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*Original article*

# On the front condition for compositional gravity currents

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**Abstract.** A two-dimensional potential flow is employed to derive the front condition of the gravity current. The derivation starts from the balance between the static pressure of the gravity current and the form drag imposed on the gravity current by the ambient fluid. After employing Bernoulli's equation along the interface of the gravity current near the head, we end up with a front condition that is in better agreement with experiment than previous theoretical models. This condition is a function of the density ratio between current and ambient fluids, which was different from previous theoretical models, while it has been widely used in experimental studies. The present front condition suggests that the form drag may account for a significant part of the resistance force applied on the current head.

**Key words:** gravity current, front condition, shallow water equation

## 1 Introduction

A gravity current is a flow of heavier fluid intruding beneath a lighter ambient fluid [1, 2]. Its driving force is mainly the density difference between the current and the ambient fluid, which may be due to temperature or concentration differences (called the compositional gravity current) or due to the presence of suspended materials (called the particle-driven gravity current) [3]. Various types of gravity currents have been observed in nature or engineering applications. In many situations the motion of the gravity current is concerned with human safety, such as the accidental release of toxic liquid gas due to the rupture of storage tank, powder snow avalanches, the spreading of hot smoke along a ceiling (an up-side-down gravity current), etc. (see the comprehensive review by Moodie [3]).

To consider a two-dimensional inviscid gravity current without surface tension, the shallow-water equations

$$\frac{\partial h}{\partial t}+\frac{\partial (uh)}{\partial x}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g_r\frac{\partial h}{\partial x}=0, \tag{2}$$

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are applied. In above equations, $u$ is the velocity in the $x$ direction, $h$ is the height of the current, and $g_r = [(\rho - \rho_a)/\rho]g$ is the so-called reduced gravity, where $\rho_a$ and $\rho$ are the densities of the ambient fluid and the current, respectively. To solve Eqs. (1) and (2), one needs the constrained condition [4] at the current source

$$\int_0^{x_f} h(x,t)dx = Q_\alpha t^\alpha, \tag{3}$$

where $x_f$ is the forefront point of the current head. This condition presents the conservation of current volume and $Q_\alpha t^\alpha$ accounts for the volume of the gravity current per unit width, $\alpha = 0$ corresponds to the constant volume current and $\alpha = 1$ corresponds to the constant flux current [5]. Another necessary condition is the front condition [6, 7] at the current front

$$u_f = \beta (g_r h_f)^{\frac{1}{2}}, \tag{4}$$

in which $u_f$ and $h_f$ account for the velocity and the height of the current front, respectively, and $\beta$ is a parameter to be determined theoretically or experimentally.

To the time-dependent hyperbolic-type shallow water equations, the use of the front condition in Eq. (4) is a logical reconciliation [8] because the equations can not take the resistance from the ambient fluid into account while the front condition actually accounts for the force balance between the static pressure of the current and the resistance force from the ambient fluid. To account for the resistance force properly, the determination of $\beta$ is a crucial issue. There have been two approaches to determine $\beta$: analytical and experimental. In analytical approaches, von Karman [9] first considered a steady-state inviscid current moving beneath an unbounded ambient fluid and ended up with $\beta = \sqrt{2}$. Later, Benjamin [1] considered the steady current in a finite-depth channel where dissipation at the interface between current and ambient fluid might exist. After solving the inviscid momentum equations together with relevant boundary conditions, he obtained a formula

$$\beta = \sqrt{\frac{2(1-\Phi)(1-0.5\Phi)}{1+\Phi}}, \tag{5}$$

wherein $\Phi = h/H$, and $H$ is the height of the channel. Note that for an unbounded ambient fluid $\Phi = 0$ (or $H \to \infty$), Eq. (5) yields $\beta = \sqrt{2}$, which is the same as the result of von Karman [9]. Benjamin [1] nevertheless argued that the reduction of Eq. (5) to von Karman's result was a coincidence because, for an inviscid flow past a semi-infinite half-body, analogous to the gravity current considered by von Karman, the drag force must vanish as $H \to \infty$. Later, Rottman and Simpson [10] modified Eq. (5) for time-dependent two-layer gravity currents in a channel, yielding

$$\beta = k\sqrt{\frac{(1-\Phi)(1-0.5\Phi)}{1+\Phi}}, \tag{6}$$

in which $k$ is a parameter that can be adjusted according to the situation. Afterwards, numerous experiments under various situations were conducted and ended up with many different values of $k$. Note that when $k = \sqrt{2}$, Eq. (6) is identical to Eq. (5) and $k = \beta$ when a semi-infinite ambient fluid ($\Phi = 0$) is considered.

In experimental approaches, two kinds of experiment have been widely implemented. One is for gravity currents observed in the atmosphere and the other is for gravity currents generated in laboratories. For atmospheric gravity currents, various uncertainties lead to a wide range of the measured $\beta$, which fall generally below $\sqrt{2}$. The uncertainties arise mainly from the estimations of the current depth, the negative buoyancy, and the front speed [11]. For laboratory gravity currents, although they can be generated under well-controlled conditions, discrepancies still remain. In addition to the factors mentioned above, the discrepancies were also attributed to the uncertainties arising from the estimation of the surface friction due to the mixing of the current with the ambient fluid and the moving-belt boundary of experiments simulating the slip boundary condition at the bottom of the current. A detailed review of the experimental results can be found in Droegemeier and Wilhelmson [11], Klemp et al. [8] and Simpson [2].

In the present study, we employ potential theory to develop a new formula of $\beta$ for a dense gravity current intruding beneath a lighter ambient fluid in a semi-infinite domain. The new feature of this formula is that it is a function of the density ratio of the two fluids $\gamma = \rho_a/\rho$, which was not seen in previous theoretical models while it has been widely used in experimental studies. This formula turns out to be in better agreement with experimental results than previous ones. In the following, we present the derivation of the formula of $\beta$ in Sect. 2, illustrate comparisons with previous results in Sect. 3, and finally provide the discussion and conclusion in Sect. 4.

## 2 The derivation of $\beta$

Consider that an inviscid gravity current moves along a horizontal smooth bed and intrudes beneath a semi-infinite ambient fluid. By assuming that there is no mixing and no surface tension at the interface, the momentum equations of the current are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x}, \tag{7}$$

$$\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - g. \tag{8}$$

Here $P$ is the pressure of the current, and $u$ and $\upsilon$ are, respectively, the velocity components in the $x$- and $y$-direction. Based on the shallow-water approximation and relevant scaling analyses, Eqs. (7) and (8) can be reduced to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}, \tag{9}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - g. \tag{10}$$

Integrating Eq. (10) across the current depth gives $P(x, y, t) = P_i(x, t) + \rho g(h - y)$, where $P_i$ is the interfacial pressure. We can then rewrite Eq. (9) into

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = -\frac{1}{\rho}\frac{\partial P_i}{\partial x}. \tag{11}$$

Since in the self-similar phase the gravity current advances with a constant speed and the shape of the current head remains unchanged during the motion, we can take Galilean transformation by fixing the coordinate on the current (Fig. 1), so that Eq. (11) can be rewritten into

$$\rho g\frac{dh}{dx} = -\frac{dP_i}{dx}. \tag{12}$$

To account for the pressure gradient term on the right hand side of Eq. (12), we apply Bernoulli's equation

$$-\frac{dP_i}{dx} = \frac{\rho_a}{2}\frac{d\left(\vec{V}_i\cdot\vec{V}_i\right)}{dx} + \rho_a g\frac{dh}{dx}, \tag{13}$$

along the interface, which is actually a streamline. Equation (13) is then used to replace the right hand side of Eq. (12), yielding

$$g_r\frac{dh}{dx} = \frac{\gamma}{2}\frac{d\left(\vec{V}_i\cdot\vec{V}_i\right)}{dx}, \tag{14}$$

where $\vec{V}_i$ is the velocity of the ambient fluid at the interface. Equation (14) accounts for the balance between the static pressure of the current and the form drag imposed on the current by the ambient fluid. We integrate Eq. (14) along the $x$-axis from the foremost point of the current to a point far behind the current head. The

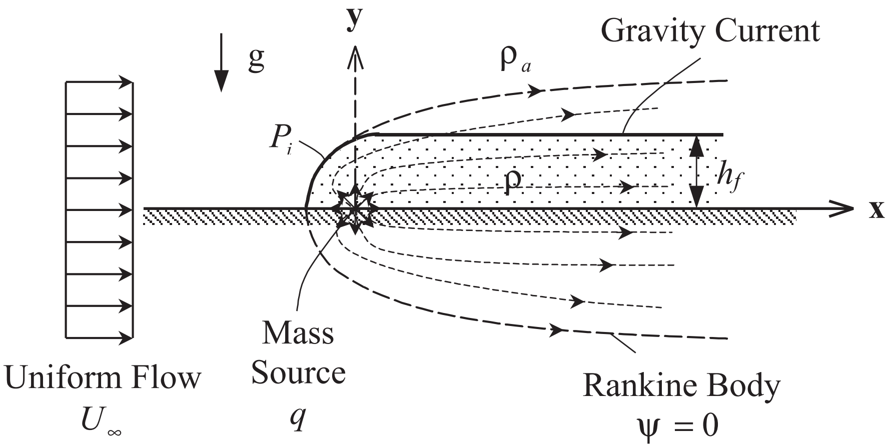


**Fig. 1.** A schematic description of the analogy between the gravity current and the Rankine body is considered. The Rankine body considered is for $\psi = 0$, which is a result of the superposition of a uniform flow and a mass source. Note that the analogy between the shape of the gravity current and that of Rankine body holds only in the front part of the current

integration constant is determined by the fact that the height of the current head at the foremost point of the current is zero, yielding

$$g_r h = \frac{\gamma}{2}(\vec{V}_i \cdot \vec{V}_i). \tag{15}$$

To find $\vec{V}_i$ in an explicit form, the potential flow theory is applied. In doing so, due to the resemblance of the shape of the current head with the front part of Rankine body (Fig. 1), we consider a superposition of a two-dimensional uniform flow of velocity $U_\infty$ passing a mass source of strength $q$. The stream function for this superposed flow is

$$\psi = \psi_{uniform} + \psi_{source} = U_\infty y + q \tan^{-1}(y/x). \tag{16}$$

According to the definition of the velocity $\vec{V} = \frac{\partial\psi}{\partial y}\vec{i} - \frac{\partial\psi}{\partial x}\vec{j}$, we have

$$\vec{V}_i \equiv \vec{V}\Big|_{y=h} = U_\infty \left[\left(1 + \frac{x}{A(x^2+h^2)}\right)\vec{i} - \left(\frac{h}{A(x^2+h^2)}\right)\vec{j}\right], \tag{17}$$

where $A \equiv U_\infty/q$, which determines the shape of the current head. Accordingly, we have

$$\vec{V}_i \cdot \vec{V}_i = U_\infty^2 \left[\left(1 + \frac{x}{A(x^2+h^2)}\right)^2 + \left(\frac{h}{A(x^2+h^2)}\right)^2\right]. \tag{18}$$

After substituting Eq. (18) into Eq. (15), we obtain the following relation

$$g_r h = \frac{\gamma}{2} U_\infty^2 \left[\left(1 + \frac{x}{A(x^2+h^2)}\right)^2 + \left(\frac{h}{A(x^2+h^2)}\right)^2\right]. \tag{19}$$

Now we need to find the shape of the current head accounted for by the relation between $x$ and $h$. To do so, we assume $\psi = 0$ to be the interface streamline of the current head, yielding

$$Ah(x) + \tan^{-1}[h(x)/x] = 0. \tag{20}$$

The above equation is transcendental and cannot be solved analytically except for special cases. A special case can be obtained when we consider $h_f = h(x = 0)$ as the characteristic height of the current head, the value of $A$ is obtained as $A = \pi/2h_f$, where the relation $\tan^{-1}(-\infty) = -\pi/2$ has been considered. Taking

$h_f = h(x=0)$ as the characteristic height is a deterministic step of the present approach because the shape of the Rankine body (and the current head) is determined solely by the value of $A$. Taking such a step is nevertheless expedient because without considering $x = 0$ Eq. (20) cannot be solved analytically, or that one would not get an explicit form of $A$ as shown above or the explicit form of $\beta$ to be shown in the following. Consequently, by considering Eq. (19) for $x = 0$, we obtain

$$g_r h_f = \frac{\gamma}{2} U_\infty^2 \left[ 1 + \frac{1}{A^2 h_f^2} \right]. \tag{21}$$

After substituting the relation $A = \pi/2h_f$ into Eq. (21), we end up with a new formula for $\beta$ as follows

$$\beta = \sqrt{\frac{2\pi^2}{\gamma(\pi^2+4)}} \approx \frac{1.19}{\sqrt{\gamma}}. \tag{22}$$

## 3 Comparisons with previous results

To show the superiority of the new formula Eq. (22), a comparison between the present and previous results is made in this section. Groebelbauer et al. [12] conducted a series of experiments for lock-exchange flows of different densities in a closed channel of a square cross-section. They used different gases to simulate gravity currents of $\gamma$ varying from 0.9 (carbon dioxide in argon) to 0.05 (Freon 22 in helium) and considered the influence of the depth ratio $\Phi = h/H$ varying from $1/2$ to $1/6$, and then took the extrapolated values for $\Phi \to 0$. To make the comparison easier, we convert their extrapolated data for $\Phi \to 0$ into the values accounted for by the present parameters. The converted relationships are

$$\gamma = \frac{1-(\rho^*)^2}{1+(\rho^*)^2}, \tag{23}$$

$$\beta = \frac{u_f}{\sqrt{g_r h_f}} = \frac{u_f}{\sqrt{(1-\gamma) g h_f}} = \frac{Fr}{\sqrt{1-\gamma}}, \tag{24}$$

where, according to Groebelbauer et al. [12],

$$\rho^* = \left( \frac{\rho - \rho_a}{\rho + \rho_a} \right)^{\frac{1}{2}}, \quad Fr = \frac{u_f}{\sqrt{g h_f}}. \tag{25}$$

The converted data are shown in Fig. 2, from which one can see that the value of $\beta$ increases exponentially with decreasing $\gamma$, leading to a result that as $\gamma \to 0$ the value of $\beta$ approaches infinity, as predicted by Eq. (22).

On the other hand, Groebelbauer et al. [12], based on their experimental data, modified the Eq. (5) of Benjamin [1] such that $\beta$ is a function of $\gamma$ and $\Phi$

$$\beta = \sqrt{\frac{1}{\gamma} \frac{(2-\Phi)(1-\Phi)}{1+\Phi}}. \tag{26}$$

As shown in Fig. 2, Eq. (22) shows a better agreement with the experimental data than Eq. (26). In the same figure, the experimental result of Simpson and Britter [13] for $\gamma \approx 1$ is also presented. It is seen that Eq. (22) approaches the result of Simpson and Britter [13] as $\gamma \to 1$. In brief, the comparisons illustrated show that Eq. (22) is in better agreement with experiments conducted under various situations [12, 13] than those proposed by previous studies [1, 12].

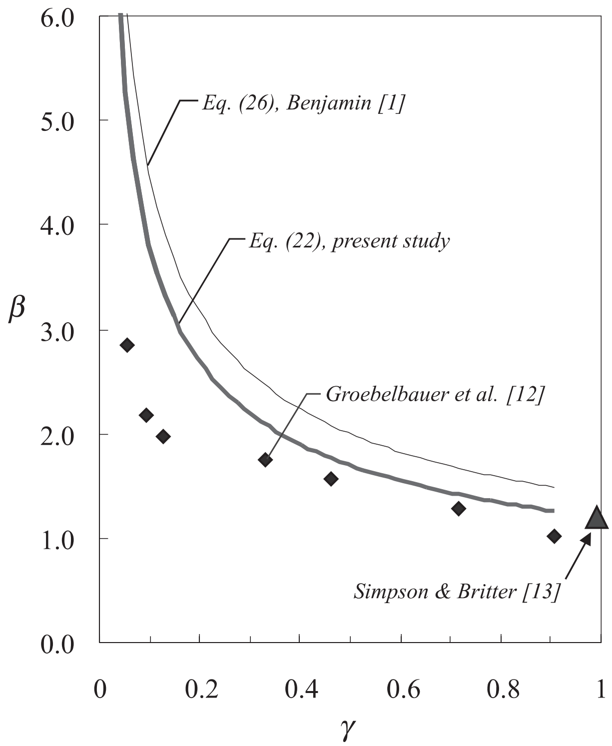


**Fig. 2.** A comparison between the present result Eq. (22), the modified formula of Benjamin [1] Eq. (26), the experimental results of Groebelbauer et al. [12] and Simpson and Britter [13]. Note that the experiments of Groebelbauer et al. [12] were implemented for the lock-exchange currents of various density ratios $\gamma$ and the data shown are for the case $\Phi = 0$, which are the extrapolational results from the cases $\Phi = 1/2$ and $1/6$. The experiment of Simpson and Britter [13] focused on the current head dynamics, which has a density ratio $\gamma \approx 1$

## 4 Discussion and conclusion

The present paper proposes a mathematical approach to lead to a new formula of $\beta$, Eq. (22), being a function of $\gamma$, which was not seen in previous theoretical models [1, 9] while has been widely used in experimental studies [12, 13]. This new formula is also shown to be in better agreement with experimental results than previous ones. The mathematical approach consists of three major steps: (1) Applying Bernoulli's equation along the current interface so that the force balance between the static pressure of the current and the form drag imposed by the ambient fluid in terms of the flow velocity can be obtained (see Eqs. (12) to (15)). (B) Applying the two-dimensional potential flow, i.e. the Rankine body, to simulate the current head so that the form drag can be expressed in terms of $h$ and $x$, or the shape of the current head (see Eqs. (16) to (19)). (C) Assuming that $h_f = h(x = 0)$ is the current height so that a new formula of $\beta$ can be obtained (see Eqs. (20) to (22)).

These three steps are mathematically straightforward. The use of a Rankine body is due to, firstly, the resemblance between the shape of the front part of the Rankine body and that of the gravity current head, and secondly and more importantly, to the easiness of applying the potential flow theory to get the explicit expression of the form drag term $\vec{V}_i \cdot \vec{V}_i$ of the right hand side of Eq. (15). The shape of the Rankine body is determined by the value of $A \equiv U_\infty/q$, while in the present study we have $A = \pi/2h_f$, which results from considering that $h_f = h(x = 0)$ is the characteristic height of the current. The determination of the value of $h_f$ was discussed extensively by previous experimental studies [8, 12, 13]. Some took the highest point of the current head to be $h_f$ while some took the height of the current behind the current head to be $h_f$, while different $h_f$ leads to a different $\beta$. The main reason for us to consider $h_f = h(x = 0)$ is simply due to the feasibility of the mathematical operation so that an explicit form of $\beta$ (Eq. (22)) can be obtained. A choice of $h_f = h(x \neq 0)$ would require a numerical computation for relevant equations, leading to a numerical solution of $\beta$, which would eliminate the significance of the present approach.

A systematic approach for the present analysis is not feasible because the two major steps, choosing Rankine body to simulate the gravity current head and using $h_f = h(x = 0)$ as the characteristic height of the current head, are straightforward and deterministic, i.e. no alternative can be made. The present approach takes advantage of the mathematical simplicity of potential flow, which turns out to be an efficient theoretical approach to get the explicit form of $\vec{V}_i \cdot \vec{V}_i$ and ends up with a new formula of $\beta$ Eq. (22). In brief, the mathematical derivation is straightforward and the resulting formula Eq. (22) is in better agreement with experimental data than previous formulae Eq. (5), (6) or (26), all of which were derived from the viewpoint of static force balance, while no detailed flow structure in terms of explicit function of $\vec{V}_i \cdot \vec{V}_i$ was considered.

Finally, we note that the comparison of Eq. (22) with experimental results [12, 13] implies that the form drag force resulting from the dynamic pressure may account for a major part of the resistance force imposed on the current head from the ambient fluid. Namely, in the similar phase, when gravity current motion is governed by the shallow water equations, neither the viscous effects [3] on the bottom boundary nor the turbulent mixing [2] at the current interface plays a significant role in the resistance force on the gravity current. This outcome could explain Benjamin's argument [1] that, "for an inviscid current intruding beneath a semi-infinite ambient fluid the *drag force* must vanish", because Benjamin's drag force means the *viscous* drag force corresponds to the viscosity, while the major drag force shall be the *form* drag force, which in fact plays a significant role even if the inviscid current is considered.

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