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Renormalization Group Analysis of Magnetohydrodynamic Turbulence with the Alfvén Effect

Chien C. CHANG* and Bin-Shei LIN

Institute of Applied Mechanics, College of Engineering, National Taiwan University
Taipei 10764, Taiwan, Republic of China

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In this study, we continue with a recursive renormalization group (RG) analysis of incompressible turbulence, aiming at investigating various turbulent properties of three-dimensional magnetohydrodynamics (MHD). In particular, we are able to locate the fixed point (i.e. the invariant effective eddy viscosity) of the RG transformation under the following conditions. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. It is found under these conditions that re-normalization does not incur an increment of the magnetic resistivity, while the coupling effect tends to reduce the invariant effective eddy viscosity. Both the velocity and magnetic energy spectra are shown to follow the Kolmogorov $k^{-5/3}$ in the inertial subrange; this is consistent with some laboratory measurements and observations in astronomical physics. By assuming further that the velocity and magnetic induction share the same specified form of energy spectrum, we are able to determine the dependence of the (magnetic) Kolmogorov constant C_K (C_M) and the model constant C_S of the Smagorinsky model for large-eddy simulation on some characteristic wavenumbers.

KEYWORDS: renormalization group analysis, magnetohydrodynamic turbulence, Alfvén effect, effective eddy viscosity, magnetic resistivity
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1. Introduction

Recently, the authors[1,2] carried out a recursive renormalization group (RG) analysis of incompressible turbulence for flow turbulence and thermal turbulent transport. In this study, we continue with this previous RG analysis for magneto-hydrodynamic (MHD) turbulence, aiming at investigating various transport properties, in particular, the coupling effects between the flow and magnetic induction fields on the kinetic energy spectrum and the effective eddy viscosity.

The plasma science is widely applied to many areas from laser skill, thin film produce, nuclear rocket, even to astronomical physics (for example, solar wind, solar flares and coronal structures). Like in ordinary Newtonian fluids, MHD turbulence is expected to arise in plasma or magnetized fluids as the Reynolds number is increased beyond some critical value. In spite of the already scarce literature, the interest of MHD turbulence may further be divided into two-dimensional and three-dimensional turbulence. Kim and Yang[3] studied the scaling behavior of the randomly stirred MHD plasma in two dimensions and were able to show existence of the scaling solution at the fixed point of the RG transformation and derive the dependence of the power exponent of the energy spectrum on the driving Gaussian noise. Liang and Diamond[4] also presented their study for two-dimensional MHD turbulence by introducing the velocity stream function and the magnetic flux function in MHD equations. However, the latter authors showed no existence of a fixed point of the RG transformation and especially suggested that the applicability of RG method to turbulent system is intrinsically limited, especially in the case of systems with dual-direction energy transfer.

In contrast to flow in two dimensions, the effect of dual-direction energy transfer becomes weak in three dimensions (cf. McComb[5]). It would therefore be legitimate to employ the RG analysis for MHD turbulence in three dimensions. In the literature, there are some measured evidences about the validity of the Kolmogorov spectrum for the three-dimensional MHD turbulence. Alemany *et al.*[6] designed an equipment in the laboratory which produced turbulence by passing magnetized fluid to a mesh under an additional magnetic induction. In the area of astronomical physics, Matthaeus *et al.*[7] measured the magnetic energy spectrum of the solar wind, while Leamon *et al.*[8] measured the MHD turbulence within the coronal mass ejection. Both of their results suggested the Kolmogorov power law for the energy spectrum. Besides, Biskamp[9] mentioned that the Kolmogorov constant depends on the precise definition of the average magnetic induction, and hence on the geometry of the large scale eddies. On the theoretical side, Hatori[10] obtained the Kolmogorov spectrum for the three-dimensional MHD turbulence, but suggested that the Kolmogorov constant is universal. Verma[11] constructed a self-consistent renormalization group procedure for MHD turbulence and also found that the energy spectrum for the velocity obeys the Kolmogorov spectrum. It is the purpose of the present study to provide a recursive renormalization group analysis for MHD turbulence in three dimensions with the specific points of interest as follows. We will obtain the energy spectra for both of the velocity and magnetic induction fields, look for the invariant effective eddy viscosity and determine the dependence of the (magnetic) Kolmogorov constant C_K (C_M) and the model constant C_S for the Smagorinsky model for large-eddy simulation (LES).

Let us give a brief description of the present work. MHD is governed by a coupling set of equations, meanwhile, the MHD turbulence considered is further assumed to be isotropic, homogeneous and stationary. It is found con-

*E-mail: changcc@gauss.iam.ntu.edu.tw

venient to introduce the Elsässer variables to write the equations for the velocity and magnetic induction fields in a symmetric form. In §3, a recursive RG analysis is carried out for the MHD equations in the wavenumber domain and a recursive relationship for the effective eddy viscosity $\nu_n(k)$ between two successive steps is established. The resulting expression is complicated enough and is apparently not amenable to further RG analysis. Instead, we restrict ourselves to the case when the following conditions hold. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. As a matter of fact, the two conditions imply a negligible effect of the subgrid cross helicity between the velocity and magnetic fields. In spite of these restrictions, the present RG analysis still warrants a sufficient interest as we investigate several observations in the area of astronomical physics. In §4, the energy spectra of the velocity and the magnetic fields are determined through use of the RG transformation, i.e. the recursive relation. Both spectra are found to follow the Kolmogorov $k^{-5/3}$ law in the inertial subrange. The results are consistent with the experimental results of Alemany *et al.*,[6] and the observational results of Matthaeus *et al.*[7] From a different approach, Chen and Montgomery[12] obtained the same power law in the inertial subrange by using some multiple-scale self-consistent calculations of turbulent MHD transport coefficients. In §5, the fixed point of the RG equation is located to give the invariant effective eddy viscosity $\nu(k)$ and magnetic resistivity $\tau(k)$. By assuming further a combination form of the energy spectra proposed respectively by Pao[13] and Quarini and Leslie,[14] the invariant effective eddy viscosity is then employed in §6 to determine the dependence of the (magnetic) Kolmogorov constant C_K (C_M) and the Smagorinsky constant C_S on the cutoff wavenumber k_c, the wavenumber k_s of the largest eddies and the wavenumber k_p that peaks in the energy spectrum. Finally, concluding remarks are drawn in §7.

2. Magnetohydrodynamic Equations

In considering MHD turbulence, we shall take the SI units. The magnetized fluid is assumed to be incompressible (constant density ρ) and have a constant permeability μ_0. It is convenient to simply set $\rho = 1$ and $\mu_0 = 1$. In doing do, the magnetic induction $\boldsymbol{B}$ and the velocity $\boldsymbol{v}$ have the same dimension because of the dimensional relationship $[\boldsymbol{B}] = \sqrt{[\mu_0][\rho]}$. Let us start with the treatment of Alfvén[16] and Cowling,[17] and write down the following magnetohydrodynamic equations.

(i) Continuity equation:

$$\nabla \cdot \boldsymbol{v} = 0,$$

where $\boldsymbol{v}$ is the velocity.

(ii) Momentum equation:

$$\frac{d\boldsymbol{v}}{dt} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} + \nu_0 \nabla^2 \boldsymbol{v} + \boldsymbol{g},$$

where $\boldsymbol{J}$ is the electric current density, $\boldsymbol{B}$ the magnetic induction, ν_0 the molecular viscosity and $\boldsymbol{g}$ the gravitation.

(iii) Electromagnetic equations:

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0,$$

and

$$\nabla \times \boldsymbol{B} = \boldsymbol{J}$$

where $\boldsymbol{E}$ is the electric field, and we have neglected the displacement current. The Ohm's law takes the form

$$\boldsymbol{J} = \sigma_0(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{1}$$

where σ_0 is the electric conductivity. Let us take $\nabla\times$ on the both hand sides of eq. (1) and reorganize the electromagnetic equations to obtain

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \tau_0 \nabla^2 \boldsymbol{B},$$

where $\tau_0 = 1/\sigma_0$ is the magnetic resistivity. In summary, we have the following MHD equations for use,

$$\begin{cases} \partial \boldsymbol{v}/\partial t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu_0 \nabla^2 \boldsymbol{v}; \\ \partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \tau_0 \nabla^2 \boldsymbol{B}, \end{cases} \tag{2}$$

with the solenoidal equations

$$\begin{cases} \nabla \cdot \boldsymbol{v} = 0; \\ \nabla \cdot \boldsymbol{B} = 0, \end{cases}$$

where the gravitation is incorporated into p. It is convenient to introduce the Elsässer variables (cf. ref. 18) for eq. (2), defined by

$$\begin{cases} \Phi = \boldsymbol{v} + \boldsymbol{B}; \\ \Psi = \boldsymbol{v} - \boldsymbol{B}. \end{cases}$$

Equation (2) can then be transformed to

$$\begin{cases} \partial \Phi/\partial t + (\Psi \cdot \nabla)\Phi = -\nabla p^* + \alpha_0 \nabla^2 \Phi + \beta_0 \nabla^2 \Psi; \\ \partial \Psi/\partial t + (\Phi \cdot \nabla)\Psi = -\nabla p^* + \alpha_0 \nabla^2 \Psi + \beta_0 \nabla^2 \Phi, \end{cases} \tag{3}$$

where $p^* = p + (\boldsymbol{B} \cdot \boldsymbol{B})/2$, and we have set

$$\begin{cases} \alpha_0 = (\nu_0 + \tau_0)/2; \\ \beta_0 = (\nu_0 - \tau_0)/2. \end{cases}$$

It is obvious from the definitions of Φ, Ψ that they are also solenoidal, i.e.

$$\nabla \cdot \Phi = \nabla \cdot \Psi = 0.$$

Since RG analysis will be performed in the wavenumber domain, the next goal is to Fourier transform eq. (3) into the wavenumber domain. First of all, take $\nabla\cdot$ on the both sides of eq. (3) to obtain

$$\nabla^2 p^* = -\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \Phi_\alpha \Psi_\beta. \tag{4}$$

Let us now introduce the following operators in the wavenumber space

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2},$$

and

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = [k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})]/2i.$$

By using the Elsässer variables with the help of eq. (4), we can transform eq. (3) into the wavenumber domain as follows,

$$\mathcal{L}_<(k,t)\begin{pmatrix}\Phi_\alpha(\boldsymbol{k},t)\\ \Psi_\alpha(\boldsymbol{k},t)\end{pmatrix}=M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j\begin{pmatrix}\Psi_\beta(\boldsymbol{j},t)\Phi_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\\ \Phi_\beta(\boldsymbol{j},t)\Psi_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\end{pmatrix}, \tag{5}$$

where the matrix $\mathcal{L}_<(k,t)$ is defined by

$$\mathcal{L}_<(k,t)=\begin{pmatrix}\partial/\partial t+\alpha_0 k^2 & \beta_0 k^2\\ \beta_0 k^2 & \partial/\partial t+\alpha_0 k^2\end{pmatrix}.$$

For later use, we shall need the following statistical correlations:

$$\begin{cases}\langle u_\alpha(\boldsymbol{k},t)u_\beta(\boldsymbol{k}',t)\rangle = D_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')Q(\boldsymbol{k});\\ \langle u_\alpha(\boldsymbol{k},t)B_\beta(\boldsymbol{k}',t)\rangle = D_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')R(\boldsymbol{k});\\ \langle B_\alpha(\boldsymbol{k},t)B_\beta(\boldsymbol{k}',t)\rangle = D_{\alpha\beta}(\boldsymbol{k})\delta(\boldsymbol{k}+\boldsymbol{k}')S(\boldsymbol{k}),\end{cases} \tag{6}$$

where $Q(\boldsymbol{k})$ is the kinetic energy spectrum, $S(\boldsymbol{k})$ the magnetic energy spectrum and $R(\boldsymbol{k})$ the cross energy spectrum. It is noted that these relationships are valid for isotropic, homogeneous and stationary turbulence; see, for example, McComb[5] for the details.

3. Renormalization Group Analysis for MHD Turbulence

The basic idea of recursive RG analysis is to divide the wavenumber space $(0,k_0)$, where k_0 is Kolmogorov's scale, to a supergrid region $(0,k_c)$ and a subgrid region (k_c,k_0). The subgrid modes are then removed shell by shell by taking the subgrid average over a spherical shell (k_{n+1},k_n), as shown in Fig. 1. At the present stage, the cutoff ratio, defined by $\Lambda = k_{n+1}/k_n$ is maintained a constant, and will be later set to tend to 1 as the differential version of the RG analysis leading to the invariant effective eddy viscosity is sought.

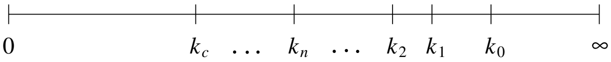

Fig. 1. The termini k_i for recursive renormalization with a fixed cutoff ratio $\Lambda = k_{n+1}/k_n$. Recursive renormalization analysis starts at the Kolmogorov's scale k_0, and ends at the cutoff wavenumber k_c.

In this section, we will follow the renormalization group analysis that we have developed in refs. 1 and 2. First of all, in order to distinguish the supergrid and subgrid modes, we introduce the following notations:

(i) for Φ_α field,

$$\Phi_\alpha(\boldsymbol{k},t)=\begin{cases}\Phi_\alpha^<(\boldsymbol{k},t) & \text{for} \quad |\boldsymbol{k}|<k_1;\\ \Phi_\alpha^>(\boldsymbol{k},t) & \text{for} \quad |\boldsymbol{k}|>k_1,\end{cases}$$

(ii) for Ψ_α field,

$$\Psi_\alpha(\boldsymbol{k},t)=\begin{cases}\Psi_\alpha^<(\boldsymbol{k},t) & \text{for} \quad |\boldsymbol{k}|<k_1;\\ \Psi_\alpha^>(\boldsymbol{k},t) & \text{for} \quad |\boldsymbol{k}|>k_1.\end{cases}$$

The momentum equations for the supergrid modes can be written

$$\mathcal{L}_<(k,t)\begin{pmatrix}\Phi_\alpha^<(\boldsymbol{k},t)\\ \Psi_\alpha^<(\boldsymbol{k},t)\end{pmatrix}=M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j$$
$$\begin{pmatrix}\left[\Psi_\beta^<(\boldsymbol{j},t)\Phi_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t)+2\Psi_\beta^<(\boldsymbol{j},t)\Phi_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t)+\Psi_\beta^>(\boldsymbol{j},t)\Phi_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t)\right]\\ \left[\Phi_\beta^<(\boldsymbol{j},t)\Psi_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t)+2\Phi_\beta^<(\boldsymbol{j},t)\Psi_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t)+\Phi_\beta^>(\boldsymbol{j},t)\Psi_\gamma^>(\boldsymbol{k}-\boldsymbol{j},t)\right]\end{pmatrix}, \tag{7}$$

and the momentum equations for the subgrid modes can be written

$$\mathcal{L}_>(j)\begin{pmatrix}\Phi_\beta^>(\boldsymbol{j},t)\\ \Psi_\beta^>(\boldsymbol{j},t)\end{pmatrix}=M_{\beta\beta'\gamma'}(\boldsymbol{j})\int \mathrm{d}^3 j'$$
$$\begin{pmatrix}\left[\Psi_{\beta'}^<(\boldsymbol{j}',t)\Phi_{\gamma'}^<(\boldsymbol{j}-\boldsymbol{j}',t)+2\Psi_{\beta'}^<(\boldsymbol{j}',t)\Phi_{\gamma'}^>(\boldsymbol{j}-\boldsymbol{j}',t)+\Psi_{\beta'}^>(\boldsymbol{j}',t)\Phi_{\gamma'}^>(\boldsymbol{j}-\boldsymbol{j}',t)\right]\\ \left[\Phi_{\beta'}^<(\boldsymbol{j}',t)\Psi_{\gamma'}^<(\boldsymbol{j}-\boldsymbol{j}',t)+2\Phi_{\beta'}^<(\boldsymbol{j}',t)\Psi_{\gamma'}^>(\boldsymbol{j}-\boldsymbol{j}',t)+\Phi_{\beta'}^>(\boldsymbol{j}',t)\Psi_{\gamma'}^>(\boldsymbol{j}-\boldsymbol{j}',t)\right]\end{pmatrix}, \tag{8}$$

and we have assumed that Markovian approximation holds for the subgrid modes (Rose[19] and McComb[20]). Its physical ground is that the subgrid modes are considered to evolve much faster than the supergrid modes, which implies that the subgrid modes relax to the steady state while the supergrid modes are still evolving. The matrix $\mathcal{L}_>(j)$ is defined by

$$\mathcal{L}_>(j)=\begin{pmatrix}\alpha_0 & \beta_0\\ \beta_0 & \alpha_0\end{pmatrix}j^2.$$

Before substantial progress can be made with the RG analysis, we shall make the following statistical hypotheses.

(i) The MHD fields have ensemble-mean-zero fluctuation,

$$\langle\Phi_\alpha^>(\boldsymbol{k},t)\rangle = \langle\Psi_\alpha^>(\boldsymbol{k},t)\rangle = 0.$$

(ii) Supergrid components are considered to be statistically independent of subgrid averaging (cf. McComb[5] and Zhou[15]),

$$\begin{cases}\langle\Phi_\alpha^<(\boldsymbol{k},t)\rangle = \Phi_\alpha^<(\boldsymbol{k},t);\\ \langle\Psi_\alpha^<(\boldsymbol{k},t)\rangle = \Psi_\alpha^<(\boldsymbol{k},t).\end{cases}$$

This assumption is simple (but not void) though its validity may be restrictive. But RG theory based on this assumption has not been explored to its full strength. Indeed, as shown in,[1,2] the RG results based on this assumption were found to be in remarkably close agreement with computed/measured data.

Performing subgrid averaging of eq. (7) with use of (i) and (ii), we may obtain the averaged equation for the supergrid modes,

$$\mathcal{L}_<(k,t)\begin{pmatrix}\Phi^<_\alpha(\boldsymbol{k},t)\\ \Psi^<_\alpha(\boldsymbol{k},t)\end{pmatrix} = M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j \begin{pmatrix}\Psi^<_\beta(\boldsymbol{j},t)\Phi^<_\gamma(\boldsymbol{k}-\boldsymbol{j},t) + \langle\Psi^>_\beta(\boldsymbol{j},t)\Phi^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle\\ \Phi^<_\beta(\boldsymbol{j},t)\Psi^<_\gamma(\boldsymbol{k}-\boldsymbol{j},t) + \langle\Phi^>_\beta(\boldsymbol{j},t)\Psi^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle\end{pmatrix}, \tag{9}$$

This subgrid averaged equation for the supergrid modes will be contrasted to eq. (5). The comparison between eqs. (5) and (9) suggests that the ensemble averaging terms on the right hand side of eq. (9) contribute respectively to the effective eddy viscosity and effective magnetic resistivity.

Now we return to the original variables $\boldsymbol{u}$ and $\boldsymbol{B}$ by rotating the matrices $\mathcal{L}_<$ and $\mathcal{L}_>$ 45 degrees counterclockwise. If we choose a set of new base vectors which are the eigenvectors of matrices $\mathcal{L}_<$ and $\mathcal{L}_>$, eq. (9) takes the following expression

$$\mathcal{L}_0(k,t)\begin{pmatrix}u^<_\alpha(\boldsymbol{k},t)\\ B^<_\alpha(\boldsymbol{k},t)\end{pmatrix} = M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j$$
$$\begin{pmatrix}u^<_\beta(\boldsymbol{j},t)u^<_\gamma(\boldsymbol{k}-\boldsymbol{j},t) - B^<_\beta(\boldsymbol{j},t)B^<_\gamma(\boldsymbol{k}-\boldsymbol{j},t) + \langle u^>_\beta(\boldsymbol{j},t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t) - B^>_\beta(\boldsymbol{j},t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle\\ B^<_\beta(\boldsymbol{j},t)u^<_\gamma(\boldsymbol{k}-\boldsymbol{j},t) - u^<_\beta(\boldsymbol{j},t)B^<_\gamma(\boldsymbol{k}-\boldsymbol{j},t) + \langle B^>_\beta(\boldsymbol{j},t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t) - u^>_\beta(\boldsymbol{j},t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle\end{pmatrix} \tag{10}$$

where

$$\mathcal{L}_0(k,t) = \begin{pmatrix}\partial/\partial t + \nu_0 k^2 & 0\\ 0 & \partial/\partial t + \tau_0 k^2\end{pmatrix}.$$

Similarly, based on the new basis, eq. (8) for the subgrid modes can be transformed into the form,

$$\begin{aligned}\mathcal{G}(j)\begin{pmatrix}u^>_\beta(\boldsymbol{j},t)\\ B^>_\beta(\boldsymbol{j},t)\end{pmatrix} &= M_{\beta\beta'\gamma'}(\boldsymbol{j})\int \mathrm{d}^3 j'\\ &\quad\times\left[\begin{pmatrix}u^<_{\beta'}(\boldsymbol{j}',t)u^<_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t) - B^<_{\beta'}(\boldsymbol{j}',t)B^<_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)\\ B^<_{\beta'}(\boldsymbol{j}',t)u^<_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t) - u^<_{\beta'}(\boldsymbol{j}',t)B^<_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)\end{pmatrix}\right.\\ &\quad+2\begin{pmatrix}u^<_{\beta'}(\boldsymbol{j}',t)u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t) - B^<_{\beta'}(\boldsymbol{j}',t)B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)\\ B^<_{\beta'}(\boldsymbol{j}',t)u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t) - u^<_{\beta'}(\boldsymbol{j}',t)B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)\end{pmatrix}\\ &\quad\left.+\begin{pmatrix}u^>_{\beta'}(\boldsymbol{j}',t)u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t) - B^>_{\beta'}(\boldsymbol{j}',t)B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)\\ B^>_{\beta'}(\boldsymbol{j}',t)u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t) - u^>_{\beta'}(\boldsymbol{j}',t)B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)\end{pmatrix}\right],\end{aligned} \tag{11}$$

where

$$\mathcal{G}(j) = \begin{pmatrix}\nu_0 j^2 & 0\\ 0 & \tau_0 j^2\end{pmatrix}.$$

The next step is to obtain the four correlations in (10) by making use of eq. (11). First of all, multiplying the factor $u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)$ on both hand sides of eq. (11), and then taking subgrid averaging yields

$$\mathcal{G}(j)\left\langle\begin{matrix}u^>_\beta(\boldsymbol{j},t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\\ B^>_\beta(\boldsymbol{j},t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\end{matrix}\right\rangle = 2M_{\beta\beta'\gamma'}(\boldsymbol{j})\int \mathrm{d}^3 j'$$
$$\begin{pmatrix}\langle u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle u^<_{\beta'}(\boldsymbol{j}',t) - \langle B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle B^<_{\beta'}(\boldsymbol{j}',t)\\ \langle u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle B^<_{\beta'}(\boldsymbol{j}',t) - \langle B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)u^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle u^<_{\beta'}(\boldsymbol{j}',t)\end{pmatrix}. \tag{12}$$

On the other hand, multiplying the factor $B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)$ on both hand sides of eq. (11), and then taking subgrid averaging yields

$$\mathcal{G}(j)\left\langle\begin{matrix}u^>_\beta(\boldsymbol{j},t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\\ B^>_\beta(\boldsymbol{j},t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\end{matrix}\right\rangle = 2M_{\beta\beta'\gamma'}(\boldsymbol{j})\int \mathrm{d}^3 j'$$
$$\begin{pmatrix}\langle u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle u^<_{\beta'}(\boldsymbol{j}',t) - \langle B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle B^<_{\beta'}(\boldsymbol{j}',t)\\ \langle u^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle B^<_{\beta'}(\boldsymbol{j}',t) - \langle B^>_{\gamma'}(\boldsymbol{j}-\boldsymbol{j}',t)B^>_\gamma(\boldsymbol{k}-\boldsymbol{j},t)\rangle u^<_{\beta'}(\boldsymbol{j}',t)\end{pmatrix}. \tag{13}$$

Next, we rewrite eq. (11) by changing the index β to γ, and changing the wavenumber $\boldsymbol{j}$ to $\boldsymbol{k}-\boldsymbol{j}$, to obtain

$$\mathcal{G}(|\boldsymbol{k}-\boldsymbol{j}|)\begin{pmatrix} u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \\ B^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \end{pmatrix} = M_{\gamma\beta'\gamma'(\boldsymbol{k}-\boldsymbol{j})}\int \mathrm{d}^3j' \\ \times\left[\begin{pmatrix} u^{<}_{\beta'}(\boldsymbol{j}',t)u^{<}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) - B^{<}_{\beta'}(\boldsymbol{j}',t)B^{<}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) \\ B^{<}_{\beta'}(\boldsymbol{j}',t)u^{<}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) - u^{<}_{\beta'}(\boldsymbol{j}',t)B^{<}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) \end{pmatrix}\right. \\ + 2\begin{pmatrix} u^{<}_{\beta'}(\boldsymbol{j}',t)u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) - B^{<}_{\beta'}(\boldsymbol{j}',t)B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) \\ B^{<}_{\beta'}(\boldsymbol{j}',t)u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) - u^{<}_{\beta'}(\boldsymbol{j}',t)B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) \end{pmatrix} \\ \left. + \begin{pmatrix} u^{>}_{\beta'}(\boldsymbol{j}',t)u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) - B^{>}_{\beta'}(\boldsymbol{j}',t)B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) \\ B^{>}_{\beta'}(\boldsymbol{j}',t)u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) - u^{>}_{\beta'}(\boldsymbol{j}',t)B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t) \end{pmatrix}\right]. \tag{14}$$

Multiplying the factor $u^{>}_{\beta}(\boldsymbol{j},t)$ on both sides of eq. (14), and then taking subgrid averaging yields

$$\mathcal{G}(|\boldsymbol{k}-\boldsymbol{j}|)\left\langle\begin{matrix} u^{>}_{\beta}(\boldsymbol{j},t)u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \\ u^{>}_{\beta}(\boldsymbol{j},t)B^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \end{matrix}\right\rangle = 2M_{\gamma\beta'\gamma'}(\boldsymbol{k}-\boldsymbol{j})\int \mathrm{d}^3j' \\ \begin{pmatrix} \langle u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)u^{>}_{\beta}(\boldsymbol{j},t)\rangle u^{<}_{\beta'}(\boldsymbol{j}',t) - \langle B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)u^{>}_{\beta}(\boldsymbol{j},t)\rangle B^{<}_{\beta'}(\boldsymbol{j}',t) \\ \langle u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)u^{>}_{\beta}(\boldsymbol{j},t)\rangle B^{<}_{\beta'}(\boldsymbol{j}',t) - \langle B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)u^{>}_{\beta}(\boldsymbol{j},t)\rangle u^{<}_{\beta'}(\boldsymbol{j}',t) \end{pmatrix}. \tag{15}$$

Multiplying the factor $B^{>}_{\beta}(\boldsymbol{j},t)$ on both sides of eq. (14), and then taking subgrid averaging yields

$$\mathcal{G}(|\boldsymbol{k}-\boldsymbol{j}|)\left\langle\begin{matrix} B^{>}_{\beta}(\boldsymbol{j},t)u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \\ B^{>}_{\beta}(\boldsymbol{j},t)B^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t) \end{matrix}\right\rangle = 2M_{\gamma\beta'\gamma'}(\boldsymbol{k}-\boldsymbol{j})\int \mathrm{d}^3j' \\ \begin{pmatrix} \langle u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)B^{>}_{\beta}(\boldsymbol{j},t)\rangle u^{<}_{\beta'}(\boldsymbol{j}',t) - \langle B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)B^{>}_{\beta}(\boldsymbol{j},t)\rangle B^{<}_{\beta'}(\boldsymbol{j}',t) \\ \langle u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)B^{>}_{\beta}(\boldsymbol{j},t)\rangle B^{<}_{\beta'}(\boldsymbol{j}',t) - \langle B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)B^{>}_{\beta}(\boldsymbol{j},t)\rangle u^{<}_{\beta'}(\boldsymbol{j}',t) \end{pmatrix}. \tag{16}$$

Let us focus on the first equations of (12) and (15). Recall the correlations defined in (6). If we take the proper rearrangement of the indices and make change of variables, it is quite straightforward to prove that the right hand sides of first equations of (12) and (15) are identical under the operation of $M_{\alpha\beta\gamma}\int \mathrm{d}^3j$. Applying the said operation and adding these two together yields

$$M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3j\langle u^{>}_{\beta}(\boldsymbol{j},t)u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle = 4M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3j\,\frac{M_{\gamma\beta'\gamma'}(\boldsymbol{k}-\boldsymbol{j})}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}\int \mathrm{d}^3j' \\ \big(\langle u^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)u^{>}_{\beta}(\boldsymbol{j},t)\rangle u^{<}_{\beta'}(\boldsymbol{j}',t) - \langle B^{>}_{\gamma'}(\boldsymbol{k}-\boldsymbol{j}-\boldsymbol{j}',t)u^{>}_{\beta}(\boldsymbol{j},t)\rangle B^{<}_{\beta'}(\boldsymbol{j}',t)\big). \tag{17}$$

(i) For $\langle u^{>}_{\beta}(\boldsymbol{j},t)u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle$, it follows from eq. (17),

$$M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3j\langle u^{>}_{\beta}(\boldsymbol{j},t)u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle = 4M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3j \\ \times\frac{M_{\gamma\beta'\gamma'}(\boldsymbol{k}-\boldsymbol{j})D_{\gamma'\beta}(\boldsymbol{j})D_{\alpha\beta'}(\boldsymbol{k})}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}\big[Q(\boldsymbol{j})u^{<}_{\alpha}(\boldsymbol{k},t) - R(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)\big] \\ = -2\int \mathrm{d}^3j\,\frac{L(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j})}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}\big[Q(\boldsymbol{j})u^{<}_{\alpha}(\boldsymbol{k},t) - R(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)\big], \tag{18}$$

where we have used the relationships:

$$M_{\alpha\beta\gamma}(\boldsymbol{k})D_{\alpha\beta'}(\boldsymbol{k}) = M_{\beta'\beta\gamma}(\boldsymbol{k}),$$

and

$$L(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j}) = -2M_{\beta'\beta\gamma}(\boldsymbol{k})M_{\gamma\beta'\gamma'}(\boldsymbol{k}-\boldsymbol{j})D_{\gamma'\beta}(\boldsymbol{j}) \\ = \frac{(k^4-2k^3j\mu+kj^3\mu)(1-\mu^2)}{|\boldsymbol{k}-\boldsymbol{j}|^2}. \tag{19}$$

A similar procedure can be applied to obtain other correlations.

(ii) For $\langle B^{>}_{\beta}(\boldsymbol{j},t)B^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle$, we have from the second equations of (13) and (16)

$$M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3j\langle B^{>}_{\beta}(\boldsymbol{j},t)B^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle \\ = -2\int \mathrm{d}^3j\,\frac{L(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j})}{\tau_0 j^2+\tau_0|\boldsymbol{k}-\boldsymbol{j}|^2}\big[R(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t) - S(\boldsymbol{j})V^{<}_{\alpha}(\boldsymbol{k},t)\big], \tag{20}$$

(iii) For $\langle B^{>}_{\beta}(\boldsymbol{j},t)u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle$, we obtain from the second equation of (12) and the first equation of (16)

$$M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j\langle B^{>}_{\beta}(\boldsymbol{j},t)u^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle$$
$$= -2\int \mathrm{d}^3 j\,\frac{L(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j})}{\tau_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}\big[Q(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)-S(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)\big], \tag{21}$$

(iv) For $\langle u^{>}_{\beta}(\boldsymbol{j},t)B^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle$, we obtain from the first equation of (13) and the second equation of (15)

$$M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j\langle u^{>}_{\beta}(\boldsymbol{j},t)B^{>}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\rangle$$
$$= -2\int \mathrm{d}^3 j\,\frac{L(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j})}{\nu_0 j^2+\tau_0|\boldsymbol{k}-\boldsymbol{j}|^2}\big[Q(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)-S(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)\big]. \tag{22}$$

Collecting the above results (i)–(iv) by substituting eqs. (18), (20), (21) and (22) in eq. (10) yields

$$\mathcal{L}_0(k,t)\begin{pmatrix} u^{<}_{\alpha}(\boldsymbol{k},t)\\ B^{<}_{\alpha}(\boldsymbol{k},t)\end{pmatrix}$$
$$= M_{\alpha\beta\gamma}(\boldsymbol{k})\int \mathrm{d}^3 j\begin{pmatrix} u^{<}_{\beta}(\boldsymbol{j},t)u^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)-B^{<}_{\beta}(\boldsymbol{j},t)B^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\\ B^{<}_{\beta}(\boldsymbol{j},t)u^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)-u^{<}_{\beta}(\boldsymbol{j},t)B^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\end{pmatrix}$$
$$-2\int \mathrm{d}^3 jL(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j})\begin{pmatrix} \dfrac{Q(\boldsymbol{j})u^{<}_{\alpha}(\boldsymbol{k},t)-R(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}-\dfrac{R(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)-S(\boldsymbol{j})u^{<}_{\alpha}(\boldsymbol{k},t)}{\tau_0 j^2+\tau_0|\boldsymbol{k}-\boldsymbol{j}|^2}\\ \dfrac{Q(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)-S(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)}{\tau_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}-\dfrac{Q(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)-S(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)}{\nu_0 j^2+\tau_0|\boldsymbol{k}-\boldsymbol{j}|^2}\end{pmatrix}. \tag{23}$$

Apparently, eq. (23) is not amenable to renormalization because of the difficulty in singling out increments of the effective eddy viscosity and of the magnetic resistivity. To alleviate this problem, we consider the two conditions: (i) the mean magnetic induction is relatively weak compared to the mean flow velocity, and (ii) the Alfvén effect holds, that is the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. Those two conditions directly imply that $Q(\boldsymbol{j})u^{<}_{\alpha}(\boldsymbol{k},t)\gg R(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)$ and $S(\boldsymbol{j})u^{<}_{\alpha}(\boldsymbol{k},t)\gg R(\boldsymbol{j})B^{<}_{\alpha}(\boldsymbol{k},t)$, and then eq. (23) can be simplified as follows:

$$\mathcal{L}_1(k,t)\begin{pmatrix} u^{<}_{\alpha}(\boldsymbol{k},t)\\ B^{<}_{\alpha}(\boldsymbol{k},t)\end{pmatrix}$$
$$= M_{\alpha\beta\gamma}(\boldsymbol{k})\int_{\Omega_1} \mathrm{d}^3 j\begin{pmatrix} u^{<}_{\beta}(\boldsymbol{j},t)u^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)-B^{<}_{\beta}(\boldsymbol{j},t)B^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\\ B^{<}_{\beta}(\boldsymbol{j},t)u^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)-u^{<}_{\beta}(\boldsymbol{j},t)B^{<}_{\gamma}(\boldsymbol{k}-\boldsymbol{j},t)\end{pmatrix}, \tag{24}$$

where

$$\mathcal{L}_1(k,t)=\begin{pmatrix} \partial/\partial t+\nu_1(k)k^2 & 0\\ 0 & \partial/\partial t+\tau_1(k)k^2\end{pmatrix}.$$

The effective eddy viscosity and magnetic resistivity after the first-step renormalization are given by $\nu_1(k)$ and $\tau_1(k)$ as follows

$$\nu_1(k)=\nu_0+\delta\nu_0(k),$$

and

$$\tau_1(k)=\tau_0+\delta\tau_0(k),$$

where

$$\delta\nu_0(k)=2\int_{\Omega_0}\mathrm{d}^3 j\,\frac{L(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j})}{k^2}\left(\frac{Q(\boldsymbol{j})}{\nu_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}+\frac{S(\boldsymbol{j})}{\tau_0 j^2+\tau_0|\boldsymbol{k}-\boldsymbol{j}|^2}\right), \tag{25}$$

and

$$\delta\tau_0(k)=2\int_{\Omega_0}\mathrm{d}^3 j\,\frac{L(\boldsymbol{k},\boldsymbol{k}-\boldsymbol{j})}{k^2}\left[\frac{Q(\boldsymbol{j})-S(\boldsymbol{j})}{\tau_0 j^2+\nu_0|\boldsymbol{k}-\boldsymbol{j}|^2}-\frac{Q(\boldsymbol{j})-S(\boldsymbol{j})}{\nu_0 j^2+\tau_0|\boldsymbol{k}-\boldsymbol{j}|^2}\right]. \tag{26}$$

The integrals in (25) or (26) are performed over the intersection of two spherical shells:

$$\Omega_0(\boldsymbol{k})=\{\boldsymbol{j}|k_1<|\boldsymbol{j}|,|\boldsymbol{k}-\boldsymbol{j}|<k_0\}.$$

Let us now discuss the validity of the conditions (i) and (ii). In light of their effects, these conditions amount to neglecting the effects of the subgrid cross helicity between the velocity and magnetic fields. Nevertheless, there are important cases of application of the present formulation.

In astronomical physics, the typical velocity is often larger than the magnetic induction, for example, in solar wind and in solar flares. For comparison in correct dimension, we have to recover the velocity $\boldsymbol{v}$ to $\boldsymbol{v}\sqrt{\mu_0\rho}$. Inside the solar interior, the typical temperature at the core is about 1.5×10^7 K, and the density about 1.5×10^5 kg/m^3. In the outer edge of the core, about 1.75×10^5 km from the center, the density drops to 2×10^4 kg/m^3. In the Radiative Zone, the density drops from 2×10^4 kg/m^3 to 2×10^2 kg/m^3. At solar surface the temperature has dropped to 5.7×10^3 K and the density is only 2×10^{-4} kg/m^3, and the magnetic induction is about 7.75×10^{-4} tesla. Near the solar surface, the solar wind bulk speed is typically from 2×10^5 m/s to 2×10^6 m/s. The hydrogen and helium are diamagnetic materials and their magnetic susceptibilities are all quite small (about 10^{-9}), thus $\mu_0 \simeq \mu_0$ (vacuum) for hydrogen and helium. From these data, the order of $\boldsymbol{v}\sqrt{\mu_0\rho}$ is therefore in the range of 1–10. At solar surface, the velocity scaled as $\boldsymbol{v}\sqrt{\mu_0\rho}$ is larger than the magnetic induction $\boldsymbol{B}$ in magnitude by 4 to 5 orders. As an another example, the highest recorded speed of solar flares is 1,500 km/s, but 100–300 km/s is more typical, with sizable variation. The velocity scaled as $\boldsymbol{v}\sqrt{\mu_0\rho}$ is still several orders larger than $\boldsymbol{B}$ in magnitude. In deriving eq. (24), we consider also the Alfvén effect[16] in the order analysis of the above discussion that assumes that the mean of magnetic induction $\boldsymbol{B}$ is not small, and the fluid takes large Reynolds number, then the Lorentz force will become important and make effect on the small scale velocity flucation $\boldsymbol{v}^>$, such that the small scale motions transform into Alfvén waves and this results in that $\boldsymbol{v}^>(\boldsymbol{x},t) \times \boldsymbol{B}^>(\boldsymbol{x},t) \simeq 0$, and thus $\boldsymbol{v}^>(\boldsymbol{x},t) \simeq \boldsymbol{B}^>(\boldsymbol{x},t)$.

4. Determination of the Energy Spectrum

For MHD turbulence, we shall consider two kinds of energy contribution with wavenumber vectors lying within the spherical shell between k and $k + \mathrm{d}k$:

$$\begin{cases} E^v(k)\mathrm{d}k = 4\pi k^2 Q(\boldsymbol{k})\mathrm{d}k \\ E^{\mathrm{M}}(k)\mathrm{d}k = 4\pi k^2 S(\boldsymbol{k})\mathrm{d}k \end{cases}. \tag{27}$$

Substituting (27) into eqs. (25) and (26) respectively yields

$$\delta\nu_0(k) = \int_{\Omega_0} \mathrm{d}^3 j\, \frac{L(\boldsymbol{k}, \boldsymbol{k}-\boldsymbol{j})}{2\pi j^2 k^2} \left(\frac{E_0^v(j)}{\nu_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2} + \frac{E_0^{\mathrm{M}}(j)}{\tau_0 j^2 + \tau_0|\boldsymbol{k}-\boldsymbol{j}|^2} \right), \tag{28}$$

and

$$\delta\tau_0(k) = \int_{\Omega_0} \mathrm{d}^3 j\, \frac{L(\boldsymbol{k}, \boldsymbol{k}-\boldsymbol{j})}{2\pi j^2 k^2} \left[\frac{E_0^v(j) - E_0^{\mathrm{M}}(j)}{\tau_0 j^2 + \nu_0|\boldsymbol{k}-\boldsymbol{j}|^2} - \frac{E_0^v(j) - E_0^{\mathrm{M}}(j)}{\nu_0 j^2 + \tau_0|\boldsymbol{k}-\boldsymbol{j}|^2} \right]. \tag{29}$$

Equations (28) and (29) are respectively the increments of the effective eddy viscosity and magnetic resistivity after the first step of renormalization. Repeating the RG procedure for $n + 1$ times, we obtain the subgrid-averaged equations for the supergrid modes,

$$\mathcal{L}_{n+1}(k,t)\begin{pmatrix} u_\alpha^<(\boldsymbol{k},t) \\ B_\alpha^<(\boldsymbol{k},t) \end{pmatrix} = M_{\alpha\beta\gamma}(\boldsymbol{k}) \int_{\Omega_{n+1}} \mathrm{d}^3 j \begin{pmatrix} u_\beta^<(\boldsymbol{j},t)u_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t) - B_\beta^<(\boldsymbol{j},t)B_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t) \\ B_\beta^<(\boldsymbol{j},t)u_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t) - u_\beta^<(\boldsymbol{j},t)B_\gamma^<(\boldsymbol{k}-\boldsymbol{j},t) \end{pmatrix},$$

and the recursive relationship is established as follows:

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k),$$

and

$$\tau_{n+1}(k) = \tau_n(k) + \delta\tau_n(k),$$

where

$$\delta\nu_n(k) = \int_{\Omega_n} \mathrm{d}^3 j\, \frac{L(\boldsymbol{k}, \boldsymbol{k}-\boldsymbol{j})}{2\pi j^2 k^2} \left(\frac{E_n^v(j)}{\nu_n(j) j^2 + \nu_n(k-j)|\boldsymbol{k}-\boldsymbol{j}|^2} + \frac{E_n^{\mathrm{M}}(j)}{\tau_n(j) j^2 + \tau_n(k-j)|\boldsymbol{k}-\boldsymbol{j}|^2} \right), \tag{30}$$

and

$$\delta\tau_n(k) = \int_{\Omega_n} \mathrm{d}^3 j\, \frac{L(\boldsymbol{k}, \boldsymbol{k}-\boldsymbol{j})}{2\pi j^2 k^2} \left[\frac{E_n^v(j) - E_n^{\mathrm{M}}(j)}{\tau_n(j) j^2 + \nu_n(k-j)|\boldsymbol{k}-\boldsymbol{j}|^2} - \frac{E_n^v(j) - E_n^{\mathrm{M}}(j)}{\nu_n(j) j^2 + \tau_n(k-j)|\boldsymbol{k}-\boldsymbol{j}|^2} \right]. \tag{31}$$

with $\Omega_n(\boldsymbol{k}) = \{\boldsymbol{j} | k_{n+1} < |\boldsymbol{j}|, |\boldsymbol{k}-\boldsymbol{j}| < k_n\}$. The typical behavior of the increment of the effective eddy viscosity $\delta\nu_n(k)$ is shown in Fig. 2, while the increment of the magnetic resistivity $\delta\tau_n(k)$ is small due to large cancellation of the two integrands. There are two physical quantities in eqs. (30) and (31); they are the wavenumber k, the kinetic energy dissipation rate ε_v and the magnetic energy dissipation rate ε_{M}. It is therefore natural to propose the following scaling laws respectively for E_n^v and E_n^{M}:

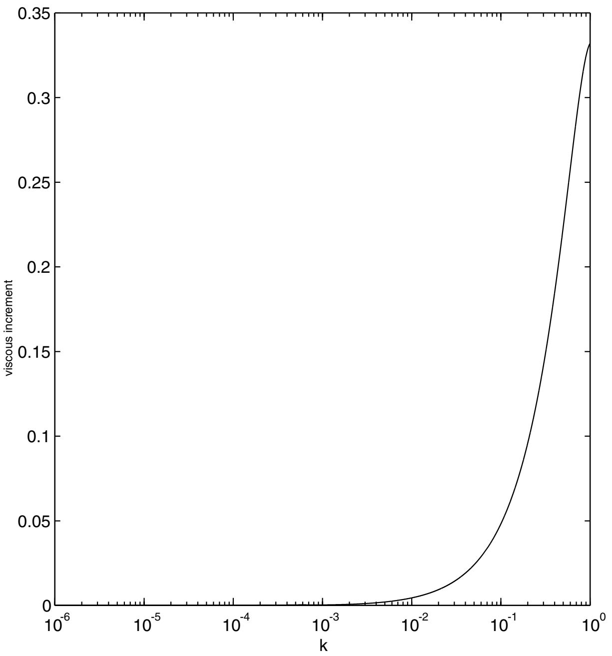

Fig. 2. The typical behavior of the increment of the effective eddy viscosity $\delta\nu$ versus the normalized wavenumber k $(0 \le k \le 1)$, as the cutoff ratio $\Lambda = k_n/k_{n+1}$ is close to 1.

$$\begin{cases} E_n^v(j) = C_K \varepsilon_v^a j^b \phi_n(j/k_p); \\ E_n^M(j) = C_M \varepsilon_M^c j^d \varphi_n(j/k_p), \end{cases} \tag{32}$$

where k_p denotes the wavenumber that peaks in the energy spectrum, C_K is the Kolmogorov constant and C_M may be termed the magnetic Kolmogorov constant. Both scaling functions of $\phi_n(j/k_p)$, $\varphi_n(j/k_p)$ will be specified more precisely later. In analogy to (32) we may assume that there are dimensionless effective eddy viscosity $\hat{\nu}_n(j/k_p)$ and effective eddy resistivity $\hat{\tau}_n(j/k_p)$, such that

$$\begin{cases} \nu_n(j) = C_K^e \varepsilon_v^f j^g \hat{\nu}_n(j/k_p); \\ \tau_n(j) = C_M^h \varepsilon_M^i j^l \hat{\tau}_n(j/k_p). \end{cases} \tag{33}$$

Next, set $\xi = j/k$ and substitute the first equation of (33) into eq. (30); this yields

$$\delta\nu_n(k) = \int_{\bar{\Omega}_n} \mathrm{d}^3\xi \, \frac{(1 - 2\xi\mu + \xi^3\mu)(1-\mu^2)}{2\pi(1+\xi^2-2\xi\mu)} \left(\frac{C_K^{1-e} \varepsilon_v^{a-f} k^{b-g-1} \xi^{b-2} \phi(j/k_p)}{\hat{\nu}_n(\xi)\xi^2 + \hat{\nu}_n(\zeta)(1+\xi^2-2\xi\mu)} + \frac{C_M^{1-h} \varepsilon_M^{c-i} k^{d-l-1} \xi^{d-2} \varphi(j/k_p)}{\hat{\tau}_n(\xi)\xi^2 + \hat{\tau}_n(\zeta)(1+\xi^2-2\xi\mu)} \right), \tag{34}$$

where we have used the shorthand $\zeta = \sqrt{1+\xi^2-2\xi\mu}$, and μ denotes the direction cosine between $\boldsymbol{k}$ and $\boldsymbol{j}$. In order that both sides of eq. (34) are consistent in dimension, we must have the following relationships:

$$\begin{cases} b - g = d - l; \\ e = h = 1/2; \\ f = a/2; \\ g = (b-1)/2. \end{cases} \tag{35}$$

The same argument is applied to eq. (31), that is,

$$\delta\tau_n(k) = \int_{\bar{\Omega}_n} \mathrm{d}^3\xi \, \frac{(1 - 2\xi\mu + \xi^3\mu)(1-\mu^2)}{2\pi(1+\xi^2-2\xi\mu)} \left(\frac{C_K \epsilon_v^a k^b \xi^b \phi(j/k_p) - C_M \varepsilon_M^c k^d \xi^d \varphi(j/k_p)}{C_M^h \varepsilon_M^i k^{l+1} \xi^l \hat{\tau}_n(\xi)\xi^2 + C_K^e \epsilon_v^f k^{g+1} \xi^g \hat{\nu}_n(\zeta)(1+\xi^2-2\xi\mu)} - \frac{C_K \epsilon_v^a k^b \xi^b \phi(j/k_p) - C_M \varepsilon_M^c k^d \xi^d \varphi(j/k_p)}{C_K^e \varepsilon_v^f k^{g+1} \xi^g \hat{\nu}_n(\xi)\xi^2 + C_M^h \epsilon_M^i k^{l+1} \xi^l \hat{\tau}_n(\zeta)(1+\xi^2-2\xi\mu)} \right). \tag{36}$$

The consistency in dimension on both sides of eq. (36) gives another three independent relationships:

$$\begin{cases} a = c; \\ b = d; \\ i = c/2. \end{cases} \tag{37}$$

So far, eqs. (35) and (37) contain eight independent relations, and we still need two more constraints to determine the overall ten exponents. For this, the eddy dissipation equations for both of the velocity and magnetic induction will be employed; they are

$$\begin{cases} \varepsilon_v = \int_0^{k_{n+1}} 2\nu k^2 E^v(k) \mathrm{d}k; \\ \varepsilon_M = \int_0^{k_{n+1}} 2\tau k^2 E^M(k) \mathrm{d}k. \end{cases} \tag{38}$$

From (32) and (33), we have

$$\begin{cases} \varepsilon_v = 2k_n^{(3b+5)/2} C_K^{3/2} \varepsilon_v^{3a/2} \int_0^{\Lambda} \hat{\nu}_n(\tilde{k}) \tilde{k}^{(3b+3)/2} \mathrm{d}\tilde{k}; \\ \varepsilon_M = 2k_n^{(3d+5)/2} C_M^{3/2} \varepsilon_M^{3c/2} \int_0^{\Lambda} \hat{\tau}_n(\tilde{k}) \tilde{k}^{(3d+3)/2} \mathrm{d}\tilde{k}, \end{cases} \tag{39}$$

where we have set $\tilde{k} = k/k_{n+1}$ and $\Lambda = k_{n+1}/k_n$. Dimensional consistency on both sides of eq. (39) gives

$$a = 2/3, \quad b = -5/3.$$

Finally, we obtain by substituting these values in eq. (37), then in (35)

$$\begin{cases} c = 2/3; \\ d = -5/3; \\ g = l = -4/3; \\ f = i = 1/3. \end{cases}$$

With all the exponents determined, the kinetic energy spectrum and the magnetic energy spectrum take respectively the following expressions:

$$\begin{cases} E_n^v(k) = C_K \varepsilon_v^{2/3} k^{-5/3} \phi_n(k/k_p); \\ E_n^M(k) = C_M \varepsilon_M^{2/3} k^{-5/3} \varphi_n(k/k_p), \end{cases} \tag{40}$$

and the effective eddy viscosity and the magnetic resistivity take respectively the expressions:

$$\begin{cases} \nu_n(k) = C_K^{1/2} \varepsilon_v^{1/3} k^{-4/3} \hat{\nu}_n(k/k_p); \\ \tau_n(k) = C_M^{1/2} \varepsilon_M^{1/3} k^{-4/3} \hat{\tau}_n(k/k_p), \end{cases} \tag{41}$$

Equation (40) shows that the energy spectrum have the dependence of the power law of $k^{-5/3}$ which is exactly the Kolmogorov energy spectrum. Compared with laboratory experiments, the result is consistent with Alemany's[6] measurement in passing a magnetized fluid to a grid mesh. Part of their experimental results are shown in Fig. 3, for which, they provided a energy spectrum of the type:

$$E^v(k,t) \approx \frac{\varepsilon_v^{2/3} k^{-5/3}}{[1+N(t)]^{2/3}}.$$

Except the time dependence, our RG result is in good agreement with their experimental results. There are some other evidences from observations in astronomical physics that also support this Kolmogorov spectrum law. Matthaeus *et al.*[7] discovered that the magnetic energy spectrum measured in the solar wind is often found to be close to $k^{-5/3}$, as shown in Fig. 4. Velli *et al.*[21] investigated a new phenomenology which involves the solar wind fluctuations near the sun and leads to a kinetic power spectrum scaling as $k^{-\alpha}$ where $\alpha \simeq 1$ for the largest scales, and $\alpha \simeq 1.5$–1.7 for the small scales. Moreover, the recent observations by Leamon *et al.*[8] (the January 1997 event which involves the solar coronal mass ejections), also showed a power law, scaled as $k^{-1.67}$.

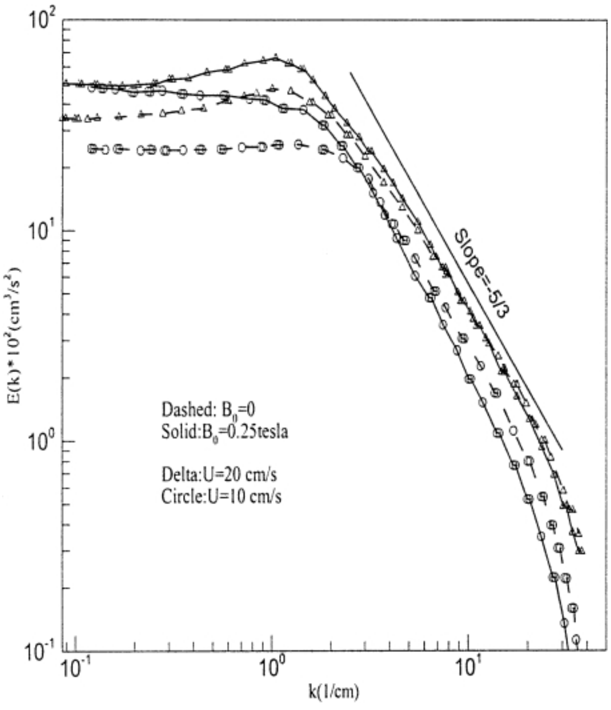

Fig. 3. The experimental results of energy spectrum of velocity from Alemany *et al.*,[6] for various values of the magnetic induction ($B = 0$ T; $B = 0.25$ T) and velocity ($V = 10$ cm/s; $V = 20$ cm/s).

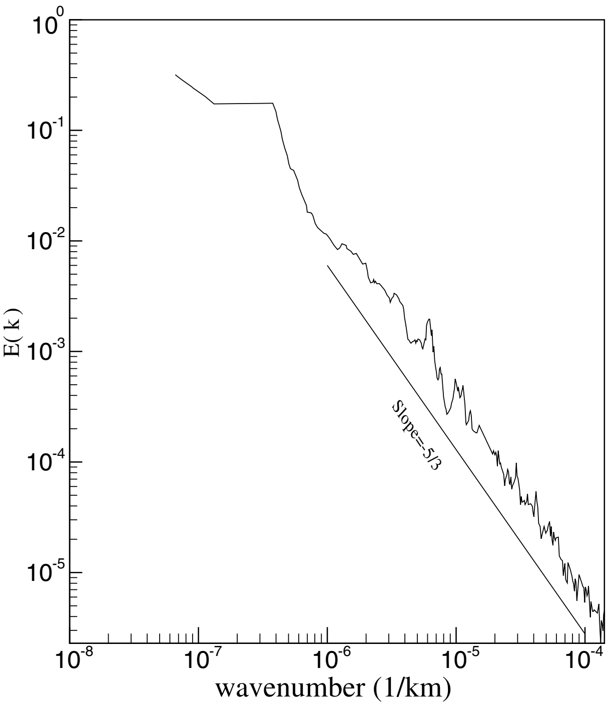

Fig. 4. The observational results of magnetic energy spectrum of the solar wind at 2.8 AU. from Matthaeus *et al.*[7] The solar wind velocity is 442 km/s, and the total fluctuation energy is 4.8×10^{-12} erg/cm^3. $E(k)$ has a power-law slope of $k^{-1.7\pm0.1}$. It is also noted that Biskamp[9] (p. 203) mentioned that the numerical simulations showed also that E_M is close to $k^{-5/3}$.

5. Equation of the Invariant Effective Eddy Viscosity

The purpose of this section is to look for the invariant effective eddy viscosity by pursuing a differential version of the recursive relationship. Recall that the basic idea underlining the recursive RG analysis is to divide the wavenumber space $(0, k_0)$ to a supergrid region $(0, k_c)$ and a subgrid region (k_c, k_0); the subgrid modes are then removed piece by piece by taking subgrid averaging over a spherical shell (k_{n+1}, k_n). The result will certainly depend on the cutoff ratio $\Lambda = k_{n+1}/k_n$; and thus the invariant (limiting) effective eddy viscosity should be sought by taking the limiting operation $\Lambda \to 1$.

First of all, we rescale the wavenumber by setting $\tilde{k} = k/k_{n+1}$ and rewrite eqs. (30) and (31) by expressing the results of (40) and (41) in the form:

$$\delta\nu_n(k) = k_{n+1}^{-8/3}\int_{\tilde{\Omega}_n} \mathrm{d}^3\tilde{j}\,\frac{(\tilde{k}^4 - 2\tilde{k}^3\mu + \tilde{k}\tilde{j}^3\mu)(1-\mu^2)}{2\pi\tilde{j}^2\tilde{k}^2(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)}$$
$$\left[\frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\tilde{j}^{-5/3}\phi(\tilde{j}/\tilde{k}_p)}{\hat{\nu}_n(j)\tilde{j}^2 + \hat{\nu}_n(|\boldsymbol{k}-\boldsymbol{j}|)(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)} + \frac{C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\tilde{j}^{-5/3}\varphi(\tilde{j}/\tilde{k}_p)}{\hat{\tau}_n(j)\tilde{j}^2 + \hat{\tau}_n(|\boldsymbol{k}-\boldsymbol{j}|)(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)}\right], \tag{42}$$

and

$$\delta\tau_n(k) = k_{n+1}^{-8/3}\int_{\tilde{\Omega}_n} \mathrm{d}^3\tilde{j}\,\frac{(\tilde{k}^4 - 2\tilde{k}^3\mu + \tilde{k}\tilde{j}^3\mu)(1-\mu^2)}{2\pi\tilde{j}^2\tilde{k}^2(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)}$$
$$\left[\frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\tilde{j}^{-5/3}\phi(\tilde{j}/\tilde{k}_p) - C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\tilde{j}^{-5/3}\varphi(\tilde{j}/\tilde{k}_p)}{\hat{\tau}_n(j)\tilde{j}^2 + \hat{\nu}_n(|\boldsymbol{k}-\boldsymbol{j}|)(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)} - \frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\tilde{j}^{-5/3}\phi(\tilde{j}/\tilde{k}_p) - C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\tilde{j}^{-5/3}\varphi(\tilde{j}/\tilde{k}_p)}{\hat{\nu}_n(j)\tilde{j}^2 + \hat{\tau}_n(|\boldsymbol{k}-\boldsymbol{j}|)(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)}\right] \tag{43}$$

According to eqs. (42) and (43), we may assume that $\nu_n(k) = k_n^t\tilde{\nu}_n(\tilde{k})$, and $\tau_n(k) = k_n^t\tilde{\tau}_n(\tilde{k})$ where t is an undetermined parameter. With this scaling law, combining the recursive relationship of viscosity and eq. (42) gives

$$\begin{cases} k_{n+1}^t\tilde{\nu}_{n+1}(\tilde{k}) = k_n^t\tilde{\nu}_n(\tilde{k}\Lambda) + k_n^{-8/3-t}\delta\tilde{\nu}_n(\tilde{k}\Lambda); \\ k_{n+1}^t\tilde{\tau}_{n+1}(\tilde{k}) = k_n^t\tilde{\tau}_n(\tilde{k}\Lambda) + k_n^{-8/3-t}\delta\tilde{\tau}_n(\tilde{k}\Lambda). \end{cases} \tag{44}$$

For consistency of the dimension on both sides of eq. (44), we must have $t = -8/3 - t$, and thus $t = -4/3$. It follows by dividing by $k_{n+1}^{-4/3}$ on both sides of eq. (44),

$$\begin{cases} \tilde{\nu}_{n+1}(\tilde{k}) - \Lambda^{\frac{-4}{3}}\tilde{\nu}_n(\tilde{k}\Lambda) = \Lambda^{\frac{-4}{3}}\delta\tilde{\nu}_n(\tilde{k}\Lambda); \\ \tilde{\tau}_{n+1}(\tilde{k}) - \Lambda^{\frac{-4}{3}}\tilde{\tau}_n(\tilde{k}\Lambda) = \Lambda^{\frac{-4}{3}}\delta\tilde{\tau}_n(\tilde{k}\Lambda). \end{cases} \tag{45}$$

Now we write $\Lambda = 1 - \lambda$, and let $n \to \infty$, equivalently, we have $\lambda \to 0$, $\tilde{\nu}_n \to \tilde{\nu}$ and $\tilde{\tau}_n \to \tilde{\tau}$. Then for $n \gg 1$, eq. (45) becomes

$$\left[\tilde{k}\frac{\mathrm{d}\tilde{\nu}(\tilde{k})}{\mathrm{d}\tilde{k}} + \frac{4}{3}\tilde{\nu}(\tilde{k})\right]\lambda = \frac{(1-\lambda)^{\frac{-4}{3}}}{2\pi}\int_{\tilde{\Omega}_n}\mathrm{d}^3\tilde{j}\,\frac{(\tilde{k}^4 - 2\tilde{k}^3\tilde{j}\mu + \tilde{k}\tilde{j}^3\mu)(1-\mu^2)}{\tilde{j}^2\tilde{k}^2(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)}$$
$$\times\left\{\frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\tilde{j}^{-5/3}\phi(\tilde{j}/\tilde{k}_p)}{\left[\tilde{\nu}(\tilde{j})|\tilde{\boldsymbol{j}}|^2 + \tilde{\nu}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|)|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|^2\right]} + \frac{C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\tilde{j}^{-5/3}\varphi(\tilde{j}/\tilde{k}_p)}{\left[\tilde{\tau}(\tilde{j})|\tilde{\boldsymbol{j}}|^2 + \tilde{\tau}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|)|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|^2\right]}\right\}$$
$$+ O(\lambda^2), \tag{46}$$

and

$$\left[\tilde{k}\frac{\mathrm{d}\tilde{\tau}(\tilde{k})}{\mathrm{d}\tilde{k}} + \frac{4}{3}\tilde{\tau}(\tilde{k})\right]\lambda = \frac{(1-\lambda)^{\frac{-4}{3}}}{2\pi}\int_{\tilde{\Omega}_n}\mathrm{d}^3\tilde{j}\,\frac{(\tilde{k}^4 - 2\tilde{k}^3\tilde{j}\mu + \tilde{k}\tilde{j}^3\mu)(1-\mu^2)}{\tilde{j}^2\tilde{k}^2(\tilde{k}^2+\tilde{j}^2-2\tilde{k}\tilde{j}\mu)}$$
$$\times\left\{\frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\tilde{j}^{-5/3}\phi(\tilde{j}/\tilde{k}_p) - C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\tilde{j}^{-5/3}\varphi(\tilde{j}/\tilde{k}_p)}{\left[\tilde{\tau}(\tilde{j})|\tilde{\boldsymbol{j}}|^2 + \tilde{\nu}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|)|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|^2\right]} - \frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\tilde{j}^{-5/3}\phi(\tilde{j}/\tilde{k}_p) - C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\tilde{j}^{-5/3}\varphi(\tilde{j}/\tilde{k}_p)}{\left[\tilde{\nu}(\tilde{j})|\tilde{\boldsymbol{j}}|^2 + \tilde{\tau}(|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|)|\tilde{\boldsymbol{k}}-\tilde{\boldsymbol{j}}|^2\right]}\right\} + O(\lambda^2), \tag{47}$$

where $\Pi(\tilde{\Omega}_n)$ denotes the measure of the set $\tilde{\Omega}_n$, under the limit of $n \to \infty$, the measure of $\tilde{\Omega}_n$ had evaluated in[1] to be

$$\Pi(\tilde{\Omega}_n) = 2\pi\tilde{k}\lambda + O(\lambda^2).$$

Therefore in the limit of $\lambda \to 0$, eqs. (46) and (47) simply become

$$\begin{cases} \tilde{k}\dfrac{\mathrm{d}\tilde{\nu}(\tilde{k})}{\mathrm{d}\tilde{k}} + \dfrac{4}{3}\tilde{\nu}(\tilde{k}) = \left[\dfrac{C_{\mathrm{K}}\varepsilon_v^{2/3}\phi(1/\tilde{k}_p)}{\tilde{\nu}(1)} + \dfrac{C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\varphi(1/\tilde{k}_p)}{\tilde{\tau}(1)}\right]\dfrac{\tilde{k}}{4}\left[1 - \left(\dfrac{\tilde{k}}{2}\right)^2\right]; \\ \tilde{k}\dfrac{\mathrm{d}\tilde{\tau}(\tilde{k})}{\mathrm{d}\tilde{k}} + \dfrac{4}{3}\tilde{\tau}(\tilde{k}) = 0. \end{cases} \tag{48}$$

Notice that the right hand side of the second equation of (48) vanishes, since the two integrands in eq. (43) will cancel out each other exactly in the limit of $n \to \infty$. The two equations in (48) can be readily solved to yield

$$\nu(k) = \left\{ \nu(k_c) k_c^{\frac{4}{3}} - \frac{135}{364} \left[\frac{C_K \varepsilon_v^{2/3} \phi(k_c/k_p)}{4\nu(k_c) k_c^{4/3}} + \frac{C_M \varepsilon_M^{2/3} \varphi(k_c/k_p)}{4\tau(k_c) k_c^{4/3}} \right] \right\} k^{\frac{-4}{3}}$$
$$- \left[\frac{C_K \varepsilon_v^{2/3} \phi(k_c/k_p)}{4\nu(k_c) k_c^{4/3}} + \frac{C_M \varepsilon_M^{2/3} \varphi(k_c/k_p)}{4\tau(k_c) k_c^{4/3}} \right] \left[\frac{3}{52} \left(\frac{k}{k_c} \right)^3 - \frac{3}{7} \left(\frac{k}{k_c} \right) \right] k_c^{\frac{-4}{3}}, \tag{49}$$

and

$$\tau(k) = C_M^{1/2} \varepsilon_M^{1/3} k_c^{4/3} k^{-4/3} \hat{\tau}_n(k_c/k_p), \tag{50}$$

where k_c denotes the cutoff wavenumber. It is appropriate to term $\nu(k)$ and $\tau(k)$ the invariant effective eddy viscosity and the invariant effective magnetic resistivity, respectively. It is notable that the RG procedure does not incur an increment of the magnetic resistivity $\tau(k)$, which obeys the second equation of (48) and must scale as in eq. (50) being proportional to $\varepsilon_M^{1/3} k^{-4/3}$. On the other hand, because of the minus sign in front of the terms containing C_M (or ϵ_M) in the expression (49), the effect of the magnetic effect on the effective eddy viscosity is to reduce the latter in magnitude, but not to change its basic behavior.

6. Evaluation of the Kolmogorov Constant and Smagorinsky Model

The results of §5 will be applied here to evaluate the Kolmogorov and Smagorinsky constants. First of all, we set $\nu(k) = C_K^{\frac{1}{2}} \varepsilon_v^{\frac{1}{3}} F(k)$, then (49) can be written

$$F(k) = \left\{ F(k_c) k_c^{\frac{4}{3}} - \frac{135}{364} \left[\frac{\phi(k_c/k_p)}{4F(k_c) k_c^{4/3}} + \frac{\chi \varphi(k_c/k_p)}{4\hat{\tau}(k_c/k_p) k_c^{4/3}} \right] \right\} k^{\frac{-4}{3}}$$
$$- \left[\frac{\phi(k_c/k_p)}{4F(k_c) k_c^{4/3}} + \frac{\chi \varphi(k_c/k_p)}{4\hat{\tau}(k_c/k_p) k_c^{4/3}} \right] \left[\frac{3}{52} \left(\frac{k}{k_c} \right)^3 - \frac{3}{7} \left(\frac{k}{k_c} \right) \right] k_c^{\frac{-4}{3}}, \tag{51}$$

where we used the result of (50), and set $\chi = \sqrt{C_M/C_K} \sqrt[3]{\varepsilon_M/\varepsilon_v}$. Let us now consider a cutoff k_c for the first expression of (38) and then substitute the first expression of (40) and (51) in it; this yields

$$\int_{k_s}^{k_c} 2k^2 \nu(k) E(k) \mathrm{d}k = 2C_K^{\frac{3}{2}} \varepsilon_v \int_{k_s}^{k_c} F(k) k^{\frac{1}{3}} \phi(k/k_p) \mathrm{d}k = \varepsilon_v,$$

where k_s denotes the wavenumber of the largest eddy existing in the flow. Canceling out ϵ on both sides, we obtain the Kolmogorov constant C_K in terms of the three characteristic wavenumbers k_c, k_p and k_s:

$$C_K = \left[2 \int_{k_s}^{k_c} F(k) k^{\frac{1}{3}} \phi(k/k_p) \mathrm{d}k \right]^{\frac{-2}{3}}. \tag{52}$$

Similarly, substituting the second expression of (40) and (50) in the second expression of (38) gives

$$C_M = \left[2k_c^{\frac{4}{3}} \hat{\tau}(k_c/k_p) \int_{k_s}^{k_c} k^{-1} \varphi(k/k_p) \mathrm{d}k \right]^{\frac{-2}{3}}. \tag{53}$$

So far, we have not given a precise form for ϕ and φ. Let us assume further that both the velocity and magnetic induction fields share the same form of the energy spectrum which is a combination form of the scaling laws proposed respectively by Pao,[13] and Leslie and Quarini,[14] that is,

$$\phi(k/k_p) = A_p\left(\frac{k}{k_p}\right) \exp\left(\frac{-3}{2} C_K^{\frac{-1}{2}} \varepsilon_v^{\frac{-1}{3}} \nu(k) k^{\frac{4}{3}} \right),$$

and

$$\varphi(k/k_p) = A_p\left(\frac{k}{k_p}\right) \exp\left(\frac{-3}{2} C_M^{\frac{-1}{2}} \varepsilon_M^{\frac{-1}{3}} \tau(k) k^{\frac{4}{3}} \right).$$

It follows immediately from (40) that

$$E^v(k) = A_p\left(\frac{k}{k_p}\right) C_K \varepsilon_v^{2/3} k^{-5/3} \exp\left(-\frac{3}{2} C_K^{\frac{-1}{2}} \varepsilon_v^{-1/3} \nu(k) k^{4/3} \right), \tag{54}$$

and

$$E^M(k) = A_p\left(\frac{k}{k_p}\right) C_M \varepsilon_M^{2/3} k^{-5/3} \exp\left(-\frac{3}{2} C_M^{\frac{-1}{2}} \varepsilon_M^{-1/3} \tau(k) k^{4/3} \right). \tag{55}$$

In these formulas, we have the factor

$$A_p(x) = \frac{x^{s+5/3}}{1 + x^{s+5/3}},$$

to take care of energy-containing eddies, where s is a flow parameter. If we consider the leading term of (51) and apply (54) to (52), we may rewrite (52) in a more precise form as follows:

$$C_K = \left\{ \frac{2\mathcal{M}}{s + 5/3} e^{-1.5\mathcal{M}} \log\left[\frac{(k_c/k_p)^{s+5/3} + 1}{(k_s/k_p)^{s+5/3} + 1} \right] \right\}^{\frac{-2}{3}}, \tag{56}$$

where we denote

$$\mathcal{M} \equiv \left\{ F(k_c)k_c^{\frac{4}{3}} - \frac{135}{364}\left[\frac{A_p(k/k_p)\exp(-1.5F(k)k^{4/3})}{4F(k_c)k_c^{4/3}} + \frac{\chi\varphi(k_c/k_p)}{4\hat{\tau}(k_c/k_p)k_c^{4/3}} \right] \right\}.$$

Following the same calculations as in the above, we may also obtain

$$C_{\mathrm{M}} = \left\{ \frac{2k_c^{\frac{3}{4}}\hat{\tau}\left(\frac{k_c}{k_p}\right)\exp\left[-1.5k_c^{\frac{3}{4}}\hat{\tau}\left(\frac{k_c}{k_p}\right)\right]}{s+5/3}\log\left[\frac{(k_c/k_p)^{s+5/3}+1}{(k_s/k_p)^{s+5/3}+1}\right] \right\}^{\frac{-2}{3}}. \tag{57}$$

As a matter of fact, Biskamp[9)] indicated that the Kolmogorov constant depends on the precise definition of the averge magnetic induction, and hence on the geometry of the large scale eddies. Here, we have provided two relationships which show how the large-scale eddies can influence not only the Kolmogorov constant C_{K} but also the magnetic Kolmogorov constant C_{M}, as shown in (56) and (57). The large-scale eddies with the wavenumbers k_p and k_s indeed play an important role in deciding both of C_{K} and C_{M}. Here we recall that, k_p denotes the 1/(geometric size) of the energy-containing eddies and k_s denotes the 1/(geometric size) of the largest eddies in fluid.

In order to carry out large eddy simulation (LES) for MHD turbulence, we need to evaluate the Smagorinsky constant for MHD turbulence by using (49). First of all, we suppose no matter whenever we perform the RG analysis, the cutoff k_c is always very close to the Kolmogorov scale k_0, that is, we may replace k_c by k_0 in (49). In doing so, we evaluate the effective eddy viscosity at $k = k_c$ which is far from k_0. Then (49) becomes

$$\begin{aligned}\nu(k_c) &= \left\{ \nu_0 k_0^{\frac{4}{3}} - \frac{135}{364}\left[\frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\phi(k_0/k_p)}{4\nu_0 k_0^{4/3}} + \frac{C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\varphi(k_0/k_p)}{4\tau_0 k_0^{4/3}} \right] \right\} k_c^{\frac{-4}{3}} \\ &\quad - \left[\frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\phi(k_0/k_p)}{4\nu_0 k_0^{4/3}} + \frac{C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\varphi(k_0/k_p)}{4\tau_0 k_0^{4/3}} \right]\left[\frac{3}{52}\left(\frac{k_c}{k_0}\right)^3 - \frac{3}{7}\left(\frac{k_c}{k_0}\right) \right] k_0^{\frac{-4}{3}}.\end{aligned} \tag{58}$$

Since $k_c/k_0 \ll 1$, we can make the following approximation

$$\begin{aligned}\nu(k_c) &\simeq \left\{ \nu_0 k_0^{\frac{4}{3}} - \frac{135}{364}\left[\frac{C_{\mathrm{K}}\varepsilon_v^{2/3}\phi(k_0/k_p)}{4\nu_0 k_0^{4/3}} + \frac{C_{\mathrm{M}}\varepsilon_{\mathrm{M}}^{2/3}\varphi(k_0/k_p)}{4\tau_0 k_0^{4/3}} \right] \right\} k_c^{\frac{-4}{3}} \\ &= C_{\mathrm{K}}^{1/2}\varepsilon_v^{1/3}\left\{ \hat{\nu}_0(k_0/k_p) - \frac{135}{364}\left[\frac{\phi(k_0/k_p)}{4\mathcal{H}_1} + \frac{\chi\varphi(k_0/k_p)}{4\mathcal{H}_2} \right] \right\} k_c^{\frac{-4}{3}},\end{aligned} \tag{59}$$

where

$$\mathcal{H}_1 = C_{\mathrm{K}}^{-1/2}\varepsilon_v^{-1/3}\nu_0 k_0^{4/3} \quad \text{and} \quad \mathcal{H}_2 = C_{\mathrm{M}}^{-1/2}\varepsilon_{\mathrm{M}}^{-1/3}\tau_0 k_0^{4/3}.$$

Next, we express ε_v in the resolvable velocity,

$$\varepsilon_v = \frac{\nu(k_c)}{2}\left(\frac{\partial \boldsymbol{u}_i^<}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j^<}{\partial \boldsymbol{x}_i} \right)^2.$$

Substituting it in eq. (59) yields

$$\nu(k_c) = \left\{ \hat{\nu}_0(k_0/k_p) - \frac{135}{364}\left[\frac{\phi(k_0/k_p)}{4\mathcal{H}_1} + \frac{\chi\varphi(k_0/k_p)}{4\mathcal{H}_2} \right] \right\} C_{\mathrm{K}}^{\frac{1}{2}}\left[\frac{\nu(k_c)}{2}\left(\frac{\partial \boldsymbol{u}_i^<}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j^<}{\partial \boldsymbol{x}_i} \right)^2 \right]^{\frac{1}{3}} k_c^{\frac{-4}{3}}.$$

Solving the above algebraic equation for $\nu(k_c)$ and replacing k_c by $2\pi/\Delta$ where Δ denotes the cutoff size, we obtain

$$\begin{aligned}\nu(k_c) &= \frac{1}{4\sqrt{2}\pi^2}\left(\left\{ \hat{\nu}_0(k_0/k_p) - \frac{135}{364}\left[\frac{\phi(k_0/k_p)}{4\mathcal{H}_1} + \frac{\chi\varphi(k_0/k_p)}{4\mathcal{H}_2} \right] \right\} C_{\mathrm{K}}^{\frac{1}{2}} \right)^{\frac{3}{2}} \Delta^2 \left| \frac{\partial \boldsymbol{u}_i^<}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j^<}{\partial \boldsymbol{x}_i} \right| \\ &\equiv C_{\mathrm{S}}\Delta^2 \left| \frac{\partial \boldsymbol{u}_i^<}{\partial \boldsymbol{x}_j} + \frac{\partial \boldsymbol{u}_j^<}{\partial \boldsymbol{x}_i} \right|,\end{aligned}$$

where

$$C_{\mathrm{S}} = \frac{1}{4\sqrt{2}\pi^2}\left\{ \hat{\nu}_0(k_0/k_p) - \frac{135}{364}\left[\frac{\phi(k_0/k_p)}{4\mathcal{H}_1} + \frac{\chi\varphi(k_0/k_p)}{4\mathcal{H}_2} \right] \right\}^{\frac{3}{2}} C_{\mathrm{K}}^{\frac{3}{4}}. \tag{60}$$

This is the Smagorinsky constant, where we have left two undetermined parameters $\mathcal{H}_1$ and $\mathcal{H}_2$, which require two additional conditions to be fully determined.

In summary, the closed-form solutions (49) and (50) for

$\nu(k)$ and $\tau(k)$ have enabled derivation of the functional dependence of C_K, C_M and C_S in (56), (57) and (60), respectively. In other words, these numbers C_K, C_M and C_S are not genuine constants but dependent upon the characteristic wavenumbers k_p and k_s of the energy-containing eddies. Namely, the theory requires an input of the large-eddy wavenumbers k_p and k_s from observations and/or experiments. The value of k_s is approximately that of k_p. This was done in our early study for incompressible flow turbulence as well as in thermal-fluid turbulence; the range of variation of the relevant Kolmogorov's and Batchelor's constants were found in close agreement with experiments (cf. Chang *et al.*[1)] and Lin *et al.*[2)]).

7. Concluding Remarks

In this study, we have extended our previous RG analysis of incompressible flow turbulence to incompressible MHD turbulence.

The Elsässer variables are introduced to write the MHD equations for the velocity and magnetic induction fields in a symmetric form. RG analysis is then performed in the wavenumber domain. Taking subgrid averaging of the equation governing the supergrid modes yields a renormalizable form of the MHD equations. To proceed further with the RG transformation, we have to impose the following two assumptions. (i) The mean magnetic induction is relatively weak compared to the mean flow velocity. (ii) The Alfvén effect holds, that is, the fluctuating velocity and magnetic induction are nearly parallel and approximately equal in magnitude. That these conditions still warrant sufficient interest are illustrated by some available data from observations in astronomical physics. Under these conditions, renormalization does not incur an increment of the magnetic resistivity τ, while the coupling effect tends to reduce the invariant effective eddy viscosity $\nu(k)$. Both the velocity and magnetic energy spectra are shown to follow the Kolmogorov $k^{-5/3}$ in the inertial subrange; this is consistent with some available laboratory measurements and observations in astronomical physics. Furthermore, by assuming that the velocity and magnetic induction fields share the same combined form of the energy spectra proposed respectively by Pao, and Leslie and Quarini, we are able to determine the dependence of the Kolmogorov constant C_K and the magnetic Kolmogorov constant C_M on the characteristic wavenumbers k_c, k_p and k_s. The results are applied to obtain the dependence of the Smagorinsky constant C_S for large-eddy simulation, which however contains two undetermined constants to be resolved.

In spite of the present success, it must be stressed upon that the imposed conditions (i) and (ii) imply a negligible effect of the subgrid cross helicity between the velocity and magnetic fields. There are cases where the effect is important and which may lead to quite different energy spectrum. In an early study, Kraichnan[22)] derived a $k^{-3/2}$ energy spectrum of the inertial subrange when the magnetic energy in the subinertial wavenumbers exceeds the total energy in the inertial subrange. Pouquet *et al.*[23)] had an intensive study on strong MHD helical turbulence and the nonlinear dynamo effect. Recently, Nakayama,[24,25)] obtained also the $k^{-3/2}$ energy spectrum in the inertial subrange by constructing a spectral theory of strong shear Alfvén turbulence anisotropized by the presence of a uniform mean magnetic field. Of particular interest, we refer to Yoshizawa *et al.*[26)] for reviewing the importance of the cross-helicity effect, and more generally for an extensive review of turbulence theories and modeling of fluids and plasmas.

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