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整合連體力學與熱動力學之變分原理(1/3)

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中文摘要

關鍵詞：連體力學、熱動力學、虛功率原理、特權座標、追蹤控制

在處理各類物體，如質點、剛體、或變形體受有約束的力學問題時，變分原理常能提供一適當的解決途徑。本計畫之主要課題，乃在已建立的基礎上，繼續探討虛功率原理(Principle of Virtual Power)的可能應用與推廣，以及是否可將該原理與熱力學定律整合成統一熱與力之變分原理。經過近一年的研究，我們在確定速度及速梯度(velocity and velocity gradient)為力學基本變數，溫度或商(temperature or entropy)及熱通率(heat flux)為熱力學基本變數後，確可統合熱力學與連體力學，完成統一虛功率原理，並據以導得變分方程，從而推出連體動量平衡及角動量平衡原理(Cauchy's laws of linear and angular momentum)及新導出之連體變分組成率(variational constitutive law)。在選取適當的內能函數(internal energy)與耗散函數(dissipating function)後，即可推出彈性固體、黏性流體之組成律及熱力學組成律 Fourier Law 及 Maxwell-Catteneo Equation。詳細討論請參考本報告第一部份。此外，在追蹤控制方面，我們將系統座標分成特權座標與非特權座標兩類，雖然各有其設計軌跡，但我們採取雙迴圈控制策略，即先控制特權座標到其軌跡，再依特權座標與非特權座標關係，調整特權座標設計軌跡，使得非特權座標亦能趨近於設計軌跡。我們將此策略運用於三輪自走車的軌跡追蹤控制上，效果良好，請參考本報告第二部份。

英文摘要

Keywords: Continuum Mechanics, Thermodynamics, Principle of Virtual Power, Privileged Coordinates, Tracking Control

In dealing with the mechanics of material bodies modeled by particles, rigid body, or deformable body subject to various types of constraint, variational principles can usually provide a suitable path to obtain the solutions. The main objective of this project is to seek additional applications and possible extensions on the basis of the Principle of Virtual Power, and to explore whether the proposed principle can be unified with the law of thermodynamics to form a unified variational principle. After nearly one-year term of endeavor, we are able to obtain the desired Principle by choosing the velocity and the velocity gradient as fundamental variables in mechanics, and the entropy and the hear flux as the fundamental quantities in theomodynamics. The Principle indeed unify the theory of thermodynamics and continuum mechanics, from which a variational equation can be established. The Cauchy's laws of linear and angular momentum and the newly developed variational constitutive law are then deduced. By selecting appropriate internal energy function and dissipating function, we could derive the constitutive equations for elastic body or viscous fluid in mechanics, and Fourier law and Maxwell-Catteneo Equation, which are treated as the constitutive equation in thermodynamics. See Part I of this report for more detailed. On the other hand, to perform tracking control, we divide the system coordinates into two sets: privileged coordinates and non-privileged coordinates. While all the coordinates have their desired trajectory, two control loops are designed. The inner loop drives the privileged coordinates to their desired value. The relations between privileged coordinates and non-privileged coordinates are then used to change the desired values of privileged coordinates so that the non-privileged ones can be tracked. We apply the strategy to the trajectory tracking control of a three-wheeled vehicle and the performance is quite well. The results are shown in Part II of this report.

PART I

Abstract

In this report is presented a unified variational principle for the motion of a heat-conducting continuum, which is based on a recent publication [1] and a paper in preparation of the same title [2]. The principles and equations of classical mechanics and thermodynamics as summarized in the first half of this report can all be recovered from the single unified variational principle, which is discussed in the second half of this report to the Workshop.

1 Introduction

In the preface to the presentation of his principle of least constraints in 1829, Gauss makes the following remark as recorded in the classical treatise on mechanics by Mach (1893, p. 441, [5]), *"No essentially new principle can now be established in mechanics; but this does not exclude the discovery of new points of view, from which mechanical phenomena may be fruitfully contemplated."* The variational principle proposed in this report, the principle of virtual power, is of no exception.

The proposed principle is a unified approach to study the motion of material bodies modeled by particles (mass with no extent), rigid bodies (mass with constant extent), and deformable continua (mass with variable extent). For continua, there are additional considerations of heat conduction and energy dissipation in material bodies. Since the proposition of three laws of motion by Newton in 1687 [3], many fundamental principles of mechanics have been proposed [6][7], including D'Alembert's principle and Gauss's principle for discrete systems of particles; Euler's principles of linear and angular momentum for rigid bodies [8], as well as for deformable continua [13]; and the energy principle based on the first and second laws of thermodynamics [14][15]. The proposed principle is independent of all aforementioned principles.

The term of virtual velocity was used interchangeably with that of virtual displacement in the original work of Bernoulli. The Principle of Virtual Displacement was later called the Principle of Virtual Work, and hence the title of Principle of Virtual Power appeared frequently in the literature in connection with virtual velocity. In this report, the virtual velocity means an imposed arbitrary, infinitesimal, and instantaneous change of the velocity of a particle or a mass element at a given position. For a discrete or isentropic continuous system constrained by whichever means, the Principle asserts that the total virtual power generated by the net applied forces in a dynamic system is balanced by the virtual change of internal power for all virtual velocities compatible with constraints. For a continuum with heat conduction, we must consider the virtual changes of several thermovariables and the virtual change of internal power is balanced by the additional virtual change of the net applied heating. Associated with the principle, a fundamental variational equation of thermomechanical process in virtual velocity and other thermokinetic variables is postulated.

By applying different conditions of constraints to the variational equation, we can derive the basic equations of motion for discrete systems of particles, including the Lagrange equation, the Appell equation, and the Gibbs-Appell equation for holonomic and nonholonomic system [7][9]; the Euler equation for the translation and rotation of rigid bodies; the Cauchy's first and second equations of motion (linear and angular momentum) for continua in local form [11]. In addition, we derive from the same variational equation the variational form of constitutive equations of the material bodies which are characterized by the internal energy function and the newly introduced dissipation function appropriate for elastic solid, ideal fluid, viscous fluid, and thermoelastic solids. In addition, the Fourier equation of heat conduction as well as the modified form of Maxwell-Catteneo equation can all be recovered from the variations of the appropriate internal energy and dissipation functions. We summary in the next two sections several familiar basic laws and the principles of mechanics, and those of thermodynamics. The former are essential to the science of motion, and the latter to the science of heat. Together they form the basis of our postulating the unified variational principle and the variational equation.

2 Laws and Principles of Mechanics

All principles of mechanics are based on Newton's law for the motion of a material body. The body is modelled by a single particle, and the law may be stated as follows.

(1) Newton's Law — The second law of motion states that the change of momentum $m\dot{\mathbf{r}}$ is proportional to the impressed force $\mathbf{F}$, i.e.

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{F}. \quad (1)$$

(2) Newtonian Principle — For a system of N particles of m_j $(j = 1, \cdots, N)$, connected by whatever fashion, the Newton's Law is modified to the following form for each particle [4]

$$m_j \ddot{\mathbf{r}}_j = \mathbf{F}_j + \sum_{\substack{k=1, \\ k \neq j}}^{N} \mathbf{f}_{jk} = \mathbf{F}_j^I + \mathbf{F}_j^C, \tag{2}$$

where $\mathbf{F}_j$ denotes the external force applied to each particle, $\mathbf{f}_{jk}$ the interactive forces among pair of particles, and the total force is decomposed into the impressed force $\mathbf{F}_j^I$ and the constraint force $\mathbf{F}_j^C$. This system of differential equations must be supplemented with conditions of constraints in order to determine the constraint forces $\mathbf{F}_j^C$ and the motion.

(3) Variational Equations of Motion for Discrete Systems — The constraint forces are removed from the equations of motion based upon different postulates of principles of mechanics:

$$\begin{aligned} \sum_j (\mathbf{F}_j^I - m_j \ddot{\mathbf{r}}_j) \cdot \delta \mathbf{r}_j = 0, &\text{(D'Alembert-Lagr. Eq.)} \\ \sum_j \left(\mathbf{F}_j^I - m_j \ddot{\mathbf{r}}_j \right) \cdot \delta_1 \dot{\mathbf{r}}_j = 0, &\ \text{(Jourdain Eq.)} \\ \sum_j \left(\mathbf{F}_j^I - m_j \ddot{\mathbf{r}}_j \right) \cdot \delta_2 \ddot{\mathbf{r}}_j = 0. &\ \text{(Gauss-Gibbs Eq.)} \end{aligned} \tag{3}$$

The virtual displacement $\delta \mathbf{r}_j$, the virtual velocity $\delta_1 \dot{\mathbf{r}}_j$, and the virtual acceleration $\delta_2 \ddot{\mathbf{r}}_j$ in respective equation are constrained by geometric or kinematic conditions of constraints for holonomic or nonholonomic systems.

(4) Conservation Law of Mass — In classical mechanics, the mass of a particle or that of a mass element in a material body is assumed to remain constant during the motion. A continuum is regarded as an aggregate of infinite number of interconnected particles, and the summation over mass m_j is replaced by the integration over the mass element dm over the body $\mathcal{B}$ in a volumn $\mathcal{V}$ enclosed by the surface $\mathcal{A}$. The mass element is further transformed to the volume element dV by the distribution of the density function ρ. The conservation law of mass is thus expressed as

$$m = \int_{\mathcal{B}} dm = \int_{\mathcal{V}} \rho dV = \text{constant}. \tag{4}$$

>From this law, the equation of continuity $\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0$ can be derived.

(5) Eulerian Principles of Momenta for a continuum — The principle of linear momentum is

$$\frac{d}{dt} \int_{\mathcal{B}} \dot{\mathbf{r}} dm = \int_{\mathcal{B}} d\mathbf{F} = \int_{\mathcal{B}} \mathbf{f}^B dm + \int_{\mathcal{A}} \mathbf{t}^{(\mathbf{n})} dA. \tag{5}$$

where $\mathbf{f}^B$ denotes the body force per unit mass, and $\mathbf{t}^{(\mathbf{n})}$ the surface force per unit area acting on dA with unit outer normal vector $\mathbf{n}$. Euler postulates the additional principle of angular momentum expressed as

$$\frac{d}{dt} \int_{\mathcal{B}} \mathbf{r} \times \dot{\mathbf{r}} dm = \int_{\mathcal{B}} \mathbf{r} \times d\mathbf{F}. \tag{6}$$

(6) Euler's Equation for a Rigid Body — A body is said to be rigid if the distance between each and every pair of particles or mass elements remain constant. For a rigid body, the two Eulerian equations of motion are

$$m \ddot{\mathbf{r}}^c = \mathbf{F}, \qquad \mathbf{I}^c \cdot \dot{\omega} - \omega \times \mathbf{I}^c \cdot \omega = \mathbf{L}^C, \tag{7}$$

where $m = \int_{\mathcal{B}} dm$, $m\mathbf{r}^c = \int_{\mathcal{B}} \mathbf{r} dm$, $\mathbf{F} = \int_{\mathcal{B}} d\mathbf{F}$, $\mathbf{L}^C = \int_{\mathcal{B}} (\mathbf{r} - \mathbf{r}^c) \times d\mathbf{F}$. In the equation of angular momentum, $\mathbf{I}^c$ is the moment of inertia of the body rotating with the angular velocity ω.

(7) The Cauchy Equations for Deformable Continuum — Cauchy introduced the stress tensor σ in the Principle of Stress,

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \sigma. \tag{8}$$

He then applied the divergence theorem to the surface integral in eqs. (5), (6) to derive the local linear and angular momentum equations for a mass element ρdV,

$$\rho \dot{\mathbf{v}} = \rho \mathbf{f}^B + \nabla \cdot \sigma, \quad \text{(Cauchy 1st eq.)} \tag{9}$$

$$\sigma = \sigma^T, \ \text{(Cauchy 2nd eq., } \sigma_{ij} = \sigma_{ji}) \tag{10}$$

where $\mathbf{v} \equiv \dot{\mathbf{r}}$ and it is given approximately by $\partial \mathbf{u}/\partial t$ in linear theory of deformation with displacement $\mathbf{u}$.
(8) Constitutive Equations — The constitutive equations for material bodies are originally established from observations. The equation for elastic solids was based on Hooke's law and extended by Cauchy to the following anisotropic form

$$\sigma = \mathbf{C} : \varepsilon = \mathbf{C} : \nabla \mathbf{u}, \text{ (elastic solid)} \tag{11}$$

where ε is the strain of the body approximated by $(\nabla \mathbf{u} + \mathbf{u}\nabla)/2$ in linear theory of elasticity, and $\mathbf{C}$ is a fourth ranked tensor of material constants. For viscous fluids, the equation was based on Newton's law of viscous flow, and extended to the following anisotropic form by Stokes:

$$\sigma^d = \mathbf{C}' : \mathbf{D} = \mathbf{C} : \nabla \mathbf{v}; \text{ (viscous fluid)} \tag{12}$$

where σ^d is the dissipative part of σ, $\mathbf{D} = (\nabla \mathbf{v} + \mathbf{v}\nabla)/2$ is the stretching tensor of a continuum, and $\mathbf{C}'$ is another tensor of material constants. Substituting the previous constitutive equations to eq. (5) gives rise to the Navier-Cauchy equation for an elastic solid [10] and the Navier-Stokes equation for a viscous fluid [12], respectively.

3 Laws of Thermodynamics

The heat is a form of energy that has the dimension of mechanical work but cannot be expressed in terms of mechanical variables. When the physical state of a material body is changed from a state of thermo equilibrium, the first law states that the increment of internal energy is balanced by the incremental work done on the body and the heat supply to the body. The second law of thermodynamics was originally developed to investigate the upper limit of heat supply that can be converted to mechanical work. If the change of states is dynamic, these laws are modified and extended to a continuum in the following forms, or having the dimension of mechanical power (work done per unit time).
(9) First Law of Thermodynamics – Let $\mathcal{U}$ denote the internal energy of the body, $\mathcal{Q}$ the heating or the rate of heat supply to the material body, and $\mathcal{W}$ the net working, which is the difference between the rate of work done by the mechanical forces and the rate of kinetic energy of the body. The first law is then expressed as [15]

$$\dot{\mathcal{U}} = \mathcal{W} + \mathcal{Q}. \tag{13}$$

For a continuum, one defines the respective quantities as follows:

$$\dot{\mathcal{U}} = \int_{\mathcal{V}} \rho\, \dot{U}\; dV, \tag{14}$$

$$\mathcal{W} = \int_{\mathcal{B}} \dot{\mathbf{r}} \cdot d\mathbf{F} - \frac{1}{2}\int_{\mathcal{V}} \rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} dV \tag{15}$$

$$= \int_{\mathcal{V}} \rho(\mathbf{f}^B - \ddot{\mathbf{r}}) \cdot \mathbf{v} dV + \int_{\mathcal{A}} \mathbf{t}^{(\mathbf{n})} \cdot \delta_1 \mathbf{v} dA, \tag{16}$$

$$\mathcal{Q} = \int_{\mathcal{V}} \rho q^b dV - \int_{\mathcal{A}} \mathbf{q} \cdot \mathbf{n}\; dA, \tag{17}$$

where q^b is the body heating per unit mass and $\mathbf{q}$ is the heat flux flowing across the surface dA with unit outer normal $\mathbf{n}$.
(10) Second Law of Thermodynamics — The second law of thermodynamics relates the heating to the rate of change of the entropy $\mathcal{S}$ at a given temperature θ in absolute scale by the following inequality

$$\theta\, \dot{\mathcal{S}} \geq \mathcal{Q}. \tag{18}$$

For a continuum, both entropy and heating must be defined in local variables and the inequality is expressed as

$$\int_{\mathcal{V}} \rho\, \dot{S}\; dV \geq \int_{\mathcal{V}} \frac{\rho q^b}{\theta} dV - \int_{\mathcal{A}} \frac{\mathbf{q}}{\theta} \cdot \mathbf{n}\; dA \tag{19}$$

$$= \int_{\mathcal{V}} (\frac{\rho q^b}{\theta} - \nabla \cdot \frac{\mathbf{q}}{\theta}) dV. \tag{20}$$

where S denotes the entropy density per unit volume. These two laws are retained with a modification in our formulation of the variational principle.

4 Principle of Virtual Power and Variational Equation

In our formulation of variational principle of thermomechanics, we retain the conservation law of mass (4), the first law of thermodynamics (13), and the modified second law by adding the internal dissipation of energy $\mathcal{R}$ to equation (18). The inequality of entropy is then modified to an equality in the following form:

$$\int_{\mathcal{V}} \rho \dot{S} dV = \int_{\mathcal{V}} (\frac{\rho q^b}{\theta} - \nabla \cdot \frac{\mathbf{q}}{\theta}) dV + \int_{\mathcal{V}} \frac{R}{\theta} dV, \tag{21}$$

where

$$\mathcal{R} \equiv \int_{\mathcal{V}} R dV \geq 0. \tag{22}$$

The combination of the equality and the inequality in the previous two equations replaces the second law of thermodynamics.

Substituting the expression of $(\rho q^b - \nabla \cdot \mathbf{q})$ from eq. (21) in local form into the integrand of Eq.(13), we obtain

$$\mathcal{W} + \mathcal{Q}' = \dot{\mathcal{U}} + \mathcal{R}, \quad \mathcal{R} \geq 0 \tag{23}$$

where $\mathcal{Q}'$ is called the net applied heating given by

$$\mathcal{Q}' = \int_{\mathcal{V}} \rho \theta \dot{S} dV - \int_{\mathcal{V}} \frac{\nabla \theta}{\theta} \cdot \mathbf{q} dV. \tag{24}$$

This is called the modified second law of thermodynamics in this paper.

On the basis of the modified second law which defines the upper bound for heating conversion and energy dissipation in a thermomechanical process, we introduce a basic principle of thermomechanics, the Principle of Virtual Power, as follows:

> In all thermo-mechanical processes, the sum-total of virtual power by the net forces and that by the net heating is balanced by the virtual change of internal power of the material body for all virtual changes of time-rate state variables compatible with constraints on the body.

This Principle can be expressed mathematically by the variational equation

$$\begin{aligned} &\int_{\mathcal{B}} \delta_1 \mathbf{v} \cdot (d\mathbf{F} - \ddot{\mathbf{r}} dm) + \int_{\mathcal{V}} (\rho \theta \delta_1 \dot{S} - \frac{1}{\theta} \nabla \theta \cdot \delta_1 \mathbf{q}) dV \\ &= \delta_1 \dot{\mathcal{U}} + \delta_1 \mathcal{R}', \end{aligned} \tag{25}$$

where $\mathcal{R}'$, which is called the dissipation function of the body, is different from $\mathcal{R}$ in general. The virtual change of the velocity and other rate variables in thermo kinetics are denoted by the symbol $\delta_1 \mathbf{v}$ with the understanding $\delta_1 \mathbf{r} = \delta_1 t = 0$ as first introduced by Jourdain [7]. The δ_1 variation operates only on rate variables of mechanics and thermokinetics such as the $\mathbf{v}$, $\nabla \mathbf{v}$, $\dot{S}$, $\mathbf{q}$.

The constraints mentioned in the Principle include the traditional condition of geometric constraints and that of kinematic constraints. The constraint on the velocity and displacement of a rigid body are regarded as one of the constitutive equations in modern theory of continuum mechanics [11]. Conversely, we shall consider all constitutive equations of material bodies as constraints in the variational principle. For example, the Fourier law of heat conduction considered as energetic constitutive equations of a heat conducting continuum may also be considered as a condition of constraints in the principle.

The familiar Hookes' law for an elastic solid is considered as a kinetic (mechanical) constitutive equation of a continuum, which relates the stress σ to the strain ε. It imposes obviously a constraint on the variation of stretching $\delta_1 \mathbf{D}$ because $\mathbf{D} \simeq \dot{\varepsilon}$ in linear theory. On the other hand, the σ which is conjugate of $\delta_1 \mathbf{D}$ in the product $\sigma : \delta_1 \mathbf{D}$ is an unknown in the variational equation. The constraint on $\delta_1 \mathbf{D}$ is thus removed to render it an independent variation. This is why the constitutive equation can be derived directly from the variational principle.

5 Dynamics of Particles

For a discrete system of particles, we omit all the considerations of heat flow ($\delta_1 \mathcal{Q}' = 0$) and set $\delta_1 \dot{\mathcal{U}}$ and $\delta_1 \mathcal{R}'$ to be zero. The variational equation (25) is reduced to the Jourdain variational equation of particles in Eq. (3).

Based on this reduced form, we can recover all dynamical equations of particles, such as Lagrange's equation, Appell's equation, Gibbs-Appell equation, and so on [1]. In addition, we can derive the equations of motion for systems with nonlinear kinematical constraints. Details will be reported by Li-Sheng Wang in this workshop along with a comparison of differential variational principles and integral variational principles. The former was based upon the point variations and the latter on the path variation of physical variables describing the motion of the system.

6 Dynamics of Rigid Bodies with Heat Conduction

For a heat-conducting material body, we convert the internal energy density U to the free energy density F in the variational equation by the definition $U \equiv F + \theta S$. The variational equation (25) is then converted to

$$\begin{aligned} &\int_{\mathcal{B}} \delta_1 \mathbf{v} \cdot (d\mathbf{F} - \ddot{\mathbf{r}} dm) + \int_{\mathcal{V}} (-\rho S \delta_1 \dot{\theta} - \frac{1}{\theta} \nabla\theta \cdot \delta_1 \mathbf{q}) dV \\ &= \int_{\mathcal{V}} (\rho \delta_1 \dot{F} + \delta_1 R') dV. \end{aligned} \tag{26}$$

Applying the condition of kinematic constraints on a rigid body, $\dot{\mathbf{r}} = \dot{\mathbf{r}}^c + \omega \times (\mathbf{r} - \mathbf{r}^c)$ and assuming that F is a function of θ only and R' is a function of θ and $\mathbf{q}$ only, we reduce the previous variational equation into a simple form of variational equation in $\delta_1 \mathbf{v}^c$, $\delta_1 \omega$, $\delta_1 \dot{\theta}$, and $\delta_1 \mathbf{q}$. From the first two variations, we recover the Euler's equations of motion for a rigid body, Eq. (7), and from the last two we recover the Fourier equation of heat conduction

$$\mathbf{q} = -\kappa \nabla \theta, \tag{27}$$

where κ denotes the coefficient of heat conduction as a constitutive equation. In addition, if we assume that F is a function of θ, $\mathbf{q}$, we can recover the Maxwell-Catteneo equation which gives rise to the wave equation for the temperature θ in heat conduction [20]. Details will be given by Kuo-Ching Chen in another report of this workshop.

7 Dynamics of Thermoelastic Solids

For a deformable material body, we first applied the variational equation (25) with $d\mathbf{F} = \rho \mathbf{f}^B dV + \mathbf{t}^{(\mathbf{n})} dA$ to a cubic element bounded by six rectangular planes ΔA_J, with unit outer normal $\mathbf{n}_J$, $J = 1, ..., 6$. and found that the relation between the stress vector $\mathbf{t}^{(\mathbf{n})}$ and the stress tensor as in the Cauchy stress principle. We then obtain another variational equation involving five virtual variables, $\delta_1 \mathbf{v}$, $\delta_1 \mathbf{D}, \delta_1 \Omega = \delta_1 (\nabla \mathbf{v} - \mathbf{v} \nabla)/2$, $\delta_1 \dot{\theta}$, and $\delta_1 \mathbf{q}$. By assuming that F is a function of θ, ε,and R' a function of θ ,$\mathbf{q}$, we can recover the Cauchy's first equation (9) and second equation (10) from the variations $\delta_1 \mathbf{v}, \delta_1 \Omega$ and establish the constitutive equations for a thermoelastic solid as follows.

$$\sigma = \frac{\partial \hat{F}}{\partial \epsilon}, \quad S = -\frac{\partial \hat{F}}{\partial \theta}, \quad \frac{1}{\theta} \nabla \theta = -\frac{\partial \hat{R}'}{\partial \mathbf{q}}.$$

The last constitutive equation reduces to the Fourier equation of heat conduction (27) if we assume that $\hat{R}' = (1/2\kappa\theta)\mathbf{q}^T \mathbf{q}$, and it can be modified to derive the Maxwell-Catteneo equation as in the previous sections.

Substituting the previous results of five equations derived from the variational principle into the first and modified second laws (13), (23), respectively, we find the reduced energy equation and the expression for the internal energy loss

$$\rho\theta\, \dot{S} = \rho q^b - \nabla \cdot \mathbf{q}, \qquad R = \mathbf{q} \cdot \frac{\partial R'}{\partial \mathbf{q}}. \tag{28}$$

These seven equations and the conservation of mass complete the formulation of the linear theory of thermoelasticity for heat waves [20].

Finally we note the complete theory of thermomechanics as discussed in this report can not be deduced from a single variational principle proposed by Piola[16], Germain[17], Maugin[19], or in a recent report by Green & Naghdi[18].

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Part II

1. Introduction

Tracking control for autonomous vehicles or mobile robots plays an essential role in the exploration of hazardous areas or planets, such as Mars. If the motion is realized through no-sliding wheels, the problem associated with nonholonomic constraints naturally arise. Based on Jourdain's variational equation, it is possible to formulate the dynamics of such system in terms of the privileged coordinates. Given the desired trajectory, the corresponding reduced Appell's equation can be used to design the control law for the privileged coordinates. On the other hand, to track the non-privileged coordinates, the conditions of constraints are re-structured from which the compensations for the desired values of privileged coordinates are computed. From the simulation results, it is shown that such hierarchical tracking control strategy which simultaneously takes kinematics and dynamics into consideration indeed gives rise to an effective algorithm for tracking problem.

2. Problem Description

The non-integrability of the nonholonomic constraints makes the problem of tracking control of mobile robot with rolling-without-sliding wheels very difficult to be managed. In the literature, cf. [1], some approaches are based on the kinematic equations and the dynamical equations of the system are not taken into consideration. Such methods ignore the mass and the moment of inertia of the system and thus the designs are deemed impractical. Some of the other methods combine both sets of kinematic and dynamic equations together and consider either enlarged or reduced set of equations to find the control law. However, the former case raises the complexity of the design, while the latter may be subject to under-actuated problem. It is then desired to develop a feasible and practical methodology of controller design which appropriately accommodates the dynamics and the kinematic constraints. The hierarchical tracking control algorithm proposed in this paper suits such need.

The practical problem to be solved in this paper is the tracking of a desired trajectory for a three-wheeled mobile robot, cf. [3], whose configuration is depicted in Figure 1, moving on a horizontal plane. The system may be modeled by a rigid body interconnected with two rolling-without-sliding wheels, cf. Figure 2. The castor wheel is ignored due to its negligible effects on the dynamics of the system. The motion of the wheels may be described by (x_j, y_j, ψ_j, θ_j), $j = 1, 2$, where (x_j, y_j) denote the coordinates of the center of mass and ψ_j, θ_j are the rotation angle and the orientation angle of the wheels, respectively. The body may be characterized by (x_3, y_3, θ_3), where θ_3 denotes the heading angle of the vehicle. There are total of 11 variables which are subject to five holonomic constraints: (1) $\theta_1 = \theta_2 \equiv \theta$, (2) $\theta_2 = \theta_3 \equiv \theta$, (3) $y_3 = y_1 - b\cos\theta + d\sin\theta$, (4) $x_3 = x_1 + b\sin\theta + d\cos\theta$, (5) $r\dot{\varphi}_2 = r\dot{\varphi}_1 + 2b\dot{\theta}$; and four nonholonomic constraints: (6) $\dot{x}_1 = r\dot{\varphi}_1\cos\theta$, (7) $\dot{y}_1 = r\dot{\varphi}_1\sin\theta$, (8) $\dot{x}_2 = r\dot{\varphi}_2\cos\theta$, (9) $\dot{y}_2 = r\dot{\varphi}_2\sin\theta$. It is assumed further that the two wheels are rotated by motors which generate torques τ_1, τ_2, respectively. The objective of the design is to obtain a controller which enables the system to track a reference trajectory (x_{jr}(t), y_{jr}(t), $\psi_{jr}(t), \theta_{jr}(t), j = 1, 2, x_{3r}(t), y_{3r}(t), \theta_{3r}(t)$) which satisfies the above-mentioned geometric and kinematic constraints.

3. The Methodology

Based on Jourdain's variational equation and Appell's approach, cf. [2], one may choose ψ_1, θ as privileged coordinates, and derive the reduced Appell's equations of motion as follows:

$$\begin{cases}(m_c+2m_w+2m')r^2\ddot{\varphi}_1+(m_c+2m_w+2m')br\ddot{\theta}-m_c dr\dot{\theta}^2=\tau_1+\tau_2,\\(m_c+2m_w+2m')br\ddot{\varphi}_1+[m_c(d^2+b^2)+\frac{1}{3}m_c'(w^2+\ell^2)+m'(r^2+4b^2)+4m_w b^2]\ddot{\theta}+m_c dr\dot{\varphi}_1\dot{\theta}=\frac{2b}{r}\tau_2,\end{cases}$$

where m_c, m_w, m' denotes the masses of the components of the system. It is noted that these two equations are decoupled from the non-privileged coordinates, and the controller can be designed independently to track $(\psi_1(t),\theta(t))$. Various methods can be applied to obtain such a controller. Here, the adaptive sliding mode controller is adopted due to its capability of dealing with parameter uncertainties.

While the privileged coordinates can be drived to the desired values without too much effort, the non-privileged coordinates may deviate from the reference values significantly if the initial conditions are not set appropriately or there are some disturbances during the motion. To solve this problem, it is noted from the practical experience of driving that we may change the reference values of the privileged coordinates to steer the non-privileged coordinates. The compensated values for the privileged coordinates may be computed from the kinematic equations: (6), (7), (8') $\dot{x}_2=r\dot{\varphi}_1\cos\theta+2b\dot{\theta}\cos\theta$, (9') $\dot{y}_2=r\dot{\varphi}_1\sin\theta+2b\dot{\theta}\sin\theta$, and those for the reference trajectory. Let $\mathbf{z}_2=(x_1,y_2)$, $\mathbf{z}_2^e=(x_{1r}-x_1,y_{2r}-y_2)$, $\mathbf{v}_r=(\dot{\varphi}_{1r},\dot{\theta}_r)$, $\mathbf{y}=(\varphi_1,\theta)$, $\mathbf{y}_r=(\varphi_{1r},\theta_r)$, and

$$\mathbf{B}_2(\mathbf{y})=\begin{bmatrix}r\cos\theta & 0\\ r\sin\theta & 2b\sin\theta\end{bmatrix},\ \mathbf{B}_2^r(\mathbf{y}_r)=\begin{bmatrix}r\cos\theta_r & 0\\ r\sin\theta_r & 2b\sin\theta_r\end{bmatrix},\ \boldsymbol{\lambda}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}.$$

It is found that the corresponding compensation is $(\delta\varphi_1,\delta\theta)^T=(\mathbf{B}_2^{-1}(\mathbf{y})(\mathbf{B}_2^r(\mathbf{y}_r)\mathbf{v}_r+\boldsymbol{\lambda}\mathbf{z}_2^e)-\mathbf{v}_r)\Delta t$, where $\boldsymbol{\lambda}$ is used to specify the rate of convergence. From these data, a new set of reference values for the privileged coordinates is computed. The adaptive sliding mode controller mentioned above is then invoked to track the new reference, which in turn drives the non-privileged coordinates to the desired values. It can be shown that with such hierarchical design, all the variables can be steered to the desired value asymptotically.

4. Simulation Result

Consider the wheeled mobile robot with the following parameters: m_w = 1, m_c = 20, b = 0.5, d = 0.25, r = 0.1. Simulations are conducted to evaluate the performance of the proposed methodology. As shown in Figure 3 and 4, the motion of the wheeled mobile robot can track the desired trajectory (including the heading) successfully, even if the initial configuration is away from the desired configuration. The results demonstrate the effectiveness of the proposed hierarchical control scheme which may be extended to general reducible mechanical systems.

References

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[2] Wang, L.-S., Y.-H. Pao, 2003, "Jourdain's Variational Equation and Appell's Equation of Motion for Nonholonomic Dynamical Systems," *American Journal of Physics,* Vol. 73, No. 1, pp.72-82.

[3] Tsai, Pu-Sheng, Li-Sheng Wang, Fan-Ren Chang and Ter-Feng Wu, "Point Stabilization Control of Car-Like Mobile Robot in Hierarchical Skew Symmetry Chained Form", IEEE International Conference on Networking, Sensing and Control, Taipei, 2004.

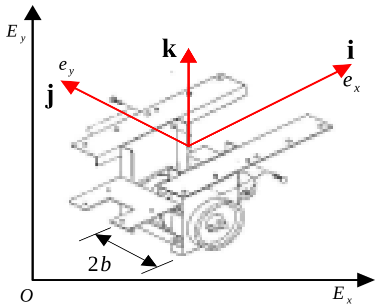

Figure 1

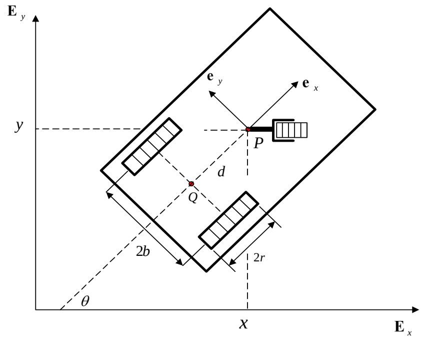

Figure 2

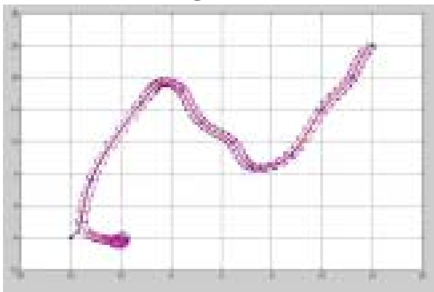

Figure 3

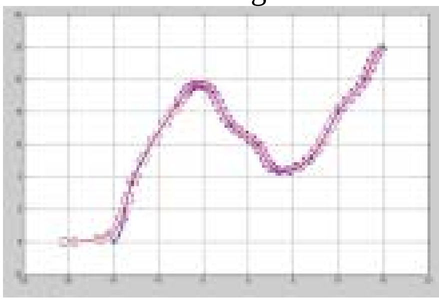

Figure 4