 4 (a-d) shows the streamline patterns at $t = 50$ for various Reynolds number; it is seen that the wake size increases gradually with increasing the Reynolds number. Figure 5 shows the streamline pattern at $t = 50$ for $\mathrm{Re} = 118$, which compares very well with the visualized result of Taneda (1956).

4. Concluding remarks

In this study, we presented a hybrid vortex method for flow about a finite body with particular application to flow about a sphere. The general agreement with data from experiment is quite good in the comparisons of drag coefficients and streamline pattern.

The method has been formulated in general coordinates and can be used for body of rather general shape. The most striking features of the present hybrid vortex method consist of (i)introducing a total vorticity for each numerical vortex in lieu of circulation in two space dimensions, (ii)designing a viscous vortex-in-cell algorithm, which relates evolution of the total vorticity of a vortex to two surface integrals, (iii) the choice of a gauge, which facilitates greatly solution for a Poisson equation for the vector stream function, and (iv) accurate reconstruction of the local vorticity field from the total vorticities by interpolating high-order polynomials.

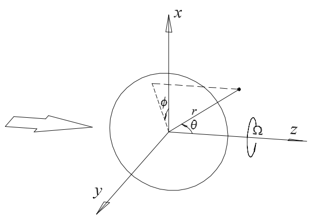


Figure 1

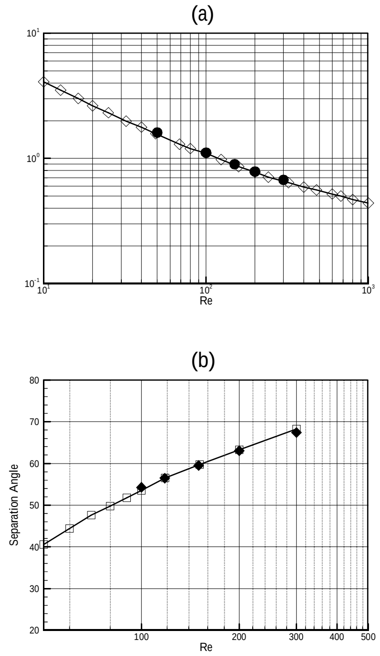


Figure 2

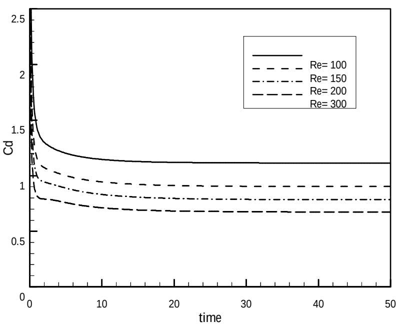


Figure 3

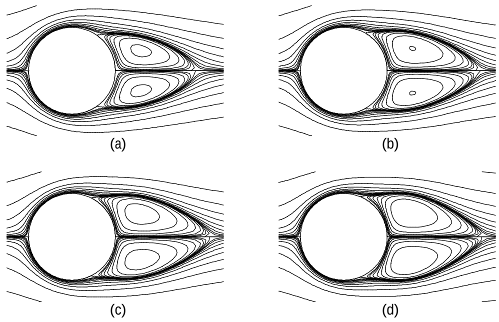
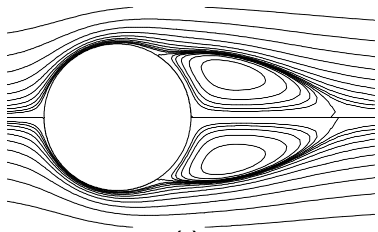


Figure 4



(a)

Figure 5

(b)

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