# 行政院國家科學委員會專題研究計畫　期中進度報告

## 整合連體力學與熱動力學之變分原理(2/3)

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# 中文摘要

關鍵詞: 連體力學、熱動力學、虛功率原理、特權座標、追蹤控制

在處理各類物體，如質點、剛體、或變形體受有約束的力學問題時，變分原理常能提供一適當的解決途徑。本計畫之主要課題，乃在探討我們過去所提出虛功率原理(Principle of Virtual Power)的可能應用與推廣，以及是否可將該原理與熱力學定律整合成統一熱與力之變分原理。經過近二年的研究，我們在確定速度及速梯度(velocity and velocity gradient)為力學基本變數，溫度或熵(temperature or entropy)及熱通率(heat flux)為熱力學基本變數後，確可統合熱力學與連體力學，完成統一虛功率原理，並據以導得變分方程，從而推出連體動量平衡及角動量平衡原理(Cauchy’s laws of linear and angular momentum)及新導出之連體變分組成率(variational constitutive law)。在選取適當的內能函數(internal energy)與耗散函數(dissipating function)後，第一年我們已推出彈性固體、黏性流體之組成律及熱力學組成律 Fourier Law 及 Maxwell-Catteneo Equation，第二年更進一步導得熱黏性流及微極性連體之組成律，詳細討論請參考本報告第一部份。此外，在追蹤控制方面，第一年我們已利用 Reduced Appell Equation 建立特權座標系統動力方程，並發現其與非特權座標分離的特性，據以設計模糊控制與滑動控制器，今年我們則採用 Chained Form 及 Backstepping Method 來完成特權座標補償量的計算，並將此方法運用在自走車的軌跡追蹤控制上，模擬效果良好，請參考本報告第二部份。

# 英文摘要

Keywords: Continuum Mechanics, Thermodynamics, Principle of Virtual Power, Privileged Coordinates, Tracking Control

In dealing with the mechanics of material bodies modeled by particles, rigid body, or deformable body subject to various types of constraint, variational principles can usually provide a suitable path to obtain the solutions. The main objective of this project is to seek additional applications and possible extensions on the basis of the Principle of Virtual Power, and to explore whether the proposed principle can be unified with the law of thermodynamics to form a unified variational principle. After nearly two-year term of endeavor, we are able to obtain the desired Principle by choosing the velocity and the velocity gradient as fundamental variables in mechanics, and the entropy and the hear flux as the fundamental quantities in theomodynamics. The Principle indeed unify the theory of thermodynamics and continuum mechanics, from which a variational equation can be established. The Cauchy’s laws of linear and angular momentum and the newly developed variational constitutive law are then deduced. By selecting appropriate internal energy function and dissipating function, we have derived the constitutive equations for elastic body or viscous fluid in mechanics, and Fourier law and Maxwell-Catteneo Equation in the first year. In the second year, we established the constitutive laws for the thermo-viscous fluid and micro-polar material. See Part I of this report for more details. On the other hand, in the aspect of tracking control, we have used the Reduced Appell Equation to construct equations of motion of the privileged coordinates, and found that the equations are decoupled from the non-privileged coordinates in the first year. The fuzzy controller and sliding mode controller were then used to design the tracking control loop. This year, we adopt the backstepping method applied to the chained form of the kinematic equation to compute the compensated values for the privileged variables. Simulation results on the tracking control of wheeled vehicle show that the performance is excellent. The results are shown in Part II of this report.

# Part I 虛功率原理在熱黏性流體及微極性連體之應用

## 一、 前言

在處理各類物體，如質點、剛體、或變形體受有約束的力學問題時，變分原理常能提供一適當的解決途徑。Pao (2003)將虛功率原理(Principle of Virtual Power) 與熱力學第一及第二定律整合，完成統一熱與力之虛功率原理，並據以推導得變分方程。Chen, Wang & Pao (2003)將虛功率原理應用在剛體(rigid body)上，利用變分方程能在剛體運動研究上，成功地推導出柯西線性動量與角動量平衡方程(Cauchy’s laws of linear and angular momentum)；並在熱傳導問題上，得到力學上之組成律及熱力學組成律。

在Bernoulli最早的文章中，虛位移與虛速度是互用的，虛位移原理後來稱為虛功原理，因此牽涉到虛速度的虛功率原理也常常被提出。但在Wang&Pao(2003)的文章中提及，虛速度乃為在特定的位置上，對物質的速度強加一任意、無限小和即時的改變。對於存在熱傳導之連體，虛功率原理說明由外力與淨熱對物體所做的虛功率會與內功率之虛變更相平衡。由此可得在所有虛速度變數之熱力學過程中的基本變分方程。

Biot(1984)提出一廣泛的統一原理：虛耗散原理(principle of virtual dissipation)，包含了力學、熱學與化學交互作用的過程。Green & Rivlin (1964)與Green & Naghdi(1995)利用架構中立原理，從熱力學系統的能量平衡方程，發展另一套統一理論。Germain(1973)與Maugin(1980)則利用虛功率方法(method of virtual power)，對連體之電-熱-力學、可逆與不可逆過程，提出一廣義的變分方程。但這些原理或方程或是依循虛功原理的架構，或是僅能從其導得部分方程。

我們則是從虛功率原理出發，探討熱黏性流體(viscous fluid with heat-conduction)之熱機(thermo-mechanical)問題與微極性連體之組成率。在傳統處理熱黏性流的問題上，是將力學與熱學之變數分別討論，而我們則可將力學的基本變數--速度及速度梯度與熱力學的基本變數--溫度或熵及熱通率統合，完成整合力學與熱學之架構。我們在虛功率原理的架構下作一虛速度變數所表示的變分方程中，選取適當之內能函數與耗散函數，從而推導出對不同虛速度變數所得之方程，包含運動方程：線動量與角動量平衡方程、應力組成律及熱學組成律，以確認虛功率原理在熱黏性流體之通用性。

在微極性連體方面，我們知道極性連體是一種可承受力矩及偶應力的材料。在古典偶應力理論中，Truesdell及 Toupin (1960)不只考慮整體力的分佈，更考慮物體表面上力偶之分佈。Toupin (1963)假設彈性材料之內能函數與第一與第二變形梯度有關，並推導出有限變形量時的組成方程。Mindlin 和 Tiersten (1963)則將Toupin對可承受偶應力之彈性體的組成方程線性化。Green & Rivlin (1964)提出利用第一類單力多極點（simple force multipoles of the first kind）特性之理論，並可推廣至不同類複合力多極點（compound force multipoles of the various kinds）之廣義理論。不同於偶應力理論，Erigen (1966)考慮微旋轉的觀念，並提出微極性彈性體之線性理論。而在古典偶應力理論中，完全缺乏此微旋轉的觀念及相對應的場方程。我們則從微粒以剛體型式表現出發，導得微極性連體之組成律，從而對微極性連體有一更深入的認識。

本文首先回顧虛功率原理與變分方程：簡介熱力學定律，並加些修正，得熱力學第二定律之另一種形式，據此描述虛功率原理及變分方程。接著將此虛功率原理與變分方程應用在熱黏性流體與微極性連體上，最後作一結論並略述未來發展方向。

## 二、 內容

我們首先依據文獻[12]及[13]，介紹熱力學定律及熱力學虛功率原理。

### 2.1 熱力學定律

(1)熱力學第一定律 – 當物質從一個熱力平衡狀態轉換至另一個平衡狀態時，熱力學第一定律說明物體內能的增加等於外界對此物體所做的功與所提供的熱之和。其可表示成下列方程式：

$$\dot{\mathcal{U}} = \mathcal{W} + \mathcal{Q}. \tag{1}$$

其中$\dot{\mathcal{U}}$為物質內能函數之變化率，$\mathcal{W}$為淨功，是外力所作的功率與物體動能變化率之差，而$\mathcal{Q}$則物質所吸收的熱變化率。上述三物理量可由以下的積分式表示：

$$\dot{\mathcal{U}} = \int_V \rho \dot{U}\, dV, \tag{2a}$$

$$\mathcal{W} = \int_B \dot{\mathbf{r}} \cdot d\mathbf{F} - \frac{1}{2}\int_V \rho \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} dV, \tag{2b}$$

$$\mathcal{Q} = \int_V \rho q^b dV - \int_A \mathbf{q} \cdot \mathbf{n}\, dA, \tag{2c}$$

其中$U$是內能密度函數，即單位質量的內能；$d\mathbf{F}$是作用在物體單位質量上所有力之和；$q^b$為物體單位質量所接受的熱率(heating)；而$\mathbf{q}$則是通過垂直向量$\mathbf{n}$的表面積$dA$之熱通量(heat flux)。

(2)熱力學第二定律 –熱力學第二定律引入一物理量--熵，並說明在固定溫度下，熵的變化率與所接受熱的變

化率的關係如下：

$$\theta\,\dot{\mathcal{S}} \geq \mathcal{Q}. \tag{3}$$

對於連體，熵及熱率可以用局部的變數表示，則上列不等式可以寫成：

$$\begin{aligned}\int_V \rho\,\dot{S}\,dV &\geq \int_V \frac{\rho q^b}{\theta}dV - \int_A \frac{\mathbf{q}}{\theta}\cdot\mathbf{n}\,dA \\ &= \int_V (\frac{\rho q^b}{\theta} - \nabla\cdot\frac{\mathbf{q}}{\theta})dV.\end{aligned} \tag{4}$$

其中 $S$ 為物質單位體積的熵密度。此二基本之熱力學定律，在處理熱力學之變分原理時，需做些小修正。

### 2.2 虛功率原理與變分方程

在處理熱力學之變分原理時，我們引入一內能耗散函數 $\mathcal{R}$，使得第二定律之不等式(4)可改寫為等式形式

$$\int_V \rho\dot{S}dV = \int_V (\frac{\rho q^b}{\theta} - \nabla\cdot\frac{\mathbf{q}}{\theta})dV + \int_V \frac{R}{\theta}dV, \tag{5}$$

令

$$\mathcal{R} \equiv \int_V RdV, \tag{6}$$

則 $\mathcal{R} \geq 0$ 可視為熱力學第二定律的另一種形式。

將方程式(5)內的第一個積分項中的( $\rho q^b - \nabla\cdot\mathbf{q}$ )代入方程式(1)的積分式，則可得

$$\mathcal{W} + \mathcal{Q}' = \dot{\mathcal{U}} + \mathcal{R}, \quad \mathcal{R} \geq 0 \tag{7}$$

其中 $\mathcal{Q}'$ 稱為淨熱率(net applied heating)，可表示成

$$\mathcal{Q}' = \int_V \rho\theta\dot{S}dV - \int_V \frac{\nabla\theta}{\theta}\cdot\mathbf{q}dV.$$

方程式(7)我們稱之為熱力學定律之修正式。在此修正式架構下，熱力學之一基本原理，虛功率原理，可敘述如下：

在真實的熱機過程中，相對於任何與系統約束條件相容之虛變數速率變分，由外力所作之虛功率與虛淨熱率之總和必與物體內功率之虛變更相平衡。

此虛功率原理可用變分方程表示為：

$$\begin{aligned}&\int_B \delta_1\mathbf{v}\cdot(d\mathbf{F} - \ddot{\mathbf{r}}dm) \\ &+ \int_V (\rho\theta\delta_1\dot{S} - \frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})dV \\ &= \delta_1\dot{\mathcal{U}} + \delta_1\mathcal{R}^W,\end{aligned} \tag{8}$$

其中 $\mathcal{R}^W$ 是物體的內耗散函數，與方程式(7)中的 $\mathcal{R}$ 不同，其關係可在運動方程式得到之後，結合熱力學第一定律推導出。在熱機系統變分過程中，速度的虛變分可用 $\delta_1\mathbf{v}$ 來表示，我們並假設 $\delta_1\mathbf{r} = \delta_1 t = 0$。另外，$\delta_1$ 只能操作在和速率有關之變數，如 $\mathbf{v}$, $\nabla\mathbf{v}$, $\dot{S}$, 及 $\mathbf{q}$.

### 2.3 連體之變分方程

連體上之作用外力和 $d\mathbf{F}$ 可分解為兩個部分：一為物體內單位質量所受的外力 $\mathbf{f}^B$ (body force)，另一為物體表面的接觸力 $\mathbf{t}^{(\mathbf{n})}$ (surface traction)。即

$$d\mathbf{F} = \mathbf{f}^B\rho dV + \mathbf{t}^{(\mathbf{n})}dA. \tag{9}$$

而柯西應力原理告訴我們 $\mathbf{t}^{(\mathbf{n})} = \mathbf{n}\cdot\sigma$，其中 $\boldsymbol{\sigma}$ 為應力張量。則變分方程式(8)可改寫成

$$\begin{aligned}&\int_{\mathcal{V}} \rho(\mathbf{f}^B - \ddot{\mathbf{r}})\cdot\delta_1\mathbf{v}dV + \int_{\mathcal{A}} \mathbf{t}^{(\mathbf{n})}\cdot\delta_1\mathbf{v}dA \\ &+ \int_{\mathcal{V}} (\rho\theta\delta_1\dot{S} - \frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})dV \\ &= \delta_1\dot{\mathcal{U}} + \delta_1\mathcal{R}^W,\end{aligned} \tag{10}$$

或是

$$
\begin{aligned}
&\int_V (\rho \mathbf{f}^B + \nabla\cdot\boldsymbol{\sigma} - \rho\ddot{\mathbf{r}})\cdot\delta_1\mathbf{v}\, dV \\
&+ \int_V \boldsymbol{\sigma} : \nabla(\delta_1\mathbf{v})\, dV \\
&+ \int_V (\rho\theta\delta_1\dot{S} - \frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})dV \\
&= \int_V (\rho\delta_1\dot{U} + \delta_1 R^W)dV.
\end{aligned}
\tag{11}
$$

因為 $\delta_1$ 和 $\nabla$ 皆是線性操作算子，可相互交換：$\nabla(\delta_1\mathbf{v}) = \delta_1(\nabla\mathbf{v})$。因此，上列方程式的第二個積分內的項可表示成

$$
\begin{aligned}
\boldsymbol{\sigma} : \nabla(\delta_1\mathbf{v}) &= \boldsymbol{\sigma} : \delta_1(\nabla\mathbf{v}) \\
&= \boldsymbol{\sigma} : \delta_1\mathbf{D} + \boldsymbol{\sigma} : \delta_1\Omega,
\end{aligned}
\tag{12}
$$

其中

$$
\mathbf{D} = \frac{1}{2}(\nabla\mathbf{v} + \mathbf{v}\nabla),
\tag{13a}
$$

$$
\boldsymbol{\Omega} = \frac{1}{2}(\nabla\mathbf{v} - \mathbf{v}\nabla) = -\mathbf{1}\times\omega.
\tag{13b}
$$

張量$\mathbf{D}$為 $\nabla\mathbf{v}$ 之對稱的部分，稱為拉伸張量(stretching tensor)，而$\boldsymbol{\Omega}$為 $\nabla\mathbf{v}$ 之反對稱部分，稱為旋轉張量(spin tensor)。因旋轉張量和渦度向量(vorticity vector) $\omega = (1/2)\nabla\times\mathbf{v}$ 有關，故可用其替代：

$$
\sigma : \delta_1\boldsymbol{\Omega} = \sigma : (-\mathbf{1}\times\delta_1\omega) = -(\sigma\dot{\times}\mathbf{1})\cdot\delta_1\omega
\tag{14}
$$

其中在兩個張量間的算子$\dot{\times}$定義為 $\mathbf{ab}\dot{\times}\mathbf{cd} \equiv (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\times\mathbf{d})$。代回方程式(11)，可得連體之變分方程：

$$
\begin{aligned}
&\int_V (\rho \mathbf{f}^B + \nabla\cdot\boldsymbol{\sigma} - \rho\ddot{\mathbf{r}})\cdot\delta_1\mathbf{v}\, dV \\
&+ \int_V \boldsymbol{\sigma} : \delta_1\mathbf{D} \text{ - } (\boldsymbol{\sigma}\dot{\times}\mathbf{1})\cdot\delta_1\omega)\, dV \\
&+ \int_V (\rho\theta\delta_1\dot{S} - \frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})dV \\
&= \int_V (\rho\delta_1\dot{U} + \delta_1 R^W)dV.
\end{aligned}
\tag{15}
$$

### 2.4 熱黏性流體之運動方程

在上述架構下，我們即可討論前述虛功率原理在熱黏性流體上之應用。由於溫度較熵易於量測，我們選用自由能(free energy) $F = U - \theta S$ 取代內能$U$，因 $\dot{F} = \dot{U} - \theta\dot{S} - S\dot{\theta}$，變分方程式(15)可改寫成

$$
\begin{aligned}
&\int_V (\rho \mathbf{f}^B + \nabla\cdot\boldsymbol{\sigma} - \rho\ddot{\mathbf{r}})\cdot\delta_1\mathbf{v}dV \\
&+ \int_V (\boldsymbol{\sigma}\cdot\delta_1\mathbf{D} - (\boldsymbol{\sigma}\dot{\times}\mathbf{1})\cdot\delta_1\omega)dV \\
&+ \int_V (-\rho S\delta_1\dot{\theta} - \frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})dV \\
&= \int_V (\rho\delta_1\dot{F} + \delta_1 R^W)dV.
\end{aligned}
\tag{16}
$$

當討論系統函數時，我們必須考慮物體之架構中立原理(Principle of Frame Indifference)，即若連體承受一剛體直線運動或轉動時，其內能函數密度 $U$ 和內耗散函數密度 $R^W$ 應保持不變，亦即 $U$ 和 $R^W$ 與 $\delta_1\mathbf{v}$ 及 $\delta_1\omega$ 無關。因此我們假設自由能是比容$\nu(=1/\rho)$ 及溫度的函數，內耗散函數是拉伸張量、溫度及熱通量的函數：

$$
F = \hat{F}(\nu,\theta), \quad R^W = \hat{R}^W(\mathbf{D},\theta,\mathbf{q}).
\tag{17}
$$

將 $\dot{F}$ 及 $R^W$ 分別做 $\delta_1$ 變分操作可得：

$$
\begin{aligned}
\delta_1\dot{F} &= \frac{\partial\hat{F}}{\partial\nu}\delta_1\dot{\nu} + \frac{\partial\hat{F}}{\partial\theta}\delta_1\dot{\theta} \\
&= \frac{1}{\rho}\frac{\partial\hat{F}}{\partial\nu}(\mathbf{1}:\delta_1\mathbf{D}) + \frac{\partial\hat{F}}{\partial\theta}\delta_1\dot{\theta},
\end{aligned}
\tag{18a}
$$

$$\delta_1 R^W = \frac{\partial \hat{R}^W}{\partial \mathbf{D}} : \delta_1 \mathbf{D} + \frac{\partial \hat{R}^W}{\partial \mathbf{q}} \cdot \delta_1 \mathbf{q}. \tag{18b}$$

由於上式中 5 個虛變數 $\delta_1 \mathbf{v}$ 、 $\delta_1 \omega$ 、 $\delta_1 \mathbf{D}$ 、 $\delta_1 \dot{\theta}$ 及 $\delta_1 \mathbf{q}$ 皆是相互獨立的，則我們可得五個方程：

$$\rho \mathbf{f} + \nabla \cdot \boldsymbol{\sigma} - \rho \dot{\mathbf{v}} = 0 \tag{19a}$$

$$\boldsymbol{\sigma} \dot{\times} \mathbf{1} = 0 \tag{19b}$$

$$\boldsymbol{\sigma} = \frac{\partial \hat{F}}{\partial \nu} \mathbf{1} + \frac{\partial \hat{R}^W}{\partial \mathbf{D}}, \tag{19c}$$

$$S = -\frac{\partial \hat{F}}{\partial \theta} \tag{19d}$$

$$\frac{1}{\theta} \nabla \theta = -\frac{\partial \hat{R}^W}{\partial \mathbf{q}} \tag{19e}$$

上列第一式及第二式為柯西第一及第二運動方程，即線動量與角動量平衡方程，第三式為應力之組成律，由最後兩式可導得熱組成方程。

若內耗散函數可進一步分成黏性 $R^V$ 及熱量 $R^T$ 兩部分獨立函數，並假設黏性部分 $R^V$ 是拉伸張量 $\mathbf{D}$ 的二次項，表示如下：

$$\begin{aligned} R^W &= \hat{R}^V(\mathbf{D}, \theta) + \hat{R}^T(\theta, \mathbf{q}) \\ &= \frac{1}{2} \mathbf{D} : \mathbf{C}'(\theta) : \mathbf{D} + \hat{R}^T(\theta, \mathbf{q}) \end{aligned} \tag{20}$$

則應力張量可表示成

$$\boldsymbol{\sigma} = \frac{\partial \hat{F}}{\partial \nu} \mathbf{1} + \mathbf{C}' : \mathbf{D}. \tag{21}$$

一般討論黏性流體時，其應力張量可分為兩部分[1]：

$$\boldsymbol{\sigma} = -p \mathbf{1} + \boldsymbol{\sigma}^d \tag{22}$$

前者是等向的靜水壓力(hydrostatic pressure)，後者則是黏性造成的耗散部分。與組成方程(21)相比較，則可得

$$p = -\frac{\partial \hat{F}}{\partial \nu}(\nu, \theta), \tag{23a}$$

$$\boldsymbol{\sigma}^d = \mathbf{C}'(\theta) : \mathbf{D}. \tag{23b}$$

若此黏性流為等向(isotropic)，則應力之耗散部分可進一步表示成

$$\boldsymbol{\sigma}^d = \lambda'(\mathbf{1} : \mathbf{D})\mathbf{1} + 2\mu' \mathbf{D}, \tag{24}$$

其中 $\lambda'$ 及 $\mu'$ 分別為流體之體及剪黏性係數(bulk and shear viscosity)。(24)式與線動量平衡方程式(19a)可推導出黏性流之運動方程

$$\begin{aligned} &\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) \\ &= \rho \mathbf{f}^B - \nabla p + (\lambda' + \mu') \nabla \nabla \cdot \mathbf{v} + \mu' \nabla^2 \mathbf{v} \end{aligned} \tag{25}$$

此方程式加上 $3\lambda' + 2\mu' = 0$ 關係式，則可得到著名的 Navier-Stokes 方程。[1]

### 2.5 熱組成律

假設自由能函數可分為僅與比容或溫度相關的獨立函數，且後者為溫度的二次函數；另假設內耗散函數的熱量部分獨立函數為熱通量之二次函數：

$$F = \hat{F}_1(\nu) + \hat{F}_2(\theta) = \hat{F}_1(\nu) - \frac{1}{2} c \theta^2, \tag{26a}$$

$$R^W = \hat{R}^V(\mathbf{D}) + \hat{R}^T(\theta, \mathbf{q}) = \hat{R}^V(\mathbf{D}) + \frac{1}{2\theta} \mathbf{q}^T \Gamma \mathbf{q}, \tag{26b}$$

其中 $c$ 為等容比熱，$\Gamma$ 為熱阻係數矩陣。則由變分組成律(19d)及(19e)可得

$$S = -\frac{\partial \hat{F}}{\partial \theta} = -\frac{\partial \hat{F}_2}{\partial \theta}(\theta) = c\theta, \tag{27a}$$

$$-\frac{1}{\theta}\nabla\theta=\frac{\partial\hat{R}^T}{\partial\mathbf{q}}(\theta,\mathbf{q})=\frac{1}{\theta}\Gamma\mathbf{q} \tag{27b}$$

此二式為熱場上的組成方程，前者是從自由能推出熵和溫度的關係，後者是從內耗散函數推得熱通量和溫度的關係，令 $\kappa=\Gamma^{-1}$ 為熱傳導係數，可得 Fourier law

$$\mathbf{q}=-\kappa\nabla\theta \tag{28}$$

在新的熱力學理論—擴展不可逆熱力學(extended irreversible thermodynamics, EIT)[9]中，廣義的自由能密度函數，除了是比容與溫度的函數外，更增加了熱通量為其變數，即

$$F=\hat{F}(\nu,\theta,\mathbf{q}), \tag{29}$$

則可得

$$\delta_1\dot{F}=\delta_1\Big(\frac{\partial\hat{F}}{\partial\nu}\dot{\nu}+\frac{\partial\hat{F}}{\partial\theta}\dot{\theta}+\frac{\partial\hat{F}}{\partial\mathbf{q}}\dot{\mathbf{q}}\Big). \tag{30}$$

接下來我們進一步的假設 $F$ 可分解成相對於 ν、θ 和 $\mathbf{q}$ 的獨立函數：

$$\hat{F}=\hat{F}_1(\nu)+\hat{F}_2(\theta)+\hat{F}_3(\mathbf{q}), \tag{31}$$

則

$$\delta_1\dot{F}=\frac{\partial\hat{F}}{\partial\nu}\delta_1\dot{\nu}+\frac{\partial\hat{F}}{\partial\theta}\delta_1\dot{\theta}+\dot{\mathbf{q}}\cdot\delta_1\mathbf{G}^{\mathbf{q}}, \tag{32}$$

其中

$$\mathbf{G}^q\equiv\frac{\partial\hat{F}}{\partial\mathbf{q}}=\mathbf{G}^q(\mathbf{q}). \tag{33}$$

令 $\mathbf{G}^q$ 為 $\mathbf{q}$ 的線性函數：

$$\mathbf{G}^q=\Lambda^q\mathbf{q},$$

內耗散函數及自由能取虛變分時，皆會出現 $\delta_1\dot{\mathbf{q}}$ 變數，因此由方程式(27b)及方程式(32)可得

$$\tau\dot{\mathbf{q}}+\mathbf{q}=-\kappa\nabla\theta, \tag{34}$$

其中 $\tau=\rho\kappa\Lambda^q\theta$ ，稱為鬆弛時間 (relaxation time)。上式即為 Maxwell-Cattaneo 方程，乃廣義的溫度與熱通量的關係式。

### 2.6 微極性連體之組成律

微極性連體是一種具有微旋轉及旋轉慣量，且能承受偶應力及分佈力矩的材料。對於微極性連體，可做功的機械力包含了四個部分：單位質量的外力 $\mathbf{F}$、單位質量的力矩 $\mathbf{l}$、正應力 $\mathbf{t}$ 及偶應力 $\mathbf{c}$。另一方面，連體動能則包含了兩個部分：線性動能 $\int\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}dV$ 及轉動動能 $\int\frac{1}{2}\rho\mathbf{I}\cdot\boldsymbol{\omega}\cdot\boldsymbol{\omega}dV$ ，其中 $\mathbf{I}$ 是旋轉慣量。對於此類物質，我們有

$$\begin{aligned}\delta_1\mathcal{W}&=\int\rho\mathbf{F}\cdot\delta_1\mathbf{v}dV+\int\rho\mathbf{l}\cdot\delta_1\boldsymbol{\omega}dV\\&\quad+\int\mathbf{t}\cdot\delta_1\mathbf{v}dS+\int\mathbf{c}\cdot\delta_1\boldsymbol{\omega}dS\\&\quad-\int\rho\ddot{\mathbf{r}}\cdot\delta_1\mathbf{v}dV-\int(\mathbf{I}\cdot\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\omega})\cdot\delta_1\boldsymbol{\omega}dV,\end{aligned} \tag{35}$$

$$\delta_1\mathcal{Q}^W=\int\rho\theta\delta_1\dot{s}dV-\int\frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q}dV. \tag{36}$$

令 $\mathbf{t}=\mathbf{n}\cdot\boldsymbol{\sigma}$ 及 $\mathbf{c}=\mathbf{n}\cdot\mathbf{C}$. 因 $\delta_1$ 和 $\nabla$ 均為線性算子並可交換，我們可得區域變分形式

$$\begin{aligned}&\rho\mathbf{F}\cdot\delta_1\mathbf{v}+\rho\mathbf{l}\cdot\delta_1\boldsymbol{\omega}+(\nabla\cdot\boldsymbol{\sigma})\cdot\delta_1\mathbf{v}+\boldsymbol{\sigma}:\delta_1(\nabla\mathbf{v})\\&+(\nabla\cdot\mathbf{C})\cdot\delta_1\boldsymbol{\omega}+\mathbf{C}:\delta_1(\nabla\boldsymbol{\omega})-\rho\ddot{\mathbf{r}}\cdot\delta_1\mathbf{v}\\&-\rho(\mathbf{I}\cdot\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\omega})\cdot\delta_1\boldsymbol{\omega}+(\rho\theta\delta_1\dot{s}-\frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})\\&=\delta_1\dot{\mathcal{U}}+\delta_1\mathcal{R}^W.\end{aligned} \tag{37}$$

利用(12)及基本張量運算，我們可得

$$\boldsymbol{\sigma}:\delta_1\Omega=\boldsymbol{\sigma}:(-\mathbf{1}\dot{\times}\delta_1\boldsymbol{\omega})=-(\boldsymbol{\sigma}\dot{\times}\mathbf{1})\cdot\delta_1\boldsymbol{\omega}=-2\boldsymbol{t}^A\cdot\delta_1\boldsymbol{\omega}.$$

其中 $\boldsymbol{t}^A$ 為 $\boldsymbol{\sigma}$ 之軸向量(axial vector), $2\boldsymbol{t}^A=-\varepsilon_{ijk}\sigma_{jk}\mathbf{e}_i$ 。因此，(37)可表為

$$
\begin{aligned}
&\rho\mathbf{F}\cdot\delta_1\mathbf{v}+[(\nabla\cdot\boldsymbol{\sigma})-\rho\ddot{\mathbf{r}}]\cdot\delta_1\mathbf{v}+\boldsymbol{\sigma}:\delta_1\mathbf{D}+\rho\mathbf{l}\cdot\delta_1\boldsymbol{\omega}\\
&+[-2\boldsymbol{t}^A+(\nabla\cdot\mathbf{C})-\rho(\mathbf{I}\cdot\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\omega})]\cdot\delta_1\boldsymbol{\omega}\\
&+\mathbf{C}:\delta_1(\nabla\boldsymbol{\omega})+(\rho\theta\delta_1\dot{s}-\frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})\\
&=\delta_1\dot{\mathcal{U}}+\delta_1\mathcal{R}^W.
\end{aligned}\tag{38}
$$

因為 $\delta_1\mathbf{v}$ 及 $\delta_1\boldsymbol{\omega}$ 表示加諸於單位體之線運動及轉動，內能 $\dot{\mathcal{U}}$ 及耗散函數 $\mathcal{R}^W$ 應不受其影響。因此 $\delta_1\dot{\mathcal{U}}$ 和 $\delta_1\mathcal{R}^W$ 應與 $\delta_1\mathbf{v}$ 、 $\delta_1\boldsymbol{\omega}$ 無關，由變分方程(38)即可推論得:

$$\rho\mathbf{F}+(\nabla\cdot\boldsymbol{\sigma})-\rho\ddot{\mathbf{r}}=0, \tag{39}$$

$$\rho\mathbf{l}-2\boldsymbol{t}^A+(\nabla\cdot\mathbf{C})-\rho(\mathbf{I}\cdot\dot{\boldsymbol{\omega}}+\boldsymbol{\omega}\times\mathbf{I}\cdot\boldsymbol{\omega})=0, \tag{40}$$

$$
\begin{aligned}
&\boldsymbol{\sigma}:\delta_1\mathbf{D}+\mathbf{C}:\delta_1(\nabla\boldsymbol{\omega})+(\rho\theta\delta_1\dot{s}-\frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q})\\
&=\delta_1\dot{\mathcal{U}}+\delta_1\mathcal{R}^{W'}.
\end{aligned}\tag{41}
$$

我們可從(41)式推出組成率。轉換為自由能 $F=U-\theta s$ 型式，(41)式可表示為

$$
\begin{aligned}
&\boldsymbol{\sigma}:\delta_1\mathbf{D}+\mathbf{C}:\delta_1(\nabla\boldsymbol{\omega})-\rho s\delta_1\dot{\theta}-\frac{1}{\theta}\nabla\theta\cdot\delta_1\mathbf{q}\\
&=\rho\delta_1F+\delta_1\mathcal{R}^W.
\end{aligned}
$$

假設自由能為比體積 $\nu$ 、溫度 $\theta$ 及 $\mathbf{K}=\frac{1}{2}\nabla\nabla\times\mathbf{u}$ 的函數，即

$$F=\hat{F}(\nu,\mathbf{K},\theta).$$

而耗散函數則假設為熱通量 $\mathbf{q}$ 的函數，

$$\mathcal{R}^{W'}=\hat{\mathcal{R}}^W(\mathbf{q}).$$

取 $\dot{F}$ 及 $\mathcal{R}^W$ 之 $\delta_1$-變分可得

$$
\begin{aligned}
\delta_1\dot{F}&=\frac{\partial\hat{F}}{\partial\nu}:\delta_1\dot{\nu}+\frac{\partial\hat{F}}{\partial\mathbf{K}}:\delta_1\dot{\mathbf{K}}+\frac{\partial\hat{F}}{\partial\theta}\delta_1\dot{\theta}\\
&=\frac{1}{\rho}\frac{\partial\hat{F}}{\partial\nu}\mathbf{1}:\delta_1\mathbf{D}+\frac{\partial\hat{F}}{\partial\mathbf{K}}:\delta_1(\nabla\boldsymbol{\omega})+\frac{\partial\hat{F}}{\partial\theta}\delta_1\dot{\theta},
\end{aligned}
$$

$$\delta_1\mathcal{R}^W=\frac{\partial\hat{\mathcal{R}}^{W'}}{\partial\mathbf{q}}:\delta_1\mathbf{q}.$$

代入(41)式，利用變分的獨立性，我們可得微極性連體之組成律為

$$\boldsymbol{\sigma}=\frac{\partial\hat{F}}{\partial\nu}\mathbf{1},\quad \mathbf{C}=\rho\frac{\partial\hat{F}}{\partial\mathbf{K}},\quad s=-\frac{\partial\hat{F}}{\partial\theta},\quad -\frac{1}{\theta}\nabla\theta=\frac{\partial\hat{\mathcal{R}}^W}{\partial\mathbf{q}}.$$

## 三、 結論

本報告探討黏性流體與微極性連體在熱機過程中之行為，熱力學第一定律，方程式(1)，說明能量平衡，即物體內能的增加等於外界對此物體所做的功與所提供的熱之和。線動量平衡方程式(25)為其運動方程，與熱力學組成律(34)，再加上質量守恆定律，則能完整描述此系統。

在上述討論中，我們成功地將虛功率原理及其變分方程應用在熱黏性流體與微極性連體上，由五個獨立的虛變數，$\delta_1\mathbf{v}$，$\delta_1\omega$，$\delta_1\mathbf{D}$ ，$\delta_1\dot{\theta}$ 及 $\delta_1\mathbf{q}$ ，可推導出五個方程：線動量平衡式、角動量平衡式、應力與自由能和耗散函數的關係、熵與自由能關係及溫度與內耗散函數的關係。前兩者可得運動方程，後三者則可推導其組成律。

虛功率原理應用於連體上，主要是將所有變數：包含力學與熱學之變數一起整合，完成一統合力學與熱學之完整架構。我們只要依據其變分方程，選取合適之內能函數與內耗散函數，即可推導出運動方程及組成律。未來，我們將推廣到熱黏彈性流體(viscoelastic fluids with heat conduction)或微極性流體(micropolar fluids)上，以擴大虛功率原理之應用範圍。

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# Part II Global Backstepping Tracking Control for Car-Like Mobile Robots

## 1. Introduction

Rolling wheels are frequently installed to enhance the mobility of a robot such that the working space can be enlarged significantly. However, due to the appearance of the nonholonomic constraints, the motion planning and the tracking control of wheeled mobile robots are difficult to be managed. In the phase of motion planning [1, 2], a suitable trajectory is designed to connect the initial posture (i.e. the position and the orientation of the robot) and the final one such that no collisions with obstacles would occur and the kinematic constraints are satisfied. Once the optimal path is obtained, the navigation and control process enters the tracking phase in which the kinematics as well as the dynamical equations must be considered. To appropriately integrate the kinematics and the dynamics such that every path can be followed efficiently and globally is the main subject of this report.

Among various schemes in performing nonholonomic motion planning and stabilization, the transformation of the kinematic equations to chained form or skew-symmetric chained form attracts the interests of many researchers [3, 4] due to their simple structures. Based on the chained form, time-varying feedback [5], discontinuous feedback [6], and hybrid strategy [7] can be applied to circumvent the under-actuated problem. In particular, the idea of backstepping [8, 9] can be adopted to systematically design recursive algorithms for the stabilization of a multi-input chained system such as a fire truck [10], the tracking of a two-input chained system such as an articulated vehicle [5], or the tracking with saturation constraint for a class of unicycle mobile robot [11]. However, in transforming the kinematic equations to the chained form according to the algorithm given in [3], singularity problems may occur. In certain postures, the transformation becomes singular and thus the corresponding controller is not global [12]. This problem needs to be solved before the backstepping controller can be applied to all possible trajectories.

Most of the methods in nonholonomic motion planning only deal with the kinematics of the wheeled mobile robot. To perform tracking control, the mass and the moment of inertia of the vehicle cannot be ignored. Therefore, for the nonholonomically constrained mechanical systems, it is desired to develop a controller which takes the kinematic model and the dynamic model into account simultaneously. In [13, 14], a kinematic controller and a neural network computed torque controller are integrated to stabilizing a nonholonomic mobile robot in which uncertainty exists. The point stabilization problems were solved in [15, 16] by first transforming the kinematical equation into a skew-symmetric chained form, and then designing the adaptive controller for the combined system. In [17], a robust adaptive control scheme is proposed for the tracking control of wheeled mobile robot. However, it shall be seen in this paper that by suitably choosing state variables, the dynamical equations can be decoupled from the kinematics. This feature has not been observed in these works so that the previously designed controllers are more complex.

In this report, based on the decoupling feature of the underlying system, a global hierarchical tracking controller using backstepping idea is proposed. By selecting a proper set of privileged variables, it is seen that the reduced Appell equations discussed in [18] is decoupled from the kinematic equations. The tracking of a desired trajectory can thus be fulfilled by a kinematic compensator, which generates updated desired values for the privileged variables, and a dynamic controller, which issues control commands such that the new set of privileged variables can be followed. To resolve the above-mentioned singularity problem, it is found that two chained systems can be used to encompass all postures of the car-like mobile robot. A recursive backstepping controller can be designed to each chained system in its region of applicability. However, the compensations of the privileged variables may be severely discontinuous when the active chained system is changed. Therefore, a switching algorithm with associated continuation method which adjusts the control parameters is proposed in this paper to yield smooth signals while maintaining the desired performance. Simulation results performed in the paper show that the proposed scheme can be used effectively to track arbitrary paths.

## 2. Problem Description

The practical problem to be attacked in this paper is the tracking of a desired trajectory for a four-wheeled mobile robot, as shown in Fig. 1, moving on a horizontal plane. The system may be modeled by a platform with mass $m_c$, width $w$, length $l$, and height $h_c$, attached by four rolling-without-sliding wheels with equal masses $m_w$ and radius $a$. To simplify the analysis, we assume that the two wheels on each axle (front or rear) can be treated as a single wheel centered at the midpoint of the axle ($Q_f$ or $Q_r$, respectively), cf. Fig. 1. The approximated model is the so-called car-like mobile robot, which consists of a platform (body $c$), a front wheel (body $f$) and a rear wheel (body $r$). The masses of the rim of each wheel and platform are denoted by $m'_w$ and $m'_c$, respectively. Let $\rho_f$, $\rho_r$ be the distance between the point $Q_f$, $Q_r$ and the mass center of platform $C$, respectively. The contacts between the wheels and the ground are assumed to be pure rolling without slipping.

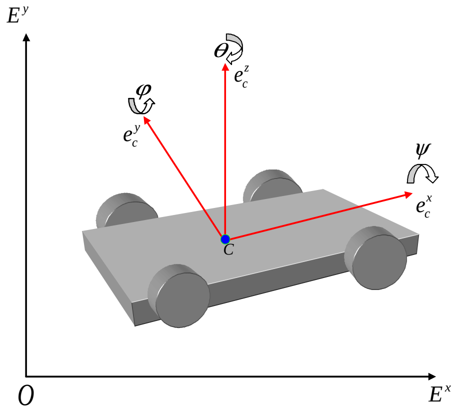


Fig. 1. A four-wheeled mobile robot.

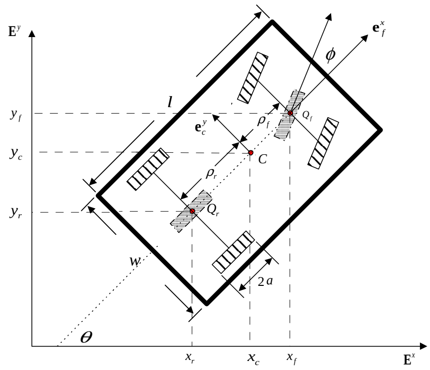


Fig. 2. The configuration of the car-like mobile robot.

The translational motion of body $i$ ($i = c, f, r$) may be described by the position of its mass center, which is expressed with respect to the inertial frame $\{\mathbf{E}^x, \mathbf{E}^y, \mathbf{E}^z\}$ as

$$\mathbf{r}_i^C = x_i\mathbf{E}^x + y_i\mathbf{E}^y + z_i\mathbf{E}^z. \tag{1}$$

To describe the rotational motion, the type (3-1-2) Eulerian angles are used to rotate the inertial frame $\{\mathbf{E}^x, \mathbf{E}^y, \mathbf{E}^z\}$ to $\{\mathbf{i}_i', \mathbf{j}_i', \mathbf{k}_i' = \mathbf{E}^z\}$ by the heading angle $\theta_i$, and then to $\{\mathbf{i}_i'' = \mathbf{i}_i', \mathbf{j}_i'', \mathbf{k}_i''\}$ by the camber angle $\psi_i$, and finally to the pre-specified body frame $\{\mathbf{e}_i^x, \mathbf{e}_i^y = \mathbf{j}_i'', \mathbf{e}_i^z\}$ of body $i$ by the spin angle $\varphi_i$. If the rotation is time-varying, the rates of change of the Eulerian angles are related to the angular velocity $\boldsymbol{\omega}_i$ by:

$$\boldsymbol{\omega}_i = \dot{\theta}_i\mathbf{E}^z + \dot{\psi}_i\mathbf{i}_i' + \dot{\varphi}_i\mathbf{j}_i''. \tag{2}$$

The six variables $(x_i, y_i, z_i, \psi_i, \varphi_i, \theta_i)$ are adopted here to describe the configuration of body $i$, and therefore there are eighteen variables to be specified for the system. However, due to physical constraints, the number required may be reduced. By the assumption that the motion is horizontal, we have (i) $z_r = a$, (ii) $z_f = a$, (iii) $z_c = h_c$. The platform is assumed to be kept horizontal as well so that (iv) $\varphi_c = 0$ and (v) $\psi_c = 0$, and hence the triad $\{\mathbf{e}_c^x, \mathbf{e}_c^y, \mathbf{e}_c^z\}$ coincides with $\{\mathbf{i}_c', \mathbf{j}_c', \mathbf{k}_c'\}$ and $\{\mathbf{i}_c'', \mathbf{j}_c'', \mathbf{k}_c''\}$ for the platform. If the wheels are properly aligned, the camber angles for the wheels vanish, (vi) $\psi_r = 0$, (vii) $\psi_f = 0$, so that $\{\mathbf{i}_i'', \mathbf{j}_i'', \mathbf{k}_i''\}$ coincides with $\{\mathbf{i}_i', \mathbf{j}_i', \mathbf{k}_i'\}$, for the wheels ($i = r, f$). Moreover, if the vehicle is FWD (front-wheel-driven), we have (viii) $\theta_r = \theta_c (\equiv \theta)$. From the geometry of the interconnected bodies (Fig. 1), we have the last four geometric constraints: (ix) $x_c = x_r + \rho_r\cos\theta$, (x) $y_c = y_r + \rho_r\sin\theta$, (xi) $x_f = x_r + \rho\cos\theta$, (xii) $y_f = y_r + \rho\sin\theta$, where $\rho = \rho_f + \rho_r$. Finally, the condition that the wheels roll without slipping is realized by the following velocity constraints:

$$\dot{x}_r = a\dot{\varphi}_r\cos\theta,\quad \dot{y}_r = a\dot{\varphi}_r\sin\theta,$$
$$\dot{x}_f = a\dot{\varphi}_f\cos(\theta+\phi),\quad \dot{y}_f = a\dot{\varphi}_f\sin(\theta+\phi),$$

where $\phi$ is the steering angle of front wheel ($\phi = \theta_f - \theta$). Applying the previous geometric constraints, the four kinematic constraints can be converted to the following independent ones:

$$\text{(xiii)}\ \dot{x}_r\sin\theta - \dot{y}_r\cos\theta = 0,$$
$$\text{(xiv)}\ \dot{x}_r\cos\theta + \dot{y}_r\sin\theta = r\dot{\varphi}_r,$$
$$\text{(xv)}\ \dot{x}_r\sin(\theta+\phi) - \dot{y}_r\cos(\theta+\phi) - d\dot{\theta}\cos\phi = 0,$$
$$\text{(xvi)}\ \dot{x}_r\cos(\theta+\phi) + \dot{y}_r\sin(\theta+\phi) + d\dot{\theta}\sin\phi = r\dot{\varphi}_f.$$

Based on these constraints, the angular velocities of the bodies in (2) can be simplified as

$$\boldsymbol{\omega}_c = \dot{\theta}\mathbf{E}^z, \ \ \boldsymbol{\omega}_r = \dot{\theta}\mathbf{E}^z + \dot{\varphi}_r \mathbf{j}_r^{''}, \ \ \boldsymbol{\omega}_f = (\dot{\theta}+\dot{\phi})\mathbf{E}^z + \dot{\varphi}_f \mathbf{j}_f^{''}. \tag{3}$$

It is desired to control the steering angle of the front wheel and the spin of the rear wheel by exerting torques $\tau_i, (i=1,2)$, respectively, such that the system is able to track a reference trajectory which satisfies the constraints. Due to the twelve geometric ones, the dimension of the system becomes six, and we may choose $\left(x_r, y_r, \theta, \varphi_r, \varphi_f, \phi\right)$ as the generalized coordinates. The desired trajectory may be then specified by ($x_{rd}(t)$, $y_{rd}(t)$, $\theta_d(t)$, $\varphi_{rd}(t)$, $\varphi_{fd}(t)$, $\phi_d(t)$), $t\in(0, t_f)$. Due to the four nonholonomic constraints, the degree of freedom of the system is further dropped to two.

# 3. Reduced Appell's Equation of Motion

The framework described in [8] for reduced Appell's equation is applied to derive the equations of motion. By choosing $\varphi_r$ and $\phi$ as the privileged coordinates, the dynamic equation can be expressed in matrix form as

$$\mathbf{M}(\mathbf{y})\ddot{\mathbf{y}} + \mathbf{C}(\mathbf{y},\dot{\mathbf{y}})\dot{\mathbf{y}} = \mathbf{B}(\mathbf{y})\boldsymbol{\tau} \tag{4}$$

where

$$\mathbf{M}(\mathbf{y}) = \begin{bmatrix} I_1\eta_a^2 \tan^2\phi + I_2 + I_w \sec^2\phi & \eta_a I_m \tan\phi \\ \eta_a I_m \tan\phi & I_m \end{bmatrix},$$

$$\mathbf{C}(\mathbf{y},\dot{\mathbf{y}}) = \begin{bmatrix} \left(I_1\eta_a^2 + I_w\right)\dot{\phi}\sec^2\phi\tan\phi & 0 \\ \eta_a I_m \dot{\phi}\sec^2\phi & 0 \end{bmatrix}, \ \mathbf{B}(\mathbf{y}) = \begin{bmatrix} 1 & \eta_a\tan\phi \\ 0 & 1 \end{bmatrix},$$

$$I_1 = (I_c + m_c\rho_r^2 + m_w\rho^2 + I_w), I_2 = (m_c a^2 + 2m_w a^2 + I_w),$$
$$I_m = m_w' a^2 \big/ 2, I_w = m_w' a^2, I_c = m_c'(w^2 + l^2)\big/3.$$

Various control schemes can be used to fulfill the objective of steering the privileged variables with (17). To deal with uncertainties on the parameters of the system, one may apply the idea of adaptive control for which some intrinsic properties of the reduced Appell equations is essential. For the dynamic systems, the following lemmas can be established.

***Lemma*** 1: $\mathbf{M}(\mathbf{y})$ is an $m\times m$ positive-definite symmetric matrix.

***Lemma*** 2. $\dot{\mathbf{M}}(\mathbf{y}) - 2\mathbf{C}(\mathbf{y},\dot{\mathbf{y}})$ is skew-symmetric.

If all the system parameters such as masses, moments of inertia, and physical specifications of the mobile robot, etc, are known, we may use traditional methods to steer the privileged coordinates. However, in many cases, some parameters are unknown or uncertain, which may form an uncertain parameter vector $\boldsymbol{\Theta} \in R^p$. By appropriate re-arrangement, we may express the left-hand side of (4) in the following linear parametric form

$$\mathbf{M}(\mathbf{y})\ddot{\mathbf{y}} + \mathbf{C}(\mathbf{y},\dot{\mathbf{y}})\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y},\dot{\mathbf{y}},\ddot{\mathbf{y}})\boldsymbol{\Theta}, \tag{5}$$

where $\mathbf{Y}(\cdot) \in R^{m\times p}$ is termed the regressor matrix [20], whose elements consist of known functions of $\mathbf{y}$, $\dot{\mathbf{y}}$, and $\ddot{\mathbf{y}}$. This form shall be used later to design the adaptive control law in the dynamic level. In particular, by choosing the unknown vector of parameters as

$$\boldsymbol{\Theta} = \begin{bmatrix} \eta_a^2 I_1 & I_2 & \eta_a I_m & I_w \end{bmatrix}^T, \tag{6}$$

the corresponding regression matrix $\mathbf{Y}$ in (5) is given by

$$\begin{bmatrix} \ddot{\varphi}_r \tan^2\phi + \dot{\varphi}_r\dot{\phi}\sec^2\phi\tan\phi & \ddot{\varphi}_r & \ddot{\phi}\tan\phi & \ddot{\varphi}_r\sec^2\phi + \dot{\varphi}_r\dot{\phi}\sec^2\phi\tan\phi \\ 0 & 0 & \ddot{\varphi}_r\tan\phi + \dot{\varphi}_r\dot{\phi}\sec^2\phi & \ddot{\phi}\big/2 \end{bmatrix}.$$

It is noted that $\mathbf{z}$ does not appear in (4), and hence the system of mobile robot is reducible so that the idea of the hierarchical control proposed in this paper is applicable.

## 4. Hierarchical Tracking Controller Design

As described in the previous section, the dynamics of a reducible mechanical system may be separated into two parts: the reduced Appell equation (4) and the corresponding kinematic equation relating the privileged velocities and the non-privileged ones. Since the reduced equations are decoupled from the kinematic equations, a controller may be designed to steer $\mathbf{y}$ to the desired $\mathbf{y}_d(t)$ independently. However, if the initial condition is not perfect or there are some disturbances during the motion, the desired non-privileged variables $\mathbf{z}_d(t)$ cannot be tracked. It is then necessary to invoke the kinematic relation to fulfill the control objective. To accommodate the constraints and take the advantage of the decoupling property of the mobile robot system, the idea of hierarchical tracking controller with three levels will be proposed. The specific structure with the backstepping controller in the kinematic level for the car-like robot is described below, with details given in the next section.

*A. Transformation to Chained Forms*

It is noted first that, in the tracking of the vehicle's motion, the front wheel is not the driven wheel and its rotation angle $\varphi_f$ is not concerned. In fact, for the planar motion of the vehicle, it is desired to track the non-privileged posture variables $(x_r, y_r, \theta)$, which is subject to the kinematic constraints

$$\dot{x}_r = a\dot{\varphi}_r \cos\theta,\ \dot{y}_r = a\dot{\varphi}_r \sin\theta,\ \dot{\theta} = \eta_a \dot{\varphi}_r \tan\phi. \tag{7}$$

The goal of the kinematic compensator is to find a set of new reference privileged velocities $\mathbf{u}_c (= \dot{\mathbf{y}}_c)$ such that desired posture variables can be followed. To construct a suitable Lyapunov function so that tracking can be assured, the idea of backstepping controller discussed in [8, 9] may be adopt, since the kinematic relation may be transformed into a chained form. To perform the transformation, the steering angle $\phi$ is added to form the state variable $\mathbf{x} = (x_1, x_2, x_3, x_4) = (x_r, y_r, \theta, \phi)$ due to its presence in (7), and the state equation becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} a\cos x_3 & 0 \\ a\sin x_3 & 0 \\ \eta_a \tan x_4 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}. \tag{8}$$

The technique of input-state linearization [19] is next used to conduct the transformation. By using the set of state transformation $\boldsymbol{\xi}(\mathbf{x}) = \boldsymbol{\Xi}(\mathbf{x})$ and input transformation $\mathbf{u} = \boldsymbol{\Psi}(\mathbf{x})\mathbf{v}$, where

$$\boldsymbol{\Xi}(\mathbf{x}) = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ \tan x_3 \\ \dfrac{1}{\rho}\tan x_4 \sec^3 x_3 \\ x_1 \end{bmatrix},$$

$$\boldsymbol{\Psi}(\mathbf{x}) = \begin{bmatrix} \dfrac{1}{a}\sec x_3 & 0 \\ \dfrac{-L_{\mathbf{g}_1}^{n-1}\xi_1}{L_{\mathbf{g}_2} L_{\mathbf{g}_1}^{n-2}\xi_1} & \dfrac{1}{L_{\mathbf{g}_2} L_{\mathbf{g}_1}^{n-2}\xi_1} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a}\sec x_3 & 0 \\ -\dfrac{3}{\rho}\sin^2 x_4 \tan x_3 \sec x_3 & \rho\cos^2 x_4 \cos^3 x_3 \end{bmatrix}, \tag{9}$$

system (8) can be transformed into the following chained form

$$\begin{cases} \dot{\xi}_1 = \xi_2 v_1 \\ \dot{\xi}_2 = \xi_3 v_1 \\ \dot{\xi}_3 = v_2 \\ \dot{\xi}_4 = v_1 \end{cases}. \tag{10}$$

The tracking problem now becomes to track the desired $\boldsymbol{\xi}_d$ by designing suitable input $\mathbf{v}$ through (10). However, singularity may occur around $\theta_d = \pm\pi/2$, where the tracking of the chain form (10) with the transformation (9) is not feasible. To overcome the above problem, we construct another set of coordinate transformation $\bar{\boldsymbol{\xi}} = \bar{\boldsymbol{\Xi}}(\mathbf{x})$ and input transformation $\mathbf{u} = \bar{\boldsymbol{\Psi}}(\mathbf{x})\bar{\mathbf{v}}$ by using the same procedures described before, as

$$\overline{\boldsymbol{\Xi}}(\mathbf{x}) = \begin{bmatrix} \overline{\xi}_1 \\ \overline{\xi}_2 \\ \overline{\xi}_3 \\ \overline{\xi}_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ \cot x_3 \\ -\dfrac{1}{\rho}\tan x_4 \csc^3 x_3 \\ x_2 \end{bmatrix},$$

$$\overline{\boldsymbol{\Psi}}(\mathbf{x}) = \begin{bmatrix} \dfrac{1}{a}\csc x_3 & 0 \\ \dfrac{3}{\rho}\sin^2 x_4 \cot x_3 \csc x_3 & -\rho\cos^2 x_4 \sin^3 x_3 \end{bmatrix}, \tag{11}$$

such that the state equation can be transformed into another set of chained form

$$\begin{cases} \dot{\overline{\xi}}_1 = \overline{\xi}_2 \overline{v}_1 \\ \dot{\overline{\xi}}_2 = \overline{\xi}_3 \overline{v}_1 \\ \dot{\overline{\xi}}_3 = \overline{v}_2 \\ \dot{\overline{\xi}}_4 = \overline{v}_1 \end{cases} \tag{12}$$

It is observed that singularity occurs when $\theta_d = 0, \pm\pi$. Thus, the set of equations (10) is complementary to that of (12), which shall be used interchangeably according to the following switching mechanism.

*B. Switching Mechanism*

To avoid the singularity arising in the tracking process, we adopt two sets of complementary chained form systems to design the backstepping controller. From the above discussion, If $\theta$ is closed to $k\pi (k = 0, \pm 1)$, it is desirable to adopt the $\boldsymbol{\xi}$ subsystem to design the kinematic compensator $\mathbf{u}_c(t)$. On the other hand, if $\theta$ is closed to $\pm(\pi/2)$, then one should change the chained form to the one with $\overline{\boldsymbol{\xi}}$. Depending on the heading angle $\theta$ of the mobile robot, a mechanism is designed to perform the switching, which is divided into two phases. As $\theta$ increases, cf. Figure 3(a), the switch from $\boldsymbol{\xi}$-system to $\overline{\boldsymbol{\xi}}$-system occurs at $\theta = \pi/3, -2\pi/3$, and that from $\overline{\boldsymbol{\xi}}$-system to $\boldsymbol{\xi}$-system at $\theta = 5\pi/6, -\pi/6$. For example, as $\theta$ increases from $\pi/6$ and enters the region $\theta \geq \pi/3$, the system which generates $\mathbf{u}_c(t)$ shall be changed from (10) to (12). On the other hand, as $\theta$ decreases, the switching occurs at $\theta = \pi/6, -\pi/3, -5\pi/6, 2\pi/3$, as shown in Figure 3(b).

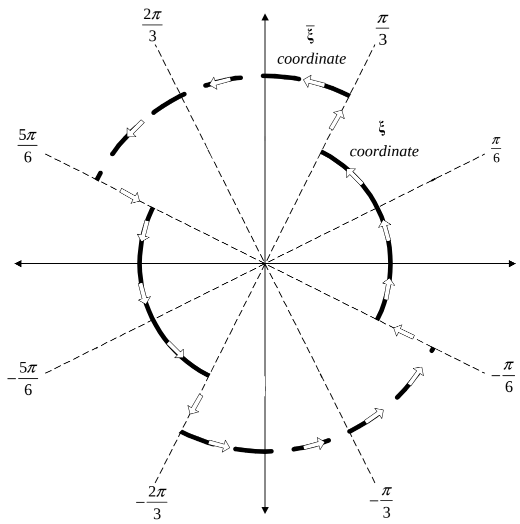


Fig. 3(a): The strategy as $\theta$ increases.

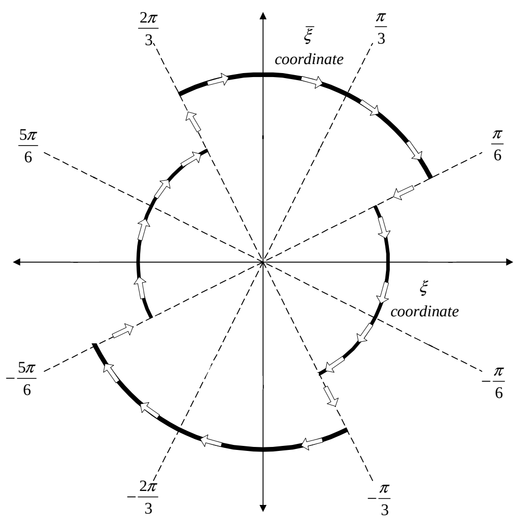


Fig. 3(b): The strategy as $\theta$ decreases.

The un-symmetric patterns are imposed to prevent the chattering phenomenon. If $\theta$ enters the region $\theta \geq \pi/3$ and moves back to the region $\pi/6 < \theta < \pi/3$, the active system remains (12) until $\theta$ goes below $\pi/6$. Therefore, if $\theta$ moves back and forth around a switching angle, no switching happens except the first one.

*C. Hierarchical Tracking Control Scheme*

Based on the switching algorithm, the overall design of the hierarchical scheme is depicted in Figure 4. On the top level, the motion planner produces the desired trajectory $\left(x_{rd}(t), y_{rd}(t)\right)$ according to task requirements and the conditions of constraints. The switching algorithm determines which set of desired values, either $\xi_{id}(t)$ or $\bar{\xi}_{id}(t), i=1,2,3,4,$ are computed from either Eq. (30) or (33), respectively. For the active chained system, the backstepping technique is then used to design suitable $\mathbf{v}_c(t)$ or $\bar{\mathbf{v}}_c(t)$, from which the compensation $\mathbf{u}_c(t)$ is obtained by using the transformation $\boldsymbol{\Psi}$ or $\bar{\boldsymbol{\Psi}}$. A continuation method is established to make the switching smooth, with details given in Section 5. The updated desired privileged coordinates $\mathbf{y}_c$ are then found, which is fed along with $\mathbf{u}_c$ to the dynamical controller in the bottom level as discussed in Section 6. It will be shown that with such a hierarchical design, including privileged or state variables, all the variables can be steered to the desired value asymptotically.

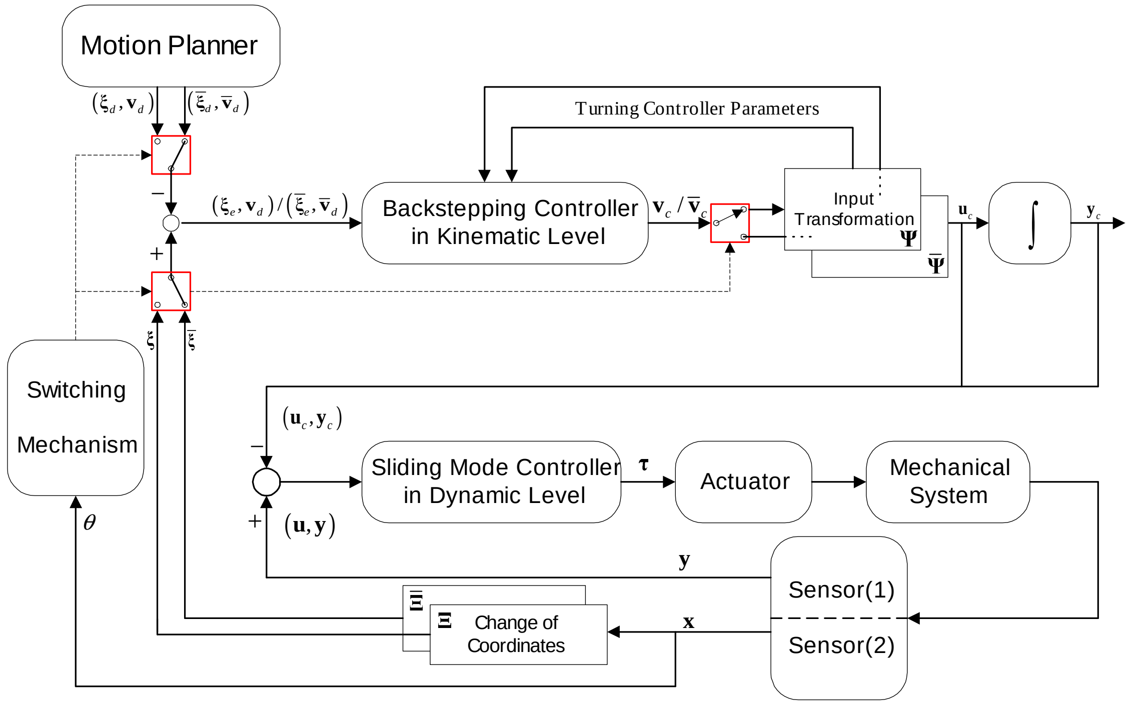


Fig. 4. The block diagram of hierarchical control design.

## 5. Continuation Method in Kinematic Compensator Design

After the state equation being transformed into the chained forms, the technique of backstepping can be applied to design an effective controller. While the method discussed in [5] may be adopted, a new algorithm is developed here to efficiently generate the controller.

*A. Backstepping Controller Design*

Consider a general 2-input chained system in the following form,

$$\begin{cases} \dot{\xi}_i = \xi_{i+1} v_1, & (1 \le i \le n-2) \\ \dot{\xi}_{n-1} = v_2, & \\ \dot{\xi}_n = v_1. & \end{cases} \tag{13}$$

To track the desired states $\xi_{id}$, $i = 1, 2, \cdots, n,$ and control inputs $v_{id}$, $i = 1, 2,$ for which (13) also holds, we first define $\xi_{ie} = \xi_i - \xi_{id}$ $(i = 1, 2, \cdots, n)$, and derive the tracking error equations as

$$\dot{\boldsymbol{\xi}}_e = \begin{bmatrix} \dot{\xi}_{1e} \\ \dot{\xi}_{2e} \\ \vdots \\ \dot{\xi}_{(n-1)e} \\ \dot{\xi}_{ne} \end{bmatrix} = \begin{bmatrix} \dot{\xi}_1 - \dot{\xi}_{1d} \\ \dot{\xi}_2 - \dot{\xi}_{2d} \\ \vdots \\ \dot{\xi}_{n-1} - \dot{\xi}_{(n-1)d} \\ \dot{\xi}_n - \dot{\xi}_{nd} \end{bmatrix} = \begin{bmatrix} \xi_{2e} v_1 + \xi_{2d}(v_1 - v_{1d}) \\ \xi_{3e} v_1 + \xi_{3d}(v_1 - v_{1d}) \\ \vdots \\ (v_2 - v_{2d}) \\ (v_1 - v_{1d}) \end{bmatrix}. \tag{14}$$

The goal is to find a time-varying controller,

$$\mathbf{v_c} = \begin{bmatrix} v_{1c} \\ v_{2c} \end{bmatrix} = \hat{\mathbf{v}}_{\mathbf{c}}(\boldsymbol{\xi}_e, v_{1d}, v_{2d}), \tag{15}$$

such that the tracking error $\boldsymbol{\xi}_e$ converges to zero asymptotically, i.e., $\lim_{t\to\infty}\|\boldsymbol{\xi}-\boldsymbol{\xi}_d\|=0$.

The idea of backstepping is used to systematically construct the Lyapunov function so that the asymptotical stability can be assured. Following the standard procedure, we obtain the control law

$$v_2 = v_{2d} - k_2\chi_{n-1} + (\beta_{n-2} - \chi_{n-2})v_1,$$
$$v_1 = v_{1d} - k_1\Delta_n,$$

where $\Delta_n = \left(\sum_{j=1}^{n-2}\chi_j\xi_{(j+1)d} - \sum_{j=1}^{n-1}\chi_j\gamma_{(j-1)} + k_3\xi_{ne}\right)$. By performing the Lyapunov stability analysis, we can show that the closed-loop system is stable.

*B. Continuation Method*

The control law developed above for general 2-input chained systems is now used to design the kinematic compensator for the car-like mobile robot. As discussed in Section IV, two chained systems (10) and (12) are used interchangeably depending on the heading of the vehicle. The respective control laws are ($n$ = 4)

$$\begin{cases} v_1 = v_{1d} - k_1(\chi_1\xi_{2d} + \chi_2\xi_{3d} + \chi_3\xi_{2d} + k_3\xi_{4e}) = v_{1d} - k_1\Delta_4, \\ v_2 = v_{2d} - k_2\chi_3 - 2\chi_2 v_1, \end{cases} \tag{16}$$

and

$$\begin{cases} \bar{v}_1 = \bar{v}_{1d} - \bar{k}_1(\bar{\chi}_1\bar{\xi}_{2d} + \bar{\chi}_2\bar{\xi}_{3d} + \bar{\chi}_3\bar{\xi}_{2d} + \bar{k}_3\bar{\xi}_{4e}) = \bar{v}_{1d} - \bar{k}_1\bar{\Delta}_4, \\ \bar{v}_2 = \bar{v}_{2d} - \bar{k}_2\bar{\chi}_3 - 2\bar{\chi}_2\bar{v}_1. \end{cases} \tag{17}$$

With appropriated designed controller parameters $k_i^*,\ \bar{k}_i^*,\ i=1,2,3,$ the updated desired values $\mathbf{u}_c$ for the privileged variables are then computed from $\bar{\boldsymbol{\Psi}}(\mathbf{x})\bar{\mathbf{v}}_c$ and $\boldsymbol{\Psi}(\mathbf{x})\mathbf{v}_c$, respectively. However, large discontinuity may appear when the active chained system is switched, which may lead to the failure of the adaptive controller in the dynamic level. To solve this problem, the following continuation method is proposed. First, we note that if $\boldsymbol{\xi}$-system is active, the computed $\mathbf{u}_c$ is related to the control parameters $k_1,\ k_2$ according to the following formula

$$\mathbf{u}_c = \boldsymbol{\Psi}(\mathbf{x})(\mathbf{FK}+\mathbf{G}), \tag{18}$$

where

$$\mathbf{F} = \begin{bmatrix} -\Delta_4 & 0 \\ 2\chi_2\Delta_4 & -\chi_3 \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} v_{1d} \\ v_{2d} - 2\chi_2 v_{1d} \end{bmatrix},\ \mathbf{K} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

On the other hand, if $\bar{\boldsymbol{\xi}}$-system is active, we have

$$\mathbf{u}_c = \bar{\boldsymbol{\Psi}}(\mathbf{x})(\bar{\mathbf{F}}\bar{\mathbf{K}}+\bar{\mathbf{G}}), \tag{19}$$

where

$$\bar{\mathbf{F}} = \begin{bmatrix} -\bar{\Delta}_4 & 0 \\ 2\bar{\chi}_2\bar{\Delta}_4 & -\bar{\chi}_3 \end{bmatrix},\ \bar{\mathbf{G}} = \begin{bmatrix} \bar{v}_{1d} \\ \bar{v}_{2d} - 2\bar{\chi}_2\bar{v}_{1d} \end{bmatrix},\ \bar{\mathbf{K}} = \begin{bmatrix} \bar{k}_1 \\ \bar{k}_2 \end{bmatrix}.$$

Recall that in the above analysis, the asymptotic stability is guaranteed if $k_1>0,\ k_2>0.$ Therefore, we may adjust the control parameters in the positive region to make $\mathbf{u}_c$ continuous during switching. If $\mathbf{F}$ or $\bar{\mathbf{F}}$ becomes singular, which means that $\chi_3$ or $\Delta_4$ is zero, both (18) and (19) lead to the original desired value of $\mathbf{u}$, and the value of $\mathbf{K}$ or $\bar{\mathbf{K}}$ becomes immaterial. We simply set $\mathbf{K}=\mathbf{K}^*$ or $\bar{\mathbf{K}}=\bar{\mathbf{K}}^*$. On the other hand, if the slack variables $\chi_i$ are away from zero, both $\mathbf{F}$ and $\bar{\mathbf{F}}$ are nonsingular. We may try to find the appropriate control parameters such that the difference between (18) and (19) is minimized. Nevertheless, the performance of the controller may be sacrificed, and a mechanism must be used after switching to drive the control parameters to their designed value $k^*$'s. Detailed process for switching from $\bar{\xi}$-system to $\xi$-system consists of the following two steps.

Step 1: Given $\bar{\mathbf{K}}$, find the control parameters $\mathbf{K}_0$.

The problem is converted to the constrained optimization program:

$$\begin{cases} \min\limits_{\mathbf{K}} \ \frac{1}{2}(\mathbf{PK}-\mathbf{Q})^T(\mathbf{PK}-\mathbf{Q}), \\ \quad subject\ to\ k_1>0,\ k_2>0, \end{cases} \tag{20}$$

where $\mathbf{P}=\mathbf{\Psi F},\ \mathbf{Q}=\bar{\mathbf{\Psi}}\bar{\mathbf{F}}\bar{\mathbf{K}}+\bar{\mathbf{\Psi}}\bar{\mathbf{G}}-\mathbf{\Psi G}.$ This is a problem of quadratic programming, and the cost function can be further simplified as

$$J(\mathbf{K})=\frac{1}{2}[\mathbf{K}-\mathbf{H}^{-1}\mathbf{N}]^T\mathbf{H}[\mathbf{K}-\mathbf{H}^{-1}\mathbf{N}],$$

where $\mathbf{H}=\mathbf{P}^T\mathbf{P}$ is a symmetric matrix and $\mathbf{N}=\mathbf{P}^T\mathbf{Q}$. The non-singularity of $\mathbf{F}$ implies that $\mathbf{H}$ is nonsingular. Let

$$\begin{bmatrix} k_1' & k_2' \end{bmatrix}^T=\mathbf{K}'=\mathbf{H}^{-1}\mathbf{N}=(\mathbf{P}^T\mathbf{P})^{-1}\mathbf{P}^T\mathbf{Q}.$$

The objective function is then expressed as

$$J(k_1,k_2)=\frac{1}{2}\left(H_{11}(k_1-k_1')^2+H_{22}(k_2-k_2')^2+2H_{12}(k_1-k_1')(k_2-k_2')\right),$$

where $H_{ij}$ denotes the $(i,j)$-component of $\mathbf{H}$. It is easily seen that if $k_1'$ and $k_2'$ are both positive, they are the solutions, i.e. $k_{10}=k_1',\ k_{20}=k_2'$. If they are both negative, the minimum occurs at $(0,0)$ and we set $k_{10}=\varepsilon,\ k_{20}=\varepsilon$, where $\varepsilon$ is a very small positive number. Alternatively, if $k_1'\le 0$ and $k_2'>0$, we set $k_{10}=\varepsilon$ and search for the optimal $k_{20}>0$ such that the cost function restricted to the axis $k_1=0$ is minimized. If the corresponding solution $k_2'+k_1'H_{12}/H_{22}$ is positive, it is set to be the value of $k_{20}$. Otherwise, we set $k_{20}=\varepsilon$. Similar treatment is applied to the case that $k_1'>0$ and $k_2'\le 0$, for which the solutions are

$$k_{20}=\varepsilon,\ \ k_{10}=\begin{cases} k_1'+k_2'\dfrac{H_{12}}{H_{11}},\ if\ k_1'+k_2'\dfrac{H_{12}}{H_{11}}>0, \\ \qquad \varepsilon,\ \text{otherwise}. \end{cases} \tag{21}$$

Step 2: Design a mechanism such that $\mathbf{K}$ starting from $\mathbf{K}_0$ approaches $\mathbf{K}^*$. A simple recursive scheme

$$\begin{cases} \mathbf{K}(0)=\mathbf{K}_0, \\ \mathbf{K}(i+1)=\mu(\mathbf{K}^*-\mathbf{K}(i))+\mathbf{K}(i), \end{cases} \tag{22}$$

may be used to drive the control parameters to $\mathbf{K}^*$ with rate of convergence μ. Larger μ implies that the desired $\mathbf{K}^*$ is reached faster but the performance may be worse due to the rapidly changing of $\mathbf{u}_c$.

Analogous process can be used to perform the switching from $\xi$-system to $\bar{\xi}$-system. It shall be shown later by simulation that the proposed scheme can solve the problems with singularity and discontinuity such that the global tracking is feasible.

## 6. Sliding Mode Control in Dynamical Level

According to the discussions in the previous two sections, the output of the global kinematic compensator using the idea of back-stepping and the continuation method is a set of updated privileged velocities $\mathbf{u}_c$, which is fed into the dynamic controller to complete the tracking process. Among various methods, the sliding mode controller is chosen here to deal with the uncertainties in the car-like mobile robot system. The decoupling of the reduced dynamics from the kinematic equations makes it possible to design the dynamical controller independently. However, if there is no kinematic compensator and $\mathbf{u}$ is driven to $\mathbf{u}_d = \Psi\mathbf{v}_d$, the tracking of the posture of the robot cannot be fulfilled due to the absence of $(x_r, y_r, \theta)$ in the reduced dynamics although the privileged variables can be tracked. The information of the posture is used in the kinematic compensator to give the direction to which the privileged coordinates should be driven. The relation of the kinematic compensator and the dynamic controller is very similar to that of the navigator and the pilot in steering an airplane.

The reduced dynamics of the privileged variables is given in (4), which may be re-written in the following form:

$$\begin{cases} \dot{\mathbf{y}} = \mathbf{u}, \\ \mathbf{M}(\mathbf{y})\dot{\mathbf{u}} + \mathbf{C}(\mathbf{y},\dot{\mathbf{y}})\mathbf{u} = \mathbf{B}(\mathbf{y})\boldsymbol{\tau}. \end{cases} \tag{23}$$

To accommodate the uncertainties in the knowledge of system parameters, an adaptive sliding mode controller is adopted to design the control torque $\boldsymbol{\tau}$ such that the tracking errors $\tilde{\mathbf{u}}(t) \equiv \mathbf{u}(t) - \mathbf{u}_c(t) \to 0$ and $\tilde{\mathbf{y}}(t) \equiv \mathbf{y}(t) - \mathbf{y}_c(t) \to 0$ as t→∞. Applying the idea of sliding mode control [20], the sliding surface is given by **s** = 0 where the sliding variable **s** is chosen as

$$\mathbf{s} = [s_1 \quad s_2]^T = \tilde{\mathbf{u}} + \Lambda\tilde{\mathbf{y}}, \tag{24}$$

where $\boldsymbol{\Lambda}$ is a positive definite matrix. The sliding variable may be further written as $\mathbf{s} = \mathbf{u} - \mathbf{u}_s$, where the auxiliary variable $\mathbf{u}_s = [\dot{\varphi}_{rs} \quad \dot{\phi}_s]^T = \mathbf{u}_c - \boldsymbol{\Lambda}\tilde{\mathbf{y}}$. With the unknown parameter vector **Θ** given in (6), it is desired to drive the system toward the sliding surface, i.e. the sliding variable **s**→0. By taking the difference between Eq. (68) and the second equation in (23), we obtain the evolution equation of the sliding variable as

$$\mathbf{M}(\mathbf{y})\dot{\mathbf{s}} + \mathbf{C}(\mathbf{y},\dot{\mathbf{y}})\mathbf{s} = \mathbf{B}(\mathbf{y})\boldsymbol{\tau} - \mathbf{Y}_s(\mathbf{y},\dot{\mathbf{y}},\mathbf{u}_s,\dot{\mathbf{u}}_s)\boldsymbol{\Theta}\,. \tag{25}$$

The control torque must depend on the unknown parameter vector **Θ**, and hence an estimation for **Θ,** i.e. $\hat{\boldsymbol{\Theta}}$, needs to be available. Let $\tilde{\boldsymbol{\Theta}} = \hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}$ be the estimation error. Since $\boldsymbol{\Theta}$ is constant, we have $\dot{\tilde{\boldsymbol{\Theta}}} = \dot{\hat{\boldsymbol{\Theta}}}$. The control and adaptive law can be chosen as

$$\begin{cases} \boldsymbol{\tau} = \mathbf{B}^{\#}(\mathbf{y})[\mathbf{Y}_s(\mathbf{y},\dot{\mathbf{y}},\mathbf{u}_s,\dot{\mathbf{u}}_s)\hat{\boldsymbol{\Theta}} - \mathbf{K}_s\mathbf{s}], \\ \dot{\hat{\boldsymbol{\Theta}}} = -\boldsymbol{\Gamma}^{-1}\mathbf{Y}(\mathbf{y},\dot{\mathbf{y}},\mathbf{u}_s,\dot{\mathbf{u}}_s)^T\mathbf{s}, \end{cases} \tag{26}$$

where $\mathbf{B}^{\#}$ is the left inverse of **B,** and $\mathbf{K}_s$ and $\boldsymbol{\Gamma}$ are positive-definite matrix. With such laws, it can be shown then that the privileged variables can be steered to their respective desired values. Details of the proof can be found in [21].

## 7. Simulation Results

To examine the effectiveness of the proposed global backstepping tracking control methodology, computer simulations for a car-like mobile robot were performed. The system parameters of a large vehicle shown in Fig. 1 were selected as $a$ = 0.3$m$, $\rho_r$ = 0.5$m$, $\rho_f$ = 0.75$m$, $\ell$ = 1.75$m$, $w$ = 1.5$m$, $m_c$ = 20$kg$, $m_c' = 6kg,\ m_w = 1\ kg,\ m_w' = 2kg$. The desired trajectory ($x_{rd}(t)$, $y_{rd}(t)$) is obtained by finding a cubic *B*-spline function (cf. [20]) passing through 12 intermediate points, {(-15,-5), (-13,4), (-10,11), (-8,13.5), (-5,15), (-2,13), (-0.5,10), (2,6), (7,8), (7,15), (2,17), (-2,13)}. Assume that the current values of the states are available so that the slack variables $\chi$ or $\bar{\chi}$ can be obtained. By choosing the sets of control parameters as $(k_1^* = 16,\ k_2^* = 24,\ k_3^* = 3)$ and $(\bar{k}_1^* = 36,\ \bar{k}_2^* = 24,\ \bar{k}_3^* = 3)$, we find the controls $\mathbf{v}_c$ and $\bar{\mathbf{v}}_c$ for the $\xi$-system and the $\bar{\xi}$-system by the laws (16) and (17), respectively, as described in Section V.

Which one of $\mathbf{v}_c$ and $\bar{\mathbf{v}}_c$ is used to generate the updated privileged velocity $\mathbf{u}_c = (\dot{\varphi}_{rc}, \dot{\phi}_c)$ depends on the heading angle $\theta$ according to the switching mechanism discussed in Section IV. To make $\mathbf{u}_c$ continuous before and after switching, the continuation method described in Section V is applied with the parameter $\mu = 0.001$. The sliding mode controller described in Section VI is then invoked to track the privileged variables to the updated desired values adaptively and asymptotically with the parameters $\mathbf{K}_s$ = *diag*{30, 30}, **Λ** = *diag*{4, 4}, and **Γ** = *diag*{100, 100, 100} in (26).

To signify the adaptive performance of the controller, the initial values for the estimator is selected as $\hat{\boldsymbol{\Theta}}(0) = [0,0,0]^T$, which is different from the true value $\boldsymbol{\Theta}_{\text{true}}$ = [0.3, 2.1, 0.4]$^T$. It is further assumed that initially $x_r(0) = -10,\ y_r(0) = -5, \theta(0) = 0^o, \phi(0) = 0^o$, which is away from the desired ones, i.e. (-15, -5, $45^o$, $35.6^o$). For the above-described scenario, simulation results are shown in Figs. 5 to 8. In Fig. 5, the solid line is the desired *B*-spline curve, and the shaded block line shows the tracking performance. The switch of active trained system between the $\xi$-system and the $\bar{\xi}$-system is shown in Fig. 6. It is shown that while the initial condition is significantly away from the desired posture, the hierarchical control scheme proposed here can successfully steer the mobile robot to the desired trajectory, with the tracking errors of state variables $(x_r, y_r)$ and $(\theta, \phi)$ being plotted in Fig. 7 and Fig. 8, respectively. Without switching mechanism, either system alone cannot be used to track the selected trajectory where the heading angle of the vehicle varies from 0 to 2π and singularity must appear at some point for either chained system.

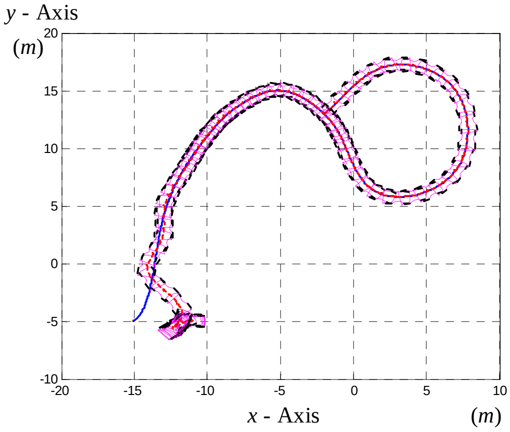


Fig. 5: *B*-spline trajectory tracking Performance.

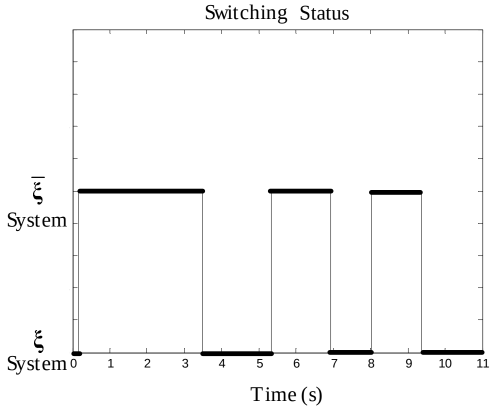


Fig. 6: Switching status.

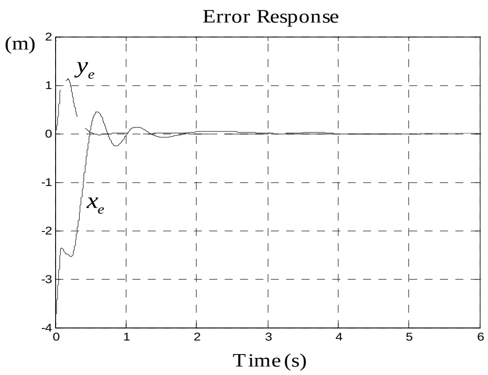


Fig. 7: Tracking error of state variables

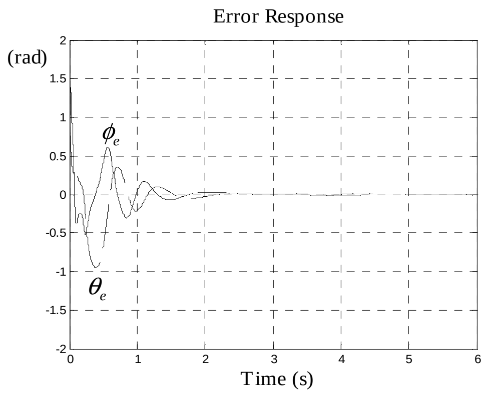


Fig. 8: Tracking error of state variables

## 8. Conclusions

A global tracking controller was designed for a car-like mobile robot after its model being established based on the reduced Appell equation, which is decoupled from the kinematics for the tacitly selected privileged variables. The advantage of the decoupling feature was taken in the hierarchical design in which the control scheme is separated into the kinematic compensator and the dynamic controller. The updated reference values for the privileged variables were obtained from the compensator were fed into a sliding mode controller for the reduced dynamics to steer the privileged variables and all the other variables shall follow suit due to the nature of the system. This concept is quite different from that of the two-stage controllers discussed in [15, 16, 17] such that it is possible to run the controller and the compensator at different sampling rates in our design. Our design scheme may be even more beneficial in dealing with more complicated systems. A mobile robot in fact consists of many components interacting with each other and the Lagrangian formulation may not be tractable. The formulation of Appell's equation is much more transparent and leads to a suitable structure for controller design as shown in this paper.

In the development of the kinematic compensator, the state equation was first transformed into chained forms such that the structure becomes simpler and the idea of backstepping can be applied directly. To deal with the singularity problem arising from the transformations, a switching mechanism based on the posture of the robot was proposed. A continuation algorithm was then invoked to yield continuous changing of the desired privileged variables by adjusting the control parameters. The performance of the controller is maintained by the approaching method such that the control parameters reach their specified values in due course. In previous works, it is more emphasized on the techniques for the controller design of chained systems, while the singularity problem has not been addressed. According to the simulation results, the proposed methodology successfully integrates the kinematic constraints and the dynamics to generate practical control command to track all the trajectories for the car-like mobile robot.

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