A Laplace transform power series solution for solute transport in a convergent flow field with scale-dependent dispersion

Jui-Sheng Chen,[1] Chen-Wuing Liu,[2] Hui-Tsung Hsu,[1] and Chung-Min Liao[2]

Received 30 April 2003; accepted 30 May 2003; published 30 August 2003.

[1] This study presents a novel mathematical model to describe solute transport in a radially convergent flow field with scale-dependent dispersion. The scale-dependent advection-dispersion equation in cylindrical coordinates derived based on the dispersivity is assumed to increase linearly with the distance of the solute transported from its input source. The Laplace transformed power series technique is applied to solve the radially scale-dependent advection-dispersion equation with variable coefficients. Breakthrough curves obtained using the scale-dependent dispersivity model are compared with those from the constant dispersivity model to illustrate the features of scale-dependent dispersion in a radially convergent flow field. The comparison results reveal that the constant dispersivity model can produce a type curve with the same shape as that from the proposed scale-dependent dispersivity model. This correspondence in type curves between the two models occurs when the product of the Peclet number used in the constant dispersivity model and the dispersivity/distance ratio used in the scale-dependent dispersivity model equals 4. Finally, the scale-dependent dispersivity model is applied to a set of previously reported field data to investigate the linearly scale-dependent dispersion effect. The analytical results reveal that the linearly scale-dependent dispersion model is applicable to this test site. *INDEX TERMS:* 1894 Hydrology: Instruments and techniques; 1832 Hydrology: Groundwater transport; 1860 Hydrology: Runoff and streamflow; *KEYWORDS:* scale-dependent dispersion, radially convergent flow filed, Laplace transformed power series technique

Citation: Chen, J.-S., C.-W. Liu, H.-T. Hsu, and C.-M. Liao, A Laplace transform power series solution for solute transport in a convergent flow field with scale-dependent dispersion, *Water Resour. Res.*, *39*(8), 1229, doi:10.1029/2003WR002299, 2003.

1. Introduction

[2] The advection-dispersion equation is the most common method of modeling solute transport in subsurface porous media. This equation includes terms that describe the physical process governing advective transport with flowing groundwater, molecular diffusion, hydrodynamic dispersion, and equilibrium or nonequilibrium exchange with the solid phase if reactive solutes are involved. The advection-dispersion equation in the radial coordinates refers to the problem of analyzing the dispersive transport of a contaminant or tracer in the steady radial flow from a recharge or extraction well that fully penetrates a homogeneous confined aquifer with uniform thickness and infinite lateral extent. Therefore this equation is important in studying solute transport in a radially convergent tracer test or aquifer decontamination by pumping. Several researchers have obtained closed-form analytical or semianalytical solutions to radial advection-dispersion equations for both conservative and sorbing solute transport in the divergent/convergent tracer test, and also for wastewater injection into an aquifer and aquifer decontamination by pumping [*Chen*, 1985, 1987; *Chen and Woodside*, 1988; *Moench*, 1989, 1991, 1995; *Goltz and Oxley*, 1991; *Welty and Gelhar*, 1994; *Harvey et al.*, 1994; *Haggerty and Gorelick*, 1995; *Chen et al.*, 1996; *Becker and Charbeneau*, 2000].

[3] Current analytical models for predicting solute transport in a radial flow field are based on advection-dispersion equations in cylindrical coordinates with constant dispersivity. However, dispersivity is well known increase with solute transport distance in groundwater systems [*Pickens and Grisak*, 1981a; *Moltz et al.*, 1983; *Domenico and Robbin*, 1984; *Gelhar et al.*, 1992; *Pang and Hunt*, 2001]. The increase of dispersion with travel distance results from the heterogeneous nature of the subsurface environment. Notably, researchers have developed analytical solutions for solute transport in a uniform flow field with scale-dependent dispersion. *Yates* [1990, 1992] derived an analytical solution for solute transport with scale-dependent dispersion by assuming that dispersivity increases linearly or exponentially with solute travel distance. Similarly, *Huang et al.* [1996] also presented an analytical solution for scale-dependent dispersion, yet assumed dispersivity to only increase with travel distance until a certain point, after which it assumes an asymptotic value. Meanwhile, *Logan* [1996] developed an analytical solution to the one-dimensional equations describing tracer transport in a heterogeneous porous medium incorporating rate-limited sorption and first-order decay under time-varying boundary conditions, and assuming an exponential increase in dispersivity. Furthermore, *Hunt* [1998] developed one-, two- and three-dimensional analyt-

[1]Department of Environmental Engineering and Sanitation, Fooyin University, Kaohsiung, Taiwan.

[2]Department of Bioenvironmental Systems Engineering, National Taiwan University, Taipei, Taiwan.

0043-1397/03/2003WR002299$09.00

ical solutions for a scale-dependent dispersion equation representing unsteady flow with an instantaneous source, and steady flow with a continuous source. Finally, *Pang and Hunt* [2001] presented exact solutions with a relatively simple form for both continuous and pulse sources in a semi-infinite domain, assuming negligible molecular diffusion and linearly increasing mechanical dispersion with distance downstream.

[4] Relatively few researchers have developed analytical solutions to advection-dispersion equations that involve scale-dependent dispersion in radial coordinates. The lack of progress in this area is attributed to deriving the analytical solutions to scale-dependent dispersion transport equation being difficult due to the dependence of dispersivity on spatial variables. *Kocabas and Islam* [2000] performed a detailed classification of the solutions of advection-dispersion equations in radial coordinates based on the velocity and/or scale-dependent form of the dispersion coefficient. Notably, *Kocabas and Islam* [2000] developed a new solution to radial advection-dispersion equation assuming spatial velocity and scale dependence of dispersivity for a radially divergent flow field. To our knowledge, few analytical or semianalytical solutions have been developed for describing scale-dependent dispersion in a radially convergent flow field. Analytical solutions for describing scale-dependent dispersion in a convergent flow field are interesting because they can offer research workers an understanding of how scale-dependent dispersion affects solute transport in a radially convergent flow field, and can also provide benchmarks for the testing of more general numerical models. Consequently, it is essential to develop an analytical solution for the problem of convergent solute transport with scale-dependent dispersion. Since no special function's solutions are presently available, to approach this problem, the Laplace transform power series (LTPS) technique [*Chen et al.*, 2002] was applied to solve the governing equation. Similar to other models of transport for approaching this problem derived the governing equation in porous media, the solution helps predict the breakthrough curves and obtain scale-dependent transport parameters by fitting field concentration breakthrough curves against the type curves obtained from the developed solution, and assuming the functional relationship between the dispersivity and distance remains constant in the domain of interest. Furthermore, the solution obtained is applied to some real world data, unlike in previous investigations which have not closely examined scale-dependent effects in their curve matching of field concentration.

2. Governing Equation

[5] To investigate solute transport in a radially convergent flow field with scale-dependent dispersion, the problem of a convergent tracer test is considered herein. Figure 1 presents the conceptual configuration. For the sake of simplicity, the following assumptions are made.

[6] 1. A vertically oriented extraction well of finite diameter is located along the vertical axis and fully penetrates a homogeneous and isotropic aquifer of constant thickness.

[7] 2. A steady state, horizontal flow field, radially convergent and axially symmetric with respect to extraction well, is established prior to the start of tracer injection. When

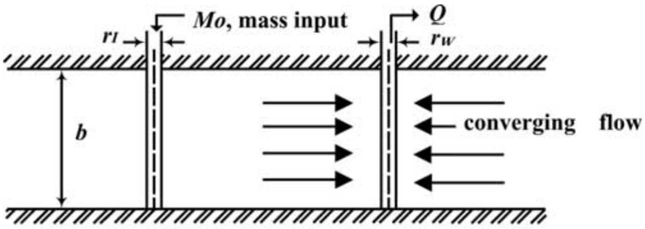

Figure 1. Schematic diagram of the radially convergent tracer test.

the tracer is injected in the injection well, the volume of injection tracer can have influence on the flow field. However, if the injection volume is small we can assume that the flow field is not affected by the injected amount and the physical and chemical behavior of injected tracer.

[8] The governing equations, and boundary conditions are derived and described below. The flow field is generated by a fully penetrating well of radius r_W located on the vertical axis at $r = 0$, and pumping fluid at a constant volume rate of Q from a homogeneous and isotropic aquifer of infinite horizontal extent. The average pore velocity V caused by extraction is described by

$$V = -\frac{A}{r}, \tag{1}$$

where $A = Q/2\pi b\phi$, and b and ϕ represent the aquifer thickness and effective porosity, respectively.

[9] The solute transport in a radial flow field can be described by the advection-dispersion equation in cylindrical coordinates as

$$R\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rD\frac{\partial C}{\partial r}\right) - V\frac{\partial C}{\partial r}, \tag{2}$$

where C is the concentration, D denotes the longitudinal dispersion coefficient, and R is retardation factor.

[10] The longitudinal dispersion coefficient, D, in equation (2) is generally considered to be a linear function of the pore velocity and diffusion coefficient can be neglected as follows:

$$D = a|V|, \tag{3}$$

where a is the longitudinal dispersivity. For a constant dispersivity, $a = a_c$, equations (2) and (3) reduce to the classical advection-dispersion equation in radial coordinates [*Moench*, 1989]:

$$\frac{a_c A}{r}\frac{\partial^2 C}{\partial r^2} + \frac{A}{r}\frac{\partial C}{\partial r} = R\frac{\partial C}{\partial t}. \tag{4}$$

Equation (4) incorporated with the associated initial and boundary conditions has been successfully solved using an Airy function to derive the analytical solution in the Laplace domain for describing solute transport in a radially convergent flow field [e.g., *Moench*, 1989; *Goltz and Oxley*, 1991].

[11] Results from field studies suggested that the dispersivity may be scale-dependent, i.e., a increases with travel distance from contaminant source. It is assumed that any growth with distance of the dispersion process is a direct consequence of heterogeneous nature of the porous medium [*Yates*, 1990]. In this paper, the growth of the dispersivity, a, as a linear function of travel distance from an injection well located at $r = r_L$ is employed herein and can be adequately describing using an equation of the form [see *Pickens and Grisak*, 1981b]

$$a(r) = e(r_L - r), \tag{5}$$

where e is dispersivity/distance ratio (i.e., the scale-proportional factor), which characterizes the scale-dependent dispersion process. Notice that a is zero at the tracer injection point $r = r_L$ and reaches a maximum, $a = a_{\max} = e(r_L - r_W)$ at the extraction well $r = r_W$.

[12] From the linear dispersivity-distance relationship, the dispersivity increases infinitely with travel distance. However, the results of field experiments at different scales indicate that dispersivity initially increases with travel distance, and eventually may approach the finite asymptotic value e. Therefore this study adopts other forms of dispersivity-distance relationships in which the finite asymptotic dispersivity is considered. For example, *Huang et al.* [1996] improved the linear distance-dependent dispersivity model [*Yates*, 1990] and assumed that dispersivity increases with travel distance until some specific distance, after which it becomes asymptotic. Huang et al. concluded the concentration distribution was constant for observation distances below the critical distance (finite asymptotic value reaches) while generally deviating substantially for distances exceeding the critical distance. This study assumes that maximum dispersivity at the extraction well does not exceed the finite asymptotic dispersivity.

[13] Substituting equations (1), (3), and (5) into equation (2), yielding the following equation.

$$R\frac{\partial C}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(re(r_L - r)\frac{A}{r}\frac{\partial C}{\partial r}\right) + \frac{A}{r}\frac{\partial C}{\partial r}. \tag{6}$$

Rewriting equation (6) in more clearly visible form as

$$R\frac{\partial C}{\partial t} = \frac{eA}{r}(r_L - r)\frac{\partial^2 C}{\partial r^2} + \frac{A(1-e)}{r}\frac{\partial C}{\partial r}. \tag{7}$$

The advective transport term of equation (7) becomes negative when $e > 1$ therefore would cause an apparent negative advective transport processes similar to that noticed by *Huang et al.* [1996] for scale-dependent dispersion in a uniform flow field. The aquifer's initial tracer concentration is assumed to be zero before starting the test:

$$C(r, t = 0) = 0. \tag{8}$$

Cady et al. [1993] performed a convergent radial tracer test and pointed that the finite volume in the extraction well bore has negligible effect on tracer transport. Therefore the outlet boundary condition, which is used to describe solute transport at the interface between the extraction well and aquifer and neglected the finite volume in the well bore is as

Table 1. Dimensionless Parameters

Dimensionless Quantity	Expression
Time[a]	$t_D = \frac{t}{t_a}$
Distance	$r_D = \frac{r}{r_L}$
Pumped well radius	$r_{WD} = \frac{r_W}{r_L}$
Injection well mixing factor	$\mu_I = \frac{r_I^2 h_I}{\phi b\left(r_L^2 - r_W^2\right)}$

[a]Here $t_a = \pi b\phi(r_L^2 - r_w^2)/Q$.

$$\frac{\partial C(r,t)}{\partial r} = 0, \qquad \text{at } r = r_W. \tag{9}$$

The boundary condition including solute transport at the injection well boundary incorporated with the finite volume in the injection well bore can be formulated [*Moench*, 1989] as

$$2\pi r_L \phi b\left[-\frac{A}{r_L}C\right] = -M\delta(t) + \pi r_I^2 h_I \frac{\partial C}{\partial t}, \qquad \text{at } r = r_L, \tag{10}$$

where r_I is injection well radius, M represents the injected dissolved tracer mass, h_I denotes mixing length of injection well.

[14] Dimensionless variables are defined in a manner similar to that used by *Moench* [1989]. Following *Moench* [1989] and substituting the definition given in Table 1 into equation (7), the dimensionless transport equation can be presented in the following form:

$$e(1 - r_D)\frac{\partial^2 C}{\partial r_D^2} + (1 - e)\frac{\partial C}{\partial r_D} = \frac{2r_D R}{1 - r_{WD}^2}\frac{\partial C}{\partial t_D}. \tag{11}$$

Consequently, the initial conditions and boundary conditions (8)–(10) become

$$C(r_D, 0) = 0, \tag{12}$$

$$\frac{\partial C(r_D, t_D)}{\partial r_D} = 0, \qquad \text{at } r_D = r_{WD}, \tag{13}$$

$$-C = -C_I\delta(t_D) + \mu_I\frac{\partial C}{\partial t_D}, \quad \text{at } r_D = 1, \tag{14}$$

where $C_I = M/[\pi b\phi(r_L^2 - r_W^2)]$. Equations (10) and (14) has been improved by *Zlotnik and Logan* [1996] based on a detailed analysis of flow and advective transport in the injection well. Revision is achieved by multiplying the injection well bore mixing factor, μ_I, by a multiplier π/δ, where δ is the aperture angle.

3. Power Series Solution

[15] Deriving the analytical solution of equation (11) with special function is very difficult due to the coefficients of first and second spatial derivative terms dependent on the

Table 2. Descriptive Simulation Conditions and Transport Parameters of the Hypothetical Tracer Test

Parameter	Test 1	Test 2
Pumping rate (Q), m^3/min	2	2
Aquifer thickness (b), m	10	10
Effective porosity (ϕ), dimensionless	0.2	0.2
Radius of extraction well (r_W), m	0.1	0.1
Radius of injection well (r_I), m	0.1	0.1
Injection well mixing length (h_I), m	10	10
Distance to the injection well (r_L), m	5	25
Injected mass (M), kg	10	40
Dispersivity/distance ratio (e), m	0.2	0.2

radial distance. Therefore the Laplace transform power series technique, LTPS, have been developed and applied to solve the initial-boundary value problem (11)–(14). The LTPS has been successfully applied to solve the variable-dependent partial differential equation [*Chen et al.*, 2002]. The LTPS technique does not suffer from the need to derive the special function's analytical solution in that the analytical solution is generally unknown priori or unavailable. The LTPS technique is parsimonious and easy to code into a program. As compared with special function's analytical solution, the LTPS technique provides an accurate solution to variable-dependent advection-dispersion equation [*Chen et al.*, 2002].

[16] First, by performing the Laplace transform of equations (11), (12) and its associated boundary conditions (13)–(14) with respect to t_D yields

$$e(1-r_D)\frac{d^2G}{dr_D^2}+(1-e)\frac{dG}{dr_D}-\frac{2r_DR}{1-r_{WD}^2}sG=0, \tag{15}$$

$$\frac{dG(r_D,s)}{dr_D}=0, \qquad \text{at } r_D=r_{WD}, \tag{16}$$

$$-G=-C_I+\mu_I sG, \quad \text{at } r_D=1, \tag{17}$$

where s denotes the Laplace transform parameter and G represents the Laplace transform of C, as defined by

$$G(r_D,s)=\int_0^\infty C(r_D,t_D)e^{-st_D}dt_D. \tag{18}$$

The solution of equation (15) is assumed to be in the form of a power series with unknown coefficients. The power series solution of governing equation (15), subject to boundary conditions (16)–(17), then can be derived as

$$G(r_D,s)=\sum_0^\infty a_m(s)r_D^m, \tag{19}$$

and substitutes this series along with the series obtained by term-wise differentiation of equation (19),

$$\frac{dG(r_D,s)}{dr_D}=\sum_1^\infty ma_m(s)r_D^{m-1}, \tag{20}$$

$$\frac{d^2G}{dr_D^2}=\sum_2^\infty m(m-1)a_m(s)r_D^{m-2}, \tag{21}$$

into equation (15), yielding the following equation.

$$e(1-r_D)\sum_2^\infty m(m-1)a_m(s)r_D^{m-2}+(1-e)\sum_1^\infty ma_m(s)r_D^{m-1} -\frac{2r_DRs}{1-r_{WD}^2}\sum_0^\infty a_m(s)r_D^m=0. \tag{22}$$

Rearranging the terms and shifting the summation indices of equation (22) yields

$$e\sum_0^\infty(m+2)(m+1)a_{m+2}(s)r_D^m-e\sum_1^\infty(m+1)ma_{m+1}(s)r_D^m +\sum_0^\infty(1-e)(m+1)a_{m+1}(s)r_D^m-\frac{2Rs}{1-r_{WD}^2}\sum_1^\infty a_{m-1}(s)r_D^m=0 \tag{23}$$

Setting the coefficients of each power of r_D to zero, for $m=0$,

$$a_2(s)=-\frac{(1-e)}{2e}a_1(s), \tag{24}$$

and in general, when m = 1,2,3$\cdots$

$$a_{m+2}(s)=\frac{1}{\varepsilon_r(m+2)(m+1)} \cdot\left[-(m+1)(1-e-em)a_{m+1}(s)+\frac{2Rs}{1-r_{WD}^2}a_{m-1}(s)\right], \tag{25}$$

the general solution can be yielded by inserting the appropriate values of the coefficients into the following equation

$$G(r_D,s)=b_1F_1(r_D,s)+b_2F_2(r_D,s), \tag{26}$$

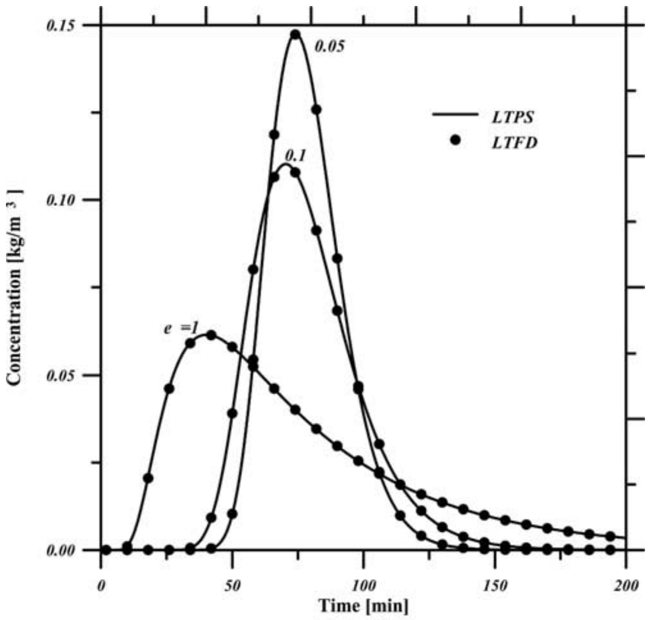

Figure 2. Comparison of breakthrough curves at the extraction well between the LTFD solution and the LTPS solution.

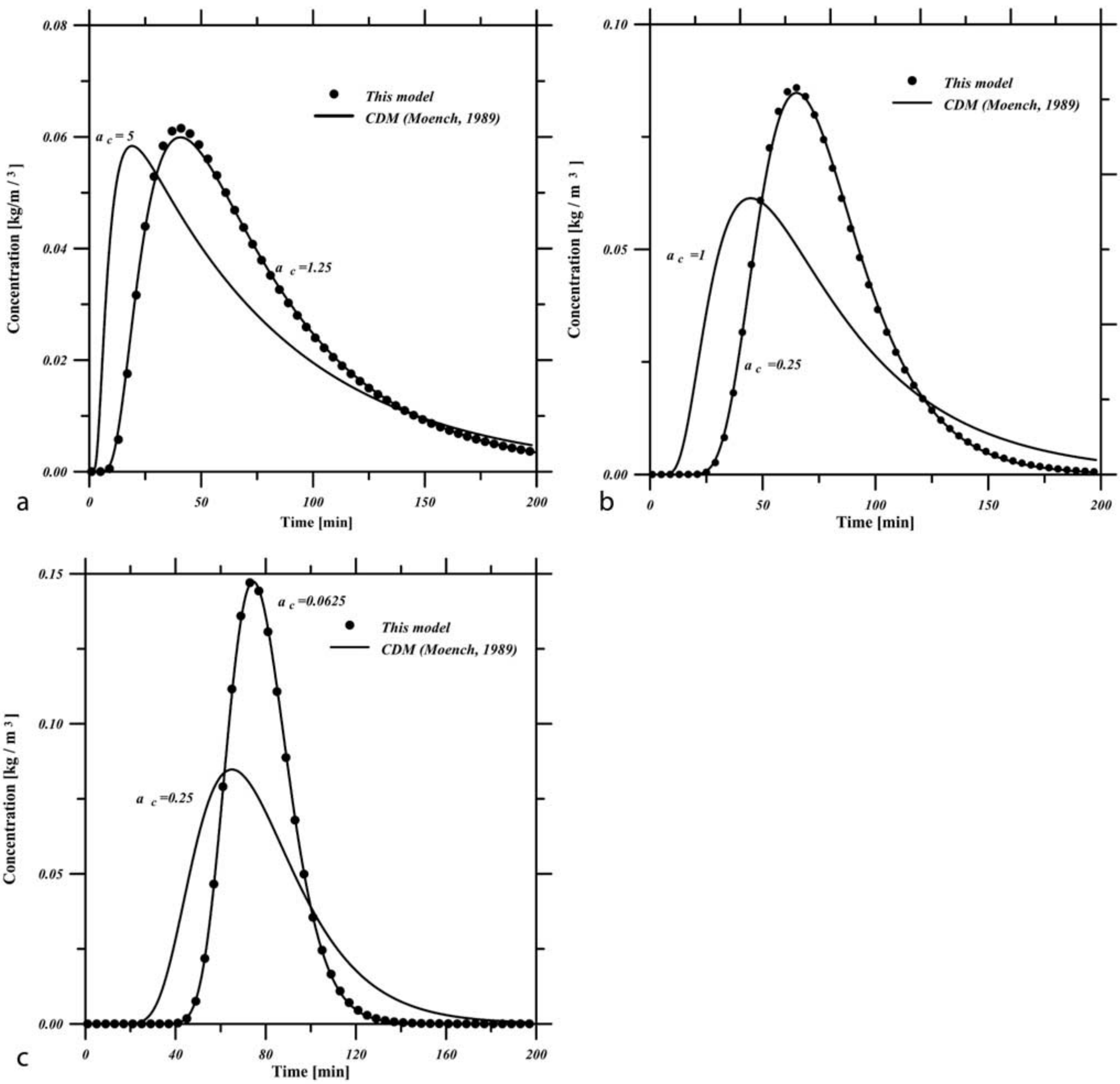

Figure 3. Comparison of breakthrough curves at the extraction well between the SDM and CDM models with 5 m interwell distance. The circle results from SDM with (a) e = 1, (b) e = 0.2 and (c) e = 0.05. Meanwhile, the solid curves are CDM from *Moench* [1989].

where $F_1(r_D, s)$ and $F_2(r_D, s)$ are two linearly independent general functions in the form of infinite series as in equation (19). The values of the series coefficients ($a_m(s)$) of $F_1(r_D, s)$ and $F_2(r_D, s)$ are determined by equations (24) and (25) in that we set $a_0(s) = 1$, $a_1(s) = 0$ and $a_0(s) = 0$, $a_1(s) = 1$, respectively. The unknown coefficients, b_1 and b_2 of equation (26) are determined by the boundary conditions (16) and (17), to obtain a particular solution of equation (26).

[17] A particular solution is obtained straightforwardly by imposing boundary conditions (16) and (17) on the general solution specified as equation (26). Therefore the Laplace transform power series solution at the extraction well may be expressed as

$$G(r_{WD}, s) = T\frac{-P_2P_5 + P_1P_6}{P_1P_4 - P_2P_3}, \tag{27}$$

where

$$P_1 = \frac{dG_1(r_{WD}, s)}{dr_D},$$

$$P_2 = \frac{dG_2(r_{WD}, s)}{dr_D},$$

$$P_3 = G_1(1.s),$$

$$P_4 = G_2(1.s),$$

$$P_5 = G_1(r_{WD}, s),$$

$$P_6 = G_2(r_{WD}, s),$$

$$T = \frac{C_I}{1 + \mu_I s}.$$

[18] The inverse Laplace transform of equation (27) provides the temporal concentration at the extraction well. In this work, a numerical Laplace inversion is adopted to yield the solution. The *de Hoog et al.* [1982] algorithm is used to execute the numerical inversion because it is accurate for a wide range of functions and it also performs reasonably well in the neighborhood of a discontinuity [*Moench*, 1991]. A FORTRAN subroutine, DINLAP/INLAP, provided by IMSL Subroutine Library [*Visual Numerics, Inc.*, 1994] and based on the de Hoog et al. algorithm, is employed to perform the numerical Laplace inversion.

4. Results and Discussion

4.1. Verification of the LTPS Solution

[19] To test the accuracy of the developed LTPS solution, it must be verified before it is applied. To date, no analytical solutions of any special functions are available for scale-dependent dispersion equations in cylindrical coordinates; therefore the solution obtained must be compared with a numerical solution. This method of verification was used by *Gelhar and Collins* [1971] and *Guvanasen and Guvanasen* [1987]. The Laplace transform finite difference (LTFD) technique was employed to generate numerical solution in this study. Furthermore, a hypothetical tracer test was considered. Table 2 summarizes the simulation conditions and transport parameters for verification (test 1). Figure 2 compares the breakthrough curves at the extraction well for various dispersivity/distance ratios, obtained using the proposed LTPS solution and the LTFD numerical solution. As expected, the two solutions agree closely with each other, thus validating the LTPS model.

4.2. Model Analysis

[20] Following validation with the numerical solution, the proposed model was applied to illustrate how the scale-dependent dispersion influences solute transport in a radially convergent flow field. The input parameters are the same as in the model verification in the previous subsection. Figures 3a–3c compare breakthrough curves at the extraction well obtained from the scale-dependent dispersivity model (SDM) described herein, where e = 1, 0.2, and 0.05, respectively, and the constant dispersivity model (CDM) [see *Moench*, 1989] has a constant dispersivity value of a_c = 5, 1 and 0.25 m. The input dispersivity values used in CDM imply that a maximum dispersivity value was assigned to the extraction well by SDM. Notably, the arrival time of the tracer concentration for the breakthrough curve determined by SDM is later than that obtained using CDM. Additionally, the spread of breakthrough curves with CDM exceeds

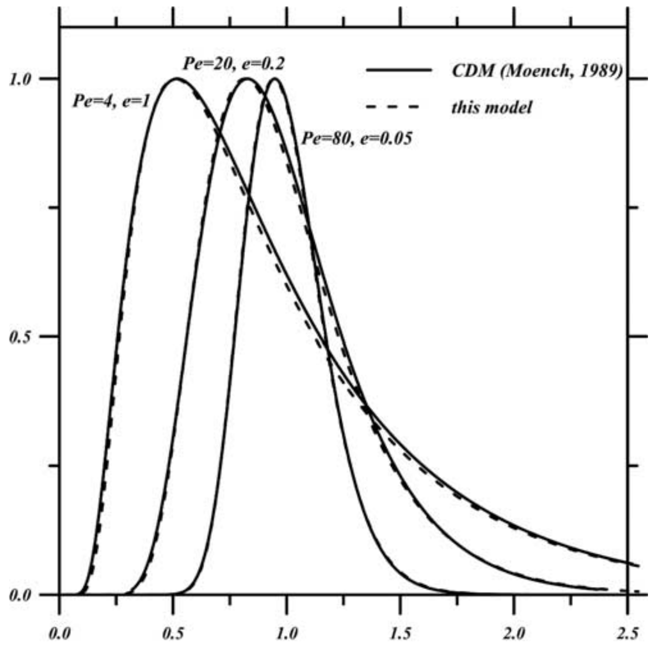

Figure 4. Comparison of types curves at the extraction well between SDM and CDM for e = 1, 0.2, and 0.05.

that from SDM. This analytical result is consistent with the assumption of linearly increasing dispersivity and can be explained as indicating that the average value of scale-dependent dispersivity over the whole travel domain used in SDM is smaller than the constant dispersivity used in CDM. To provide further evidence of this phenomenon, the input dispersivity used in the CDM was reduced and the breakthrough curves obtained with CDM were fitted against the breakthrough curves for SDM. The comparison reveals that the breakthrough curves obtained with CDM were shaped roughly the same as those obtained with SDM for rising limbs and with spreading tailing, when the input parameters of the advection-dispersion equation in CDM were reduced to the constant dispersivities of 1.25, 0.25, and 0.0625 m. However, a discrepancy occurs between the two models at peak concentration of the breakthrough curve when e = 1 and 0.2. The constant dispersivity value used in CDM is approximately one-fourth of the dispersivity in the extraction well used in SDM. *Pang and Hunt* [2001] used breakthrough curves obtained from both the SDM and CDM methods to fit the breakthrough curves at a point in a one dimensional uniform flow field determined by a laboratory experiment, and noted that the dispersivity values estimated using SDM are approximately two times greater than those from CDM.

[21] By normalizing the concentration with the peak concentration, C_{peak}, and expressing real time as dimensionless time, t_D, which was used for the type curve developed by *Sauty* [1980], Figure 4 is drawn up based on from Figures 3a–3c for e = 1, 0.2, and 0.05 in SDM and Pe = 4, 20, and 80 in CDM. In CDM, a Peclet number, $Pe = \frac{r_L}{a}$, is used to characterize advection-dominated (large Peclet number) or dispersion dominated (small Peclet number) transport, and can be obtained by fitting concentration breakthrough curves measured at the extraction well against type curves deter-

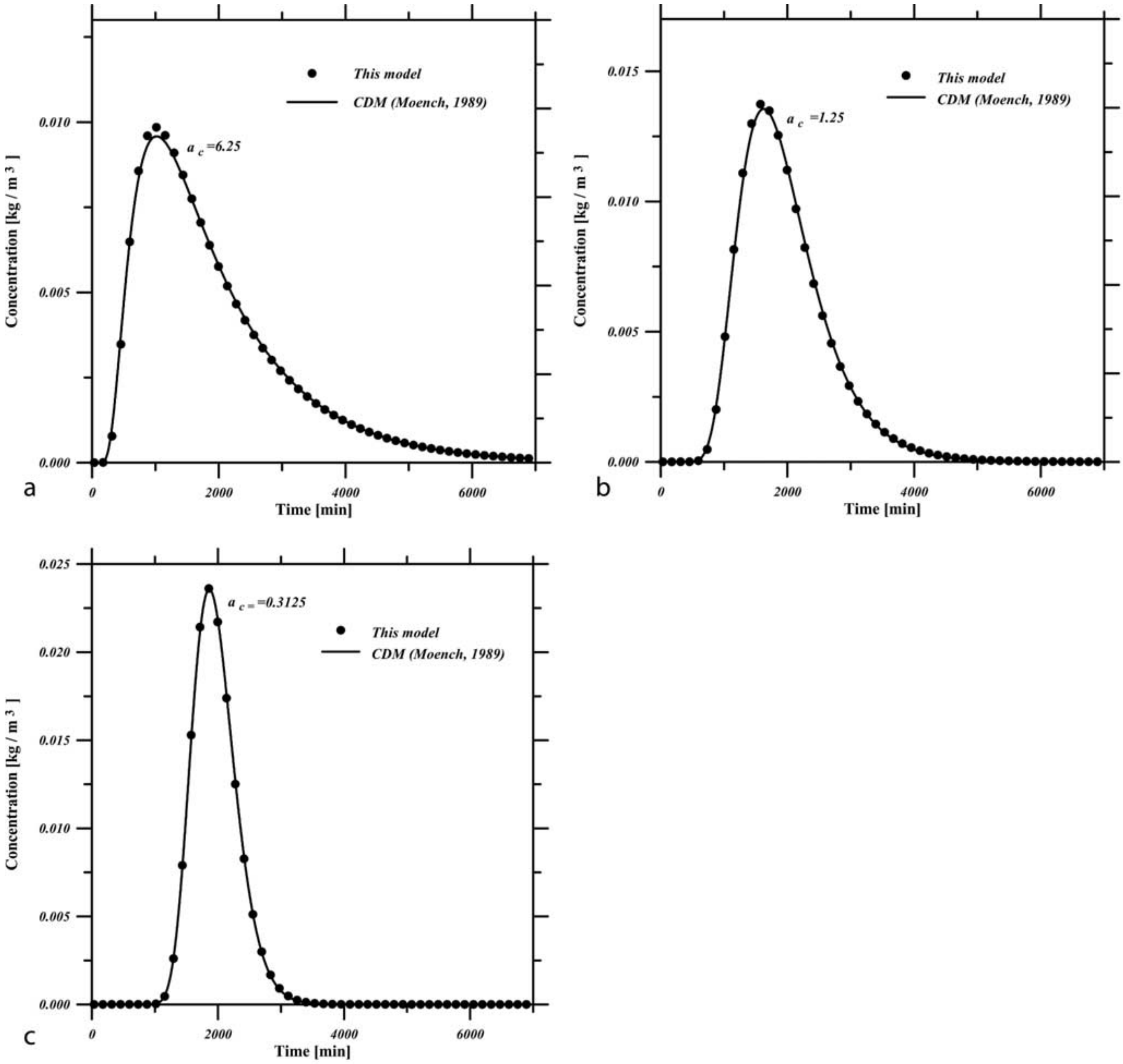

Figure 5. Comparison of breakthrough curves at the extraction well between the SDM and CDM models with 25 m interwell distance. The circle results from SDM with (a) e = 1, (b) e = 0.2 and (c) e = 0.05. Meanwhile, the solid curves are CDM from *Moench* [1989].

mined using CDM. Notably, the two curves are nearly identical, and the product of e used in SDM and Pe used in CDM equals 4.

[22] To further examine the generality of the correlative relationship ($Pe \cdot e = 4$), a case involving 25 m interwell distance is examined. Figures 5a–5c compare breakthrough curves at the extraction well obtained from the scale-dependent dispersivity model (SDM) described herein, where e = 1, 0.2 and 0.05, respectively, and the constant dispersivity model (CDM) has a constant dispersivity value of a_c = 6.25, 1.25 and 0.3125 m (i.e. Peclet number of 1, 20 and 80) in a tracer test with 25 m interwell distance (test 2 in Table 2). The result is the same as in test 1, demonstrating that the correlative relationship ($Pe \cdot e = 4$) is generally valid for $0.05 \leq e \leq 1$ of tracer tests at different interwell distances.

[23] If the dispersion process in solute transport is more accurately characterized by the SDM than by the CDM, any constant aquifer dispersivity (or Peclet number) obtained from the inverse fitting approach using the CDM can readily be transformed to yield the scale-dependent aquifer parameter (dispersivity/distance ratio) by using this relationship.

4.3. Model Application

[24] The developed solution given by equation (27) is applied to a field case to test the validity and the applicability of the proposed linearly scale-dependent dispersion model. The case study used here involves a convergent

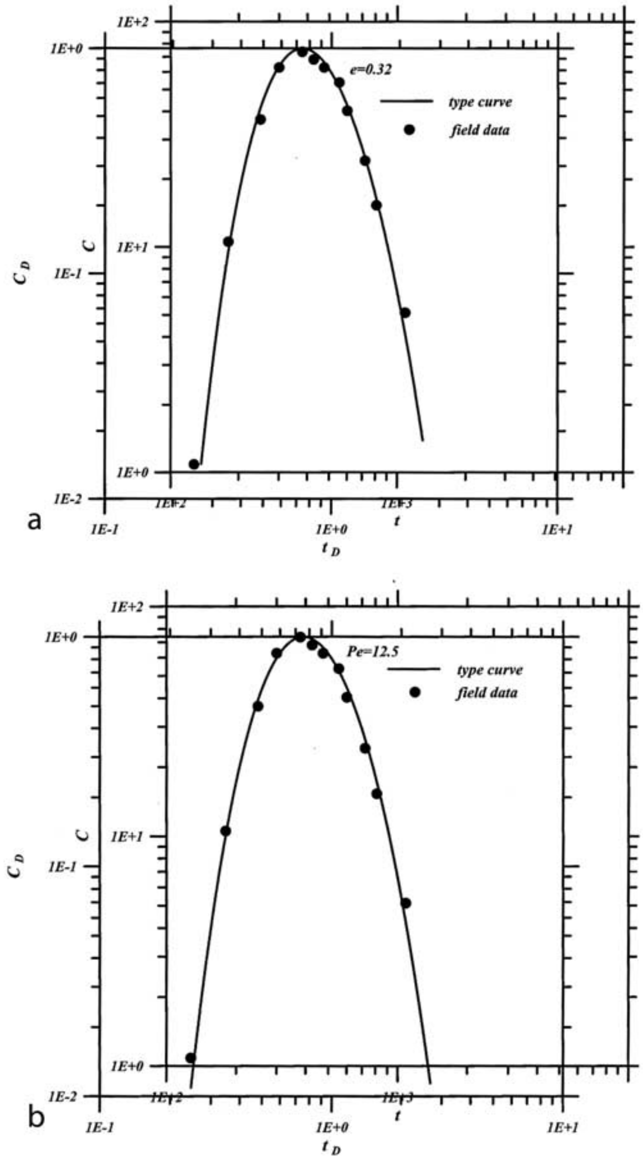

Figure 6. Breakthrough data at R = 150 m from a convergent radial tracer test conducted in Corbas, France, superimposed on the breakthrough curves from (a) SDM and (b) CDM.

radial tracer test performed in a sand and gravel aquifer with a thickness of approximately 12 m in France. *Sauty* [1977] previously presented a detailed analysis of the field data for longitudinal dispersivity. Moreover, *Welty and Gelhar* [1994] also examined the same field data. The data from this test are particularly useful for examining the scale-dependent dispersion assumption because breakthrough concentration was measured at three distances from the injection borehole (25, 50, and 150 m). To examine the applicability of the proposed SDM and compare the results with those from CDM, this study employed type curves obtained from the SDM and CDM to fit the breakthrough concentration at the extraction well with r_L = 150 m.

[25] The fitting of the field-derived breakthrough curves were performed using the following steps: (1) The type curves obtained from SDM and CDM were plotted logarithmically for various *e* or *Pe* values, respectively. (2) The field concentration at the extraction well, *C*, was then plotted against real time, *t*, on logarithmic paper of the same scale as the type curves. (3) The graph paper containing the field data was then laid over the type curves, with the sets of axes being kept parallel. The position of the field-data graph was then adjusted so as the data points lay over the type curve, with the axes of both graph sheets being kept parallel. (4) The *e* or *Pe* values, when the experimental data were matched against the type curves, were then obtained. Figures 6a and 6b present the experimental results and, specifically, the matches obtained using SDM and CDM. The *e* value from curve-matching using SDM is estimated to be 0.32, giving an approximate dispersivity value of 48 m at the extraction well. The determined *Pe* value was 12.5 and the average dispersivities value of 12 m was computed from curve-matching using CDM. This constant dispersivity from curve-matching using CDM agrees closely with Sauty's reported average dispersivity value of 12.5 m.

[26] For the breakthrough data of r_L = 25 and r_L = 50 *m*, *Welty and Gelhar* [1994] estimated that the constant dispersivities values are 2.4 m at r_L = 25 *m* and 4.6 m at r_L = 50 *m*, respectively, using a pulse width method of analysis based on the CDM. Consequently, adopting the developed relationship, the product of *e* used in SDM and *Pe* used in CDM equals 4, and the corresponding dispersivities at the extraction well are 9.6 m with r_L = 25 *m*, and 18.4 m with r_L = 50 *m* for SDM. Figure 7 plots the dispersivities at the extraction well for SDM and the constant dispersivity for CDM versus interwell (transport) distances. The comparative results clearly show that dispersivities increase linearly with transport distance for both SDM and CDM. The dispersivity/distance ratio of about 0.32 derived using SDM coincides with the *e* value derived from breakthrough

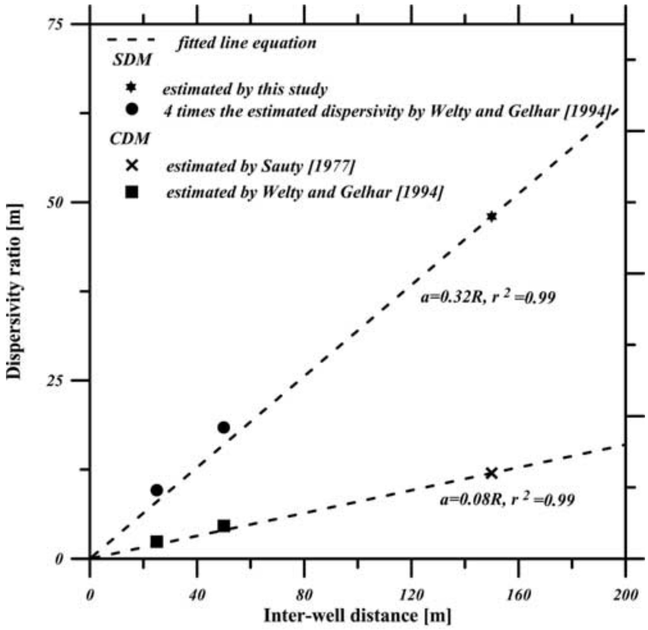

Figure 7. Dispersivity (*a*)-distance (*R*) relationship calculated using R = 25, 50, and 150 m compared with dispersivity calculated using SDM of *a* = 0.32 *R* and CDM of *a* = 0.08*R*, respectively.

concentration curve matching at r_L = 150 m using SDM. Moreover, the estimated dispersivity/distance ratio of 0.08 was yielded using CDM. This estimate agrees closely with the reported dispersivity/distance ratio of about 0.1 for a field scale in a few studies [*Pickens and Grisak*, 1981a, 1981b], but is considerably lower than the e value derived from the matching of breakthrough concentration at r_L = 150 m using SDM.

[27] The dispersivity derived from CDM underestimates the scale-dependent dispersivity, while that derived from SDM correctly yields an estimated scale-dependent dispersivity. Thus the correct method of estimating the scale-dependent dispersivity is to use the type curve derived from SDM. Notably, when applying the previously reported dispersivity/distance ratio derived from the CDM in a radially convergent tracer test to a scale-dependent dispersivity problem, the dispersivity/distance ratio should be revised by multiplying the previously reported dispersivity/distance ratio of 4. If the previous reported dispersivity/distance ratio is derived from CDMs in a one dimensional tracer test, then the application of the dispersivity/distance ratio in a scale-dependent dispersivity problem should be revised by multiplying the previously reported dispersivity/distance ratio of 2 [*Pang and Hunt*, 2001]. Failure to revise the dispersivity/distance ratio may cause the dispersivity to be underestimated by a factor of 4 or 2 for a radial or a one dimensional solute transport problem with scale-dependent dispersion, respectively.

5. Conclusions

[28] This study has presented an semianalytical solution for describing the radial convergent transport of dissolved substances in porous media with scale-dependent dispersion. The proposed model assumes that the dispersivity is a linear function of solute travel distance from the injected source. The Laplace transform power series technique is applied herein to solve the scale-dependent advection-dispersion equation. Moreover, a Laplace transform finite difference solution to the transport equation has also been derived to verify the accuracy of the proposed semianalytical solution. Comparing the breakthrough curves obtained from SDM and CDM reveals that given appropriately chosen parameters the solutions obtained with the two techniques are the same for breakthrough curves with rising limbs and spreading tails, but insignificant differences occur at an intermediate time. The type curves obtained using the two models are nearly identical. Notably, the product of the Peclet number used in CDM and the dispersivity/distance ratio used in SDM for the same type curves approximately equals 4. The proposed solution is applied to a field case to examine the applicability of the linearly scale-dependent dispersion model. The application results reveal that the solute transport process at the test site obeys the linearly scale-dependent dispersion model and that the linearly scale-dependent assumption is valid in this real world example. The previously reported dispersivity/distance ratio derived from CDMs in a radially convergent tracer test that should be revised by multiplying the above dispersivity/distance ratio of 4, and closely approximates the dispersivity for the scale-dependent dispersion problem. Further effect on theoretical analysis of the correlation ($Pe \cdot e = 4$) is suggested for future study.

References

Becker, M. W., and R. J. Charbeneau, First-passage-time transfer functions for groundwater tracer tests conducted in radially convergent flow, *J. Contam. Hydrol.*, *40*, 299–310, 2000.

Cady, C. C., S. E. Silliman, and E. Shaffern, Variation in aperture estimate ratios from hydraulic and tracer tests in a single fracture, *Water Resour. Res.*, *29*(9), 2975–2982, 1993.

Chen, C. S., Analytical and approximate solutions to radial dispersion from an injection well to a geological unit with simultaneous diffusion into adjacent strata, *Water Resour. Res.*, *21*(8), 1069–1076, 1985.

Chen, C. S., Analytical solution for radial dispersion problem with Cauchy boundary at injection well, *Water Resour. Res.*, *23*(7), 1217–1224, 1987.

Chen, C. S., and G. D. Woodside, Analytical solution for aquifer decontamination by pumping, *Water Resour. Res.*, *24*(8), 1329–1338, 1988.

Chen, J. S., C. W. Liu, C. S. Chen, and H. D. Yeh, A Laplace transform solution for tracer test in a radially convergent flow with upstream dispersion, *J. Hydrol.*, *183*, 263–275, 1996.

Chen, J. S., C. W. Liu, and C. M. Liao, A novel analytical power series solution for solute transport in a radially convergent flow field, *J. Hydrol.*, *266*, 120–138, 2002.

de Hoog, F. R., J. H. Knight, and A. N. Stokes, An improved method of Laplace transforms using a Fourier series approximation, *SIAM J. Sci. Stat. Comput.*, *3*(3), 357–366, 1982.

Domenico, P. A., and G. A. Robbin, A dispersion scale effect in model calibration and field tracer experiment, *J. Hydrol.*, *70*, 123–132, 1984.

Gelhar, L. W., and M. A. Collins, General analysis of longitudinal dispersion in nonuniform flow, *Water Resour. Res.*, *7*(6), 1511–1521, 1971.

Gelhar, L. W., C. Welty, and K. R. Rehfeldt, A critical review of data on field-scale dispersion in aquifers, *Water Resour. Res.*, *28*, 1955–1974, 1992.

Goltz, M. N., and M. E. Oxley, Analytical modeling of aquifer decontamination by pumping when transport is affected by rate-limited sorption, *Water Resour. Res.*, *27*(2), 547–556, 1991.

Guvanasen, V., and V. M. Guvanasen, An approximate semi-analytical solution for tracer injection tests in a confined aquifer with a radially convergent flow field and finite volume of tracer and chase fluid, *Water Resour. Res.*, *23*(8), 1607–1619, 1987.

Haggerty, R., and S. M. Gorelick, Multiple-rate mass transfer for modeling diffusion and surface reactions in media with pore-scale heterogeneity, *Water Resour. Res.*, *31*(10), 2383–2400, 1995.

Harvey, C. F., R. Haggerty, and S. M. Gorelick, Aquifer remediation: A method for estimating mass transfer rate coefficients and an evaluation of pulsed pumping, *Water Resour. Res.*, *30*(7), 1979–1991, 1994.

Huang, K. M., T. Van Genuchten, and R. Zhang, Exact solutions for one-dimensional transport with asymptotic scale-dependent dispersion, *Appl. Math. Modell.*, *20*(4), 297–308, 1996.

Hunt, B., Contaminant source solutions with scale-dependent dispersivity, *J. Hydrol. Eng.*, *3*(4), 268–275, 1998.

Kocabas, I., and M. R. Islam, Concentration and temperature transients in heterogeneous porous media part II: Radial transport, *J. Pet. Sci. Eng.*, *26*, 221–233, 2000.

Logan, J. D., Solute transport in porous media with scale-dependent dispersion and periodic boundary conditions, *J. Hydrol.*, *184*, 261–276, 1996.

Moench, A. F., Convergent radial dispersion: A Laplace transform solution for aquifer tracer testing, *Water Resour. Res.*, *25*(3), 439–447, 1989.

Moench, A. F., Convergent radial dispersion: A note on evaluation of the Laplace transform solution, *Water Resour. Res.*, *27*(12), 3261–3264, 1991.

Moench, A. F., Convergent radial dispersion in a double-porosity aquifer with fracture skin: Analytical solution and application to a field experiment in fractured chalk, *Water Resour. Res.*, *31*(8), 1823–1835, 1995.

Molz, F. J., O. Güven, and J. G. Melville, An examination of scale-dependent dispersion coefficients, *Ground Water*, *21*(6), 1701–1711, 1983.

Pang, L., and B. Hunt, Solutions and verification of a scale-dependent dispersion model, *J. Contam. Hydrol.*, *53*, 21–39, 2001.

Pickens, J. F., and G. E. Grisak, Scale-dependent dispersion in a stratified granular aquifer, *Water Resour. Res.*, *17*(4), 1191–1211, 1981a.

Pickens, J. F., and G. E. Grisak, Modelling of scale-dependent dispersion in a hydrogeological system, *Water Resour. Res.*, *17*(6), 1701–1711, 1981b.

Sauty, J.-P., Contribution à l'identification des paramèters de dispersion dans les aquiférs par interprétation des experiences de tracage, diploma ing. thesis, Univ. Sci. et Med. de Grenoble, Grenoble, France, 1977.

Sauty, J.-P., An analysis of hydrodispersive transfer in aquifers, *Water Resour. Res.*, *16*(1), 145–158, 1980.

Visual Numerics, Inc., IMSL user's manual, vol. 1, pp. 159–161, Houston, Tex., 1994.

Welty, C., and L. W. Gelhar, Evaluation of longitudinal dispersivity from nonuniform flow tracer tests, *J. Hydrol.*, *153*, 71–102, 1994.

Yates, S. R., An analytical solution for one-dimensional transport in heterogeneous porous media, *Water Resour. Res.*, *26*(10), 2331–2338, 1990.

Yates, S. R., An analytical solution for one-dimensional transport in heterogeneous porous media with an exponential dispersion function, *Water Resour. Res.*, *28*(8), 2149–2154, 1992.

Zlotnik, V. A., and J. D. Logan, Boundary conditions for convergent radial tracer tests and the effect of well bore mixing volume, *Water Resour. Res.*, *32*(7), 2323–2328, 1996.

J.-S. Chen and H.-T. Hsu, Department of Environmental Engineering and Sanitation, Fooyin University, 831 Kaohsiung, Taiwan.

C.-M. Liao and C.-W. Liu, Department of Bioenvironmental Systems Engineering, National Taiwan University, 106 Taipei, Taiwan. (lcw@gwater.agec.ntu.edu.tw)