***Journal of the American Water Resources Association*, 37(3): 723-736, 2001.**

A Scale-Invariant Gauss-Markov Model for Design Storm Hyetographs

Ke-Sheng Cheng[1], Irene Hueter[2], En-Ching Hsu[3], Hui-Chung Yeh[3]

Abstract. Hyetographs are essential to many hydrological designs. Many studies have shown that hyetographs are specific to storm types and durations. Recent work presented evidence that *dimensionless* hyetographs are scale invariant. We show that the *simple scaling property* of rainfall guarantees that the normalized rainfall rates of different storm durations are identically distributed and propose a *nonstationary* Gauss-Markov model based on the annual maximum events that arise from the dominant storm type. We derive the unique estimators for the parameters of the Gauss-Markov model under two constraints that (a) the typical peak rainfall rate is preserved, and (b) the most likely hyetograph is obtained. One attractive feature of this model is that it allows translating hyetographs between storms of different durations. Two examples illustrate our model.

Key words: hyetograph, simple scaling, Gauss-Markov model, design storm.

1. Introduction

Design storms are routinely used for designing stormwater management facilities and delineating floodplains. A design storm is a hypothetical storm with specific duration *D* and recurrence interval *T*. The information of design storms is conveniently presented in the form of depth-duration-frequency (DDF) curves or intensity-duration-frequency (IDF) curves. In addition, many engineering designs also rely on the *hyetographs*, i.e. the time distribution of design storm point rainfall, for runoff calculation. Because researchers over 30 years ago noticed that the time distribution of rainfall within a storm event has a significant effect on the peak runoff, the study of design storm hyetographs has a long tradition. Although the shapes of storm hyetographs vary significantly, many studies have shown that dimensionless

---

[1] Agricultural Engineering Department and Hydrotech Research Institute, National Taiwan University, Taipei, Taiwan, R.O.C.

[2] Mathematics Department, University of Florida, Gainesville, Florida 32611, U.S.A.

[3] Agricultural Engineering Department, National Taiwan University, Taipei, Taiwan, R.O.C.

hyetographs are storm-type specific (Huff, 1967; Eagleson, 1970). In contrast to earlier hyetograph models that are duration-specific (Keifer and Chu, 1957; Pilgrim and Cordery, 1975; Yen and Chow, 1980; SCS, 1986), Koutsoyiannis and Foufoula-Georgiou (1993) presented evidence that dimensionless hyetographs are scale invariant. The goal of this study is to propose a hyetograph modeling approach that is based on the dominant storm type and also allows translating the dimensionless hyetographs between storms of different durations.

We propose a *nonstationary* Gauss-Markov model for normalized rainfalls from severe storms of different durations. A key ingredient to our analysis is a simple scaling relation between instantaneous rainfall intensities from storms of different durations. Several other scaling relations are derived that hold for nonstationary processes. We restrict ourselves to the annual maximum events that are responsible for the annual maximum rainfalls to capture the essence of the dominant storm type for the model. Two important features are imposed on the Gauss-Markov model: (1) it represents the characteristics of the peak rainfall rate; and (2) it yields the hyetograph that is most likely to occur. We derive the estimators for the parameters of the Gauss-Markov model and illustrate the model with two examples.

2. Hyetographs from Different Storm Types

Huff (1967) studied rainfall data of many storm events in Illinois and normalized the incremental rainfall rates with respect to event-total-depths and storm durations. He classified the storms into four groups, depending on whether the heaviest rainfall occurred in the first, second, third, or the fourth quarter of the storm duration. We shall refer to hyetographs that are based on normalized rainfall rates as *dimensionless* hyetographs. Huff found that there is a trend for longer, heavier storms to dominate the fourth-quartile group whereas short-duration storms account for a major portion of the first and second quartile groups. Eagleson (1970) pointed out that for given climatic conditions, storm events of a given scale (microscale, mesoscale, or synoptic scale) exhibit similar time distributions when normalized with respect to total rainfall depths and storm durations. Convective cells and thunderstorms are dominant types of storms at the microscale and the mesoscale, respectively. Events of synoptic scale include frontal systems and cyclones that are typically several hundred miles in extent and often have series of mesoscale subsystems. In general, convective and frontal-type storms tend to have their peak rainfall rates near the beginning of the rainfall processes, while cyclonic events have the peak rainfall somewhere in the central third of the storm duration.

Pilgrim and Cordery (1975) developed a hyetograph model based on the average rainfall percentages of ranked rainfalls and the average rank of each time interval in a storm. Bras and Rodriguez-Iturbe (1976) and Woolhiser and Osborn (1985) also adopted the concept of dimensionless hyetograph in temporal rainfall modeling. More recently, Koutsoyiannis and Foufoula-Georgiou (1993) and Garcia-Guzman and Aranda-Oliver (1993) also developed stochastic models for dimensionless hyetographs. Koutsoyiannis and Foufoula-Georgiou proposed a simple scaling model to characterize the time distribution of instantaneous rainfall intensity and incremental rainfall depth within a storm event.

Prominently, for many hydrologic designs, major concern lies in the dominant storm types that cause the heaviest precipitation. For example, in Taiwan most severe storms happen in the period between May and November, during which heavy rainfall amounts are usually caused by convective storms and tropical cyclones. Typical hyetographs of convective storms and tropical cyclones are different. Tropical cyclones tend to produce higher rainfall intensities and larger total rainfall amounts. Thus, rainfall data from tropical cyclones should enter the design of storm hyetographs. If rainfall data from both storm types were simultaneously utilized in order to develop design storm hyetographs, it is quite likely that an average hyetograph results that characterizes the temporal rainfall variation of neither storm type.

In hydrologic design, two parameters of the hyetograph, the peak rainfall rate and the time-to-peak, i.e. the time between the beginning and the peak of a storm event, are particularly important for hyetograph modeling. Observed rainfall events have shown that, similar to the total rainfall depth, the peak rainfall rate and the time-to-peak also exhibit wide variations. Therefore, stochastic models that consider the temporal variations of storm events as random processes have become popular in hyetograph modeling.

3. Simple Scaling Model for Storm Events

The past decade has seen an increasing number of applications focusing on scaling characteristics of geophysical processes. The landform process (Turcotte, 1989), spatial and temporal rainfall processes (Gupta and Waymire, 1990; Olsson et al., 1993; Burlando and

Rosso, 1996), natural channel network (La Barbera and Rosso, 1989; Veltri, et al., 1996), and spatio-temporal distribution of earthquakes (Godano, et al., 1999) are just a few examples. Scaling or scale-invariant models enable us to transform data from one temporal or spatial model to another one, and thus, help to overcome the difficulty of inadequate data.

A natural process fulfills the simple scaling property if the underlying probability distribution of some physical measurements at one scale is identical to the distribution at another scale, multiplied by a factor that is a power function of the ratio of the two scales (Gupta and Waymire, 1991). Let $X(t)$ and $X(\lambda t)$ denote measurements at two distinct time or spatial scales t and λt, respectively. We say that the process $\{X(t), t \geq 0\}$ has the *simple scaling property* if there is some real number H such that

$$\{X(t)\} \stackrel{d}{=} \{\lambda^{-H} X(\lambda t)\} \tag{1}$$

for every real $\lambda > 0$. The $\stackrel{d}{=}$ denotes equality in distribution, and H is called the scaling exponent.

Quite often storm events have different durations, which may range from a few hours to several days. Traditionally, design storm hyetographs of various storm durations have been developed. For example, the 6-hr and 24-hr hyetographs of the Soil Conservation Service (SCS) both are commonly used in the practical engineering design (SCS, 1986). However, if the rainfall process exhibits the simple scaling property, then a scale-invariant hyetograph is desirable. Since the parameter of storm duration does not appear in the simple scaling relation of Eq. (1), it is obvious that translating data between storms of different durations is not feasible.

Koutsoyiannis and Foufoula-Georgiou (1993) introduced a simple scaling relation that includes the parameter of storm duration in the scaling relation. Let $\xi(t, D)$ represent the instantaneous rainfall intensity at time t of a storm with duration D. Since this random variable depends on the time t and the storm duration D; and thus, it shall be indexed by t and D. The proposed simple scaling relation for $\{\xi(t, D), t \geq 0, D > 0\}$ is

$$\{\xi(t,D)\} \overset{d}{=} \{\lambda^{-H}\xi(\lambda t, \lambda D)\} \tag{2}$$

for every $\lambda > 0$. In practice, cumulative rainfall depths over a fixed time interval Δ, for example one hour, are recorded and we shall refer to them as the incremental rainfall depths. The *i*-th incremental rainfall depth $X_\Delta(i,D)$ can be expressed by

$$X_\Delta(i,D) = \int_{(i-1)\Delta}^{i\Delta} \xi(t,D)\,dt \tag{3}$$

Similarly, the cumulative rainfall depth at time *t*, i.e. $h(t,D)$, and the total rainfall depth of the storm event $h(D,D)$ are respectively expressed by

$$h(t,D) = \int_0^t \xi(s,D)\,ds \tag{4}$$

$$h(D,D) = \int_0^D \xi(t,D)\,dt \tag{5}$$

. Koutsoyiannis and Foufoula-Georgiou (1993) proved that $X_\Delta(i,D)$, $h(t,D)$, and $h(D,D)$ all have the simple scaling property with scaling exponent $H+1$, i.e.,

$$\{X_\Delta(i,D)\} \overset{d}{=} \{\lambda^{-(H+1)} X_{\lambda\Delta}(i,\lambda D)\} \tag{6}$$

$$\{h(t,D)\} \overset{d}{=} \{\lambda^{-(H+1)} h(\lambda t,\lambda D)\} \tag{7}$$

$$\{h(D,D)\} \overset{d}{=} \{\lambda^{-(H+1)} h(\lambda D,\lambda D)\} \tag{8}$$

Observe that Eqs. (6), (7), and (8) are valid even for *nonstationary* random processes.

4. Selecting Storm Events for Design Hyetograph Development

Previous work on the time distribution of rainfall data drawn from: (a) only certain months or a single season (for example, Koutsoyiannis and Foufoula-Georgiou (1993)); or (b) the entire year (for example, Huff (1967) and Garcia-Guzman and Aranda-Oliver (1993)). Restricting attention to rainfall data in certain seasons allows one to focus on specific storm types and gain better understanding of the dominant and generic storms, as opposed to relying on a hyetograph created by merging rainfall from all storms within a year.

Because the design storm hyetograph represents the time distribution of the total storm depth determined by annual maximum rainfall data, the design storm hyetograph is optimally modeled when based on observed storm events that actually produced the annual maximum rainfall. Therefore, our strategy is to select the observed storms that give rise to the annual maximum rainfall, the so-called *annual maximum events,* to accurately develop design hyetographs. We shall refer to the *event duration* to represent the duration of an observed storm event and to the *design duration* to represent the designated time interval for a design storm. We emphasize that; in general, the *design durations* do not coincide with the actual durations of historic storm events (referred to as the *event durations*). The *design durations* are artificially designated durations used to determine the corresponding annual maximum depths; whereas *event durations* are actual raining periods of historic storm events. Figure 1 illustrates difference between the design duration and event duration of an annual maximum event.

Annual maximum events tend to occur in certain periods of the year (such as a few months or a season) and tend to emerge from the same storm type. Moreover, annual maximum rainfall data in Taiwan strongly indicate that a single annual maximum event often is responsible for the annual maximum rainfall depths of different design durations. In some situations, single annual maximum event even produced annual maximum rainfalls for many nearby raingauge stations. For example, Table 1 shows that a single storm event on October 22, 1987 was responsible for the annual maximum rainfall depths of almost all design durations for the two raingauge stations. All storms listed in Table 1 were major events. This observation emphasizes the importance of using only annual maximum events for design hyetograph development. Such a consideration enables us not only to focus on events of the same dominant storm type, but it also has the advantage of relying on almost the same annual maximum events that are employed to construct IDF curves.

5. Gauss-Markov Model of Dimensionless Hyetographs

Dimensionless hyetographs are popular models to represent the time distribution of rainfall intensities for storms of all durations in a common scale. In fact, storm durations of annual maximum events may range from a few hours to several days. Our key observation here is that, in view of simple scaling characteristics, the normalized rainfall rates of storms of different event durations are identically distributed. Because the annual maximum events are independent, we combine them into a random sample drawn from a nonstationary Gauss-Markov process at a fixed time. Subsequently, we estimate the mean, variance and serial correlation from the random samples.

Dimensionless hyetographs represent the temporal variation of storm events and can be expressed as the time distribution of either the cumulative or the incremental percentages of the total rainfall depths. We refer to the former as the *cumulative* dimensionless hyetograph, and to the latter as the *incremental* dimensionless hyetograph. For convenience of exposition, our derivations are based on the incremental dimensionless hyetograph.

In view of the simple scaling assumption, the incremental rainfall depth $X_\Delta(i,D)$ and the total rainfall depth $h(D,D)$ have same simple scaling exponent $H+1$ (Eqs. (6) and (8)). Therefore,

$$\left\{\frac{X_\Delta(i,D)}{h(D,D)}\right\} \stackrel{d}{=} \left\{\frac{X_{\lambda\Delta}(i,\lambda D)}{h(\lambda D,\lambda D)}\right\} \tag{9}$$

Let $Y_\Delta(i,D) = \dfrac{X_\Delta(i,D)}{h(D,D)}$ denote the *i*-th incremental rainfall percentage, i.e. the normalized rainfall rate, of a storm with duration D and incremental time interval Δ. Then Eq.(9) can be expressed as follows

$$Y_\Delta(i,D) \stackrel{d}{=} Y_{\lambda\Delta}(i,\lambda D) \tag{10}$$

for all $\lambda > 0$ and $i = 1, 2, \ldots, [D/\Delta]$ which indicates that the *i*-th incremental rainfall percentages of storms of durations D and λD are identically distributed if the time intervals are Δ and $\lambda\Delta$, respectively. Since the annual maximum events are independent, the identical distribution property allows us to combine the *i*-th ($i = 1, 2, \ldots, [D/\Delta] = [\lambda D/\lambda\Delta]$) incremental rainfall percentages of any storm durations to form a random sample, and the parameters (e.g. mean and variance) of the underlying distribution can be easily estimated.

Because the peak rainfall depth is a key element of hydrologic design, an ideal hyetograph should not only describe the random nature of the rainfall process but also the extreme characteristics of the peak rainfall. Therefore, our objective is to find the incremental dimensionless hyetograph that not only represents the peak rainfall characteristics but also has the maximum likelihood of occurrence. Our approach to achieve this objective includes two steps: (1) determine the peak rainfall rate of the dimensionless hyetograph and its time of occurrence, and (2) find the most likely realization of the normalized rainfall process with the given peak characteristics.

Let n =[D/Δ] be the number of incremental rainfall intervals in a hyetograph. Assume that the random process {Y(i), i = 1, 2, …, n} is a nonstationary Gauss-Markov process. Suppose that N realizations {y(i,j): i = 1, 2, …, n; j =1, 2, …, N } of the random process Y are available. Note that we have dropped Δ and D from $Y(i)$ since $Y_{\Delta}(i,D)$ and $Y_{\lambda\Delta}(i,\lambda D)$ are identically distributed.

Furthermore, let $y_p(j)$ and $t_p(j)$, respectively, denote the peak rainfall rate y_p and the time-to-peak t_p of the j-th realization. Then the peak rainfall rate y^* and the time-to-peak t^* of the dimensionless hyetograph are estimated as the average values of $y_p(j)$ and $t_p(j)$, respectively; that is,

$$y^* = \frac{1}{N}\sum_{j=1}^{N} y_p(j) \tag{11}$$

and

$$t^* = \frac{1}{N}\sum_{j=1}^{N} t_p(j) \tag{12}$$

The value of t^* is likely to be non-integer and should be rounded to the nearest integer.

Assume that the process {Y(i): i = 1, 2, …, n} is a nonstationary Gauss-Markov process with means μ_i, variances σ_i^2, and lag-1 correlations $\rho_1(i)$, that is, the conditional distribution of Y(i) given Y(i-1) = y_{i-1} is the conditional normal distribution given y_{i-1} with parameters

μ_i, $\sigma_i^{\ 2}$, and $\rho_1(i)$:

$$Y(i) \sim \mathrm{N}(\mu_i, \sigma_i^{\ 2}) \tag{13}$$

$$\rho_1(i) = autocorrel\ (\ Y(i), Y(i\text{-}1)\ ) \tag{14}$$

for i = 1,2,…, n. Assume that $y_0 = \mu_0 = 0$ and $\rho_1(1) = 0$. If $f(y_i \mid y_{i-1}) = f_{Y(i)|Y(i-1)}(y_i \mid y_{i-1})$ denotes the probability density function of $Y(i)$ given $Y(i$-1) = $y_{i\text{-}1}$, then

$$f(y_i \mid y_{i-1}) = [2\pi\sigma_i^2(1-\rho_1^2(i))]^{-\frac{1}{2}} \cdot \exp\Big\{ -\frac{1}{2}\Big[\frac{(y_i-\mu_i)-\rho_1(i)\dfrac{\sigma_i}{\sigma_{i-1}}(y_{i-1}-\mu_{i-1})}{\sigma_i\sqrt{1-\rho_1^2(i)}}\Big]^2\Big\}. \tag{15}$$

By the Markov property, the joint density $f(y_1, y_2, \ldots, y_n)$ of Y(1), Y(2),…, $Y(n)$ factorizes

$$f(y_1, y_2, \ldots, y_n) = f_{Y(1)}(y_1) \cdot f_{Y(2)|Y(1)}(y_2 \mid y_1) \cdots f_{Y(n)|Y(n-1)}(y_n \mid y_{n-1}) \tag{16}$$

If we write $L = \prod_{i=1}^{n} f(y_i \mid y_{i-1})$ for the joint density of Y(1), Y(2), …, $Y(n)$, then we obtain

$$\begin{aligned} lnL &= \sum_{i=1}^{n} ln\ f(y_i \mid y_{i-1}) \\ &= -\frac{n}{2} ln(2\pi) - \sum_{i=1}^{n} ln\,\sigma_i - \frac{1}{2}\sum_{i=1}^{n} ln[1-\rho_1^2(i)] \\ &\quad -\frac{1}{2}\sum_{i=1}^{n} \Big[\frac{(y_i-\mu_i)-\rho_1(i)\dfrac{\sigma_i}{\sigma_{i-1}}(y_{i-1}-\mu_{i-1})}{\sigma_i\sqrt{1-\rho_1^2(i)}}\Big]^2. \end{aligned} \tag{17}$$

Our objective is to find the hyetograph $\{y_i, i = 1,2,\ldots, n\}$ that maximizes the log-likelihood (lnL) and also satisfies the two constraints:

$$\sum_{i=1}^{n} y_i = 1 \tag{18-a}$$

and

$$y_{t^*} = y^* . \tag{18-b}$$

This can be achieved by introducing the two Lagrange multipliers ℓ and m , and by minimizing the expression:

$$M \equiv \sum_{i=1}^{n} \left[\frac{(y_i - \mu_i) - \rho_1(i)\dfrac{\sigma_i}{\sigma_{i-1}}(y_{i-1} - \mu_{i-1})}{\sigma_i\sqrt{1-\rho_1^2(i)}} \right]^2 - 2\ell(\sum_{i=1}^{n} y_i - 1) - 2m(y_{t^*} - y^*) \tag{19}$$

with respect to y_i ($i = 1,2,\ldots, n$), ℓ, and m. If we write $C_i = \rho_1(i)\dfrac{\sigma_i}{\sigma_{i-1}}$ and $D_i =$ $\sigma_i^2[1-\rho_1^2(i)]>0$, it follows that

$$M = \sum_{i=1}^{n} \frac{[(y_i - \mu_i) - C_i(y_{i-1} - \mu_{i-1})]^2}{D_i} - 2\ell(\sum_{i=1}^{n} y_i - 1) - 2m(y_{t^*} - y^*) \tag{20}$$

$$\frac{\partial M}{\partial y_i} = \frac{2[(y_i - \mu_i) - C_i(y_{i-1} - \mu_{i-1})]}{D_i} - \frac{2[(y_{i+1} - \mu_{i+1}) - C_{i+1}(y_i - \mu_i)]C_{i+1}}{D_{i+1}} - 2\ell = 0$$

$$(i = 1,2,\ldots, n;\ i \neq t^*) \tag{21-a}$$

$$\frac{\partial M}{\partial y_{t^*}} = \frac{2[(y_{t^*} - \mu_{t^*}) - C_{t^*}(y_{t^*-1} - \mu_{t^*-1})]}{D_{t^*}}$$

$$-\frac{2[(y_{t^*+1}-\mu_{t^*+1})-C_{t^*+1}(y_{t^*}-\mu_{t^*})]C_{t^*+1}}{D_{t^*+1}}-2\ell-2m=0 \tag{21-b}$$

Equivalently,

$$\frac{C_i}{D_i}y_{i-1}-(\frac{1}{D_i}+\frac{C_{i+1}^2}{D_{i+1}})y_i+\frac{C_{i+1}}{D_{i+1}}y_{i+1}+\ell \;=\; \frac{C_i}{D_i}\mu_{i-1}-(\frac{1}{D_i}+\frac{C_{i+1}^2}{D_{i+1}})\mu_i+\frac{C_{i+1}}{D_{i+1}}\mu_{i+1}$$

$$(\ i=1,2,\ldots,n,\ i\neq t^*\ C_{n+1}=0) \tag{22-a}$$

$$\frac{C_{t^*}}{D_{t^*}}y_{t^*-1}-(\frac{1}{D_{t^*}}+\frac{C_{t^*+1}^2}{D_{t^*+1}})y_{t^*}+\frac{C_{t^*+1}}{D_{t^*+1}}y_{t^*+1}+\ell+m=\frac{C_{t^*}}{D_{t^*}}\mu_{t^*-1}-(\frac{1}{D_{t^*}}+\frac{C_{t^*+1}^2}{D_{t^*+1}})\mu_{t^*}+\frac{C_{t^*+1}}{D_{t^*+1}}\mu_{t^*+1} \tag{22-b}$$

$$\frac{\partial M}{\partial \ell}=-2(\sum_{i=1}^{n}y_i-1)=0 \quad \Leftrightarrow \quad \sum_{i=1}^{n}y_i=1 \tag{23}$$

$$\frac{\partial M}{\partial m}=-2(y_{t^*}-y^*)=0 \quad \Leftrightarrow \quad y_{t^*}=y^* \tag{24}$$

Equations (22), (23) and (24) may be expressed in matrix form:

$$
\begin{bmatrix}
-(\frac{1}{D_1}+\frac{C_2^2}{D_2}) & \frac{C_2}{D_2} & 0 & & & & \cdots & & & 0 & 1 & 0 \\
\frac{C_2}{D_2} & -(\frac{1}{D_2}+\frac{C_3^2}{D_3}) & \frac{C_3}{D_3} & 0 & & & \cdots & & & 0 & 1 & 0 \\
0 & \frac{C_3}{D_3} & -(\frac{1}{D_3}+\frac{C_4^2}{D_4}) & \frac{C_4}{D_4} & 0 & & \cdots & & & 0 & 1 & 0 \\
 & & & \ddots & & & & & & & \vdots & \\
 & & & \frac{C_{t^*-1}}{D_{t^*-1}} & -(\frac{1}{D_{t^*-1}}+\frac{C_{t^*}^2}{D_{t^*}}) & \frac{C_{t^*}}{D_{t^*}} & & & & 0 & 1 & 0 \\
 & & & & \frac{C_{t^*}}{D_{t^*}} & -(\frac{1}{D_{t^*}}+\frac{C_{t^*+1}^2}{D_{t^*+1}}) & \frac{C_{t^*+1}}{D_{t^*+1}} & & & 0 & 1 & 1 \\
 & & & & & \frac{C_{t^*+1}}{D_{t^*+1}} & -(\frac{1}{D_{t^*+1}}+\frac{C_{t^*+2}^2}{D_{t^*+2}}) & \frac{C_{t^*+2}}{D_{t^*+2}} & & 0 & 1 & 0 \\
 & & \vdots & & & & & \ddots & & & \vdots & \\
 & & \vdots & & & & 0 & \frac{C_{n-1}}{D_{n-1}} & -(\frac{1}{D_{n-1}}+\frac{C_n^2}{D_n}) & \frac{C_n}{D_n} & 1 & 0 \\
 & & \vdots & & & & & 0 & \frac{C_n}{D_n} & -\frac{1}{D_n} & 1 & 0 \\
1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0
\end{bmatrix}
\bullet
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{t^*-1} \\ y_{t^*} \\ y_{t^*+1} \\ \vdots \\ y_{n-1} \\ y_n \\ \ell \\ m
\end{bmatrix}
$$

$$
= \begin{bmatrix}
-(\frac{1}{D_1}+\frac{C_2^2}{D_2})\mu_1 + \frac{C_2}{D_2}\mu_2 \\
\frac{C_2}{D_2}\mu_1 - (\frac{1}{D_2}+\frac{C_3^2}{D_3})\mu_2 + \frac{C_3}{D_3}\mu_3 \\
\frac{C_3}{D_3}\mu_2 - (\frac{1}{D_3}+\frac{C_4^2}{D_4})\mu_3 + \frac{C_4}{D_4}\mu_4 \\
\vdots \\
\frac{C_{t^*-1}}{D_{t^*-1}}\mu_{t^*-2} - (\frac{1}{D_{t^*-1}}+\frac{C_{t^*}^2}{D_{t^*}})\mu_{t^*-1} + \frac{C_{t^*}}{D_{t^*}}\mu_{t^*} \\
\frac{C_{t^*}}{D_{t^*}}\mu_{t^*-1} - (\frac{1}{D_{t^*}}+\frac{C_{t^*+1}^2}{D_{t^*+1}})\mu_{t^*} + \frac{C_{t^*+1}}{D_{t^*+1}}\mu_{t^*+1} \\
\frac{C_{t^*+1}}{D_{t^*+1}}\mu_{t^*} - (\frac{1}{D_{t^*+1}}+\frac{C_{t^*+2}^2}{D_{t^*+2}})\mu_{t^*+1} + \frac{C_{t^*+2}}{D_{t^*+2}}\mu_{t^*+2} \\
\vdots \\
\frac{C_{n-1}}{D_{n-1}}\mu_{n-2} - (\frac{1}{D_{n-1}}+\frac{C_n^2}{D_n})\mu_{n-1} + \frac{C_n}{D_n}\mu_n \\
\frac{C_n}{D_n}\mu_{n-1} - \frac{1}{D_n}\mu_n \\
1 \\
y^*
\end{bmatrix}
\qquad (25)
$$

Observe that

$$\frac{\partial^2 M}{\partial {y_i}^2} = 2(\frac{1}{D_i} + \frac{C_{i+1}^2}{D_{i+1}}) > 0 \tag{26}$$

The dimensionless hyetograph $\{y_i, i = 1,2,\ldots, n\}$ is determined by solving Eq. (25). Note that the constraint of Eq. (18-b) ensures peak rainfall characteristics of the annual maximum events will be preserved in the dimensionless hyetograph. If we do not impose the constraint on peak rainfall characteristics, then the above matrix equation is modified accordingly, and the hyetograph that is most likely to occur is nothing else than the mean hyetograph, i.e.,

$$y_i = \mu_i \qquad i = 1, 2, \ldots, n. \tag{27}$$

The parameters μ_i are estimated by means of the available realizations $\{y(i,j), i = 1,2,\ldots, n; j =1,2,\ldots, N \}$, i.e.,

$$\hat{\mu}_i = \frac{1}{N}\sum_{j=1}^{N} y(i, j) \qquad i = 1,2,\ldots, n \tag{28}$$

6. Model Application

We illustrate the scale-invariant Gauss-Markov model by applying it to develop design storm hyetographs for two raingauge stations Hosoliau and Wutuh, located in Northern Taiwan.

6.1. Annual maximum events

Annual maximum events that produced annual maximum rainfall depths of 6, 12, 18, 24, 48, and 72-hour design durations were collected. These events were identified using the minimum inter-event-time of 3 hours. Table 2 lists the numbers of selected annual maximum events and their range of event durations. The selected events include the annual maximum events shown in Table 1. All events occurred in the period between June and October and were classified as tropical depressions, tropical storms, or cyclones. All event durations were first divided into twenty-four equal periods $i\Delta$ (i=1,2,…,24, D = event duration, $\Delta = D/24$). Rainfalls of each annual maximum event were normalized, with respect to the total rainfall depth and event duration. For events of durations not exactly equal to 24 hours, we first interpolated the normalized cumulative rainfalls at the re-scaled time instances $i\Delta$, and then calculated the normalized incremental rainfalls $\{Y(i), i = 1,2,\ldots,24\}$ as the difference of two successive normalized cumulative rainfalls. Table 3 presents the sample means, standard deviations, and lag-1 correlation coefficients for $Y(i)$. The means of $Y(i)$ range from less than 2% to higher than 6% for both Hosoliau and Wutuh, indicating nonstationarity for the dimensionless hyetographs.

6.2. Normality check for normalized rainfalls

The Gauss-Markov model of dimensionless hyetographs considers the normalized rainfalls $\{Y(i), i=1,2,\ldots, n\}$ as a multivariate normal distribution. The Kolmogorov-Smirnov test was used to check the normality of $Y(i)$'s. The results of the Kolmogorov-Smirnov test indicate that at α = 0.05 significance level, the null hypothesis was not rejected for most of $Y(i)$'s (Table 4). The few rejected normalized rainfalls occur in the beginning or near the end of an event, and have less rainfall rates.

The lag-k correlation coefficients $\rho_k(i)$ =*autocorrel* ($Y(i)$, $Y(i-k)$) of the normalized rainfalls were estimated by using the following equation:

$$r_k(i) = \frac{\sum_{j=1}^{N}[y_j(i)-m_i][y_j(i-k)-m_{i-k}]}{\sqrt{\sum_{j=1}^{N}[y_j(i)-m_i]^2\sum_{j=1}^{N}[y_j(i-k)-m_{i-k}]^2}} \quad k=1,2,\ldots,n-1,\ i=k+1,\ k+2,\ldots,n. \tag{29}$$

where m_i is the sample mean of the i-th normalized rainfall $Y(i)$ and N represents the total number of annual maximum events. Figure 2 shows $r_1(i)$ and $r_2(i)$ of the normalized rainfalls for Hosoliau and Wutuh. The average values of $r_1(i)$ for Hosoliau and Wutuh are 0.57 and 0.58, respectively. Average values of $r_2(i)$ are 0.25 and 0.26, respectively. If $\rho_k(i)$ =0, then

$$t = r_k(i)\sqrt{\frac{N-2}{1-r_k^2(i)}} \tag{30}$$

has a t-distribution with (N-2) degree of freedom. At significance level α, the null hypothesis $H_O : \rho_k(i) = 0$ is rejected if $|t| > t_{1-\alpha/2, N-2}$. As illustrated in Figure 2, at α=0.025 the null hypothesis for lag-1 correlation coefficient $H_O : \rho_1(i) = 0$ was rejected in most cases except for $\rho_1(9)$ of the Wutuh Station. Null hypothesis for lag-2 correlation coefficient $H_O : \rho_2(i) = 0$ was mostly not rejected. The results of the normality check and test of significance for $\rho_1(i)$ and $\rho_2(i)$ evidently justify our Gauss-Markov assumption for modeling the dimensionless hyetographs.

6.3. Translating hyetographs between storms of different durations

The dimensionless hyetographs of Hosoliau and Wutuh established by solving Eq. (25) are shown in Figure 3. Hyetographs of the two stations are similar since they are located within 20 km of each other, and in the same watershed. With the constraint on peak rainfall characteristics (Eq. (18-b)), the peak rainfall rates are 15.3% and 16.2% for Hosoliau and Wutuh, respectively. Without the peak rainfall constraint, the peak rainfall rates are much lower, approximately 6%, thus greatly underestimated.

Equation (10) indicates that the random process of normalized rainfalls, $\{Y(i), i=1, 2, \ldots, n\}$, is independent of the storm duration and total rainfall depth. Some hyetograph models in the literature are duration-specific (for example, the SCS 6-hr and 24-hr duration hyetographs, (SCS, 1986)) and recurrence-interval-specific (for example, the alternating block method (Chow, et al., 1988)). The simple scaling property enables us to translate the dimensionless hyetographs between design storms of different durations. Translating the dimensionless hyetographs between design storms of durations D and λD is accomplished by changing the incremental time intervals by the duration ratio. Values of the normalized rainfalls $Y(i)$ (i=1,2,…,n) remain unchanged. For example, the incremental time intervals (Δ) of the design storms of 2-hr and 24-hr durations are five (120/24) and sixty (1440/24) minutes, respectively. The changes in the incremental time intervals are important since they require the subsequent rainfall-runoff modeling to be performed based on the “designated” incremental time intervals. A significant advantage of the simple scaling model is that developing two separate dimensionless hyetographs for design storms of 2-hr and 24-hr durations can be avoided. The incremental rainfall depths of a design storm are calculated by multiplying the y-coordinates of the dimensionless hyetograph by the total depth from IDF or DDF curves. Parameters of the IDF curves of Hosoliau and Wutuh are shown in Table 5. Table 6 presents a comparison of the peak rainfall depths determined by the design storm hyetographs and the corresponding rainfall depths from the IDF curves for the design storm events of a 100-yr recurrence interval. Entries in Table 6 are obtained as follows.

For Wutuh station, the total rainfall depth of 100-yr return period, 24-hr design duration is calculated by

$$\bar{i}_{T=100\,yrs}(D=24hrs)=\frac{330.36\times 100^{0.1823}}{(24\times 60)^{0.4380}}=31.64(mm/hr)$$

Total depth = 31.64 (*mm/hr*) x 24 (*hrs*) = 759.34 *mm*

Peak rainfall rate of the dimensionless hyetograph is 16.21 %, and peak rainfall depth from the design storm hyetograph is 759.34 x 16.21% = 123 *mm* (in 1-hr interval). Since the 123 *mm* rainfall depth is the peak rainfall in *one-hour interval*, therefore we compare it with rainfall depth of one-hour duration from the IDF curves.

Calculation of the 100-yr, 1-hr storm depth from IDF curves yields

$$\bar{i}_{T=100\,yrs}(D=1hr)=\frac{330.36\times 100^{0.1823}}{(1\times 60)^{0.4380}}=127.28(mm/hr)$$

The two rainfall depths, 123 *mm* and 127 *mm*, are close. Other entries are obtained in a similar way.

It is evident that, by imposing the constraint on the peak rainfall characteristics, the design storm hyetographs yield peak rainfall depths that are close to the corresponding rainfall depths given by the IDF curves. This is an important feature since it demonstrates that our hyetograph model preserves the maximum rainfall depths of different design durations.

6.4. Benefits of the simple scaling Gauss-Markov hyetograph model

We summarize the benefits of the proposed model as follows:

(1) The model uses annual maximum events that are mostly the same data as used for

developing the IDF curves, thus, it is most likely to represent the temporal variation of rainfall events that are responsible for annual maximum rainfalls.

(2) Restricting ourselves to only the annual maximum events reduces the number of events being analyzed and enables us to focus model development on the dominant storm type. In this study the dimensionless hyetographs of tropical storms were developed. However, it may still be necessary to develop the dimensionless hyetographs for convective storms by using short-duration annual maximum events.

(3) The proposed model yields hyetographs that not only have peak rainfall depths consistent with the corresponding rainfall depths from IDF curves; it is also the single realization that, under the constraint of peak rainfall characteristics, is most likely to occur. Furthermore, the hyetograph model also preserves the Markov property of the rainfall process.

(4) The scaling property of the rainfall process has two significant features. First, it shows that normalized rainfalls $Y_{\Delta}(i, D)$ and $Y_{\lambda\Delta}(i, \lambda D)$ are identically distributed, and thus, provides the theoretical basis for estimating parameters of the underlying Gauss-Markov model by means of the normalized rainfalls from storm events of different durations. Secondly, it provides an easy way to translate dimensionless hyetographs between design storms of different durations.

(5) In contrast to other hyetograph models that have regular shapes (for examples, the triangular hyetographs (Yen and Chow, 1980) and the regular curved hyetographs (Keifer and Chu, 1957), design hyetographs from the proposed model have irregular shapes that resemble those of real rainfall hyetographs.

7. Conclusions

We have shown that, due to a simple scaling property, dimensionless hyetographs can be represented by a nonstationary Gauss-Markov model. The Kolmogorov-Smirnov test and t-test indicated that the assumptions of normality and the Markov property were fulfilled for the normalized rainfall rates of the two analyzed data sets. We restricted our attention to annual maximum events, and demonstrated that this approach focuses on the dominant storm type, which in our paper is the tropical storm. The proposed model yields a design storm hyetograph that is readily translated between different storm durations. Peak rainfall depths from our model are consistent with corresponding rainfall depths from the IDF curves. In addition, given the constraint on peak rainfall characteristics, the design hyetograph developed from the proposed model is most likely to occur. Further studies on the hyetographs of short-duration convective storms and extratropical cyclones are needed.

Acknowledgements

We gratefully acknowledge the Council of Agriculture and the National Science Council of Taiwan, R.O.C., for supporting this research. The first author also greatly appreciates the Mathematics Department of the University of Florida for providing excellent research environment during his one-year visit. We are also grateful to anonymous reviewers for their constructive comments that were very helpful for an improved and more accurate presentation.

Literature Cited

Bras, R., and Rodriguez-Iturbe, I., 1976. Rainfall generation: A nonstationary time-varying multidimensional model. Water Resources Research, 12(3):450-456.

Burlando, P. and Rosso, R., 1996. Scaling and multiscaling models for depth-duration-frequency curves for storm precipitation. Journal of Hydrology, 187: 45-64.

Chow, V.T., Maidment, D.R., and Mays, L.W., 1988. Applied Hydrology. McGraw-Hill Co., New York.

Eagleson, P.S., 1970. Dynamic Hydrology. McGraw-Hill, New York.

Garcia-Guzman, A. and Aranda-Oliver, E., 1993. A stochastic model of dimensionless hyetograph. Water Resources Research, 29(7): 2363-2370.

Godano, C., Tosi, P., Derubeis, V., and Augliera, P., 1999. Scaling properties of the spatio-temporal distribution of earthquakes: a multifractal approach applied to a Californian catalogue. Geophysical Journal International, 136: 99-108.

Gupta, V.K. and Waymire, E., 1990. Multiscaling properties of spatial rainfall and river flow distributions. Journal of Geophysical Research, 95(D3): 1999-2009.

Gupta, V.K. and Waymire, E., 1991. On lognormality and scaling in spatial rainfall averages. *In*: Non-linear variability in geophysics, Schertzer, D. and Lovejoy, S. (Editors). Kluwer

Academic Publishers, Dordrecht, Netherland, pp. 175-183.
Huff, F.A., 1967. Time distribution of rainfall in heavy storms. Water Resources Research, 3(4): 1007-1019.
Keifer, C.J. and Chu, H.H., 1957. Synthetic storm pattern for drainage design. Journal of the Hydraulics Division, ASCE, 83(HY4): 1-25.
Koutsoyiannis , D. and Foufoula-Georgiou, E., 1993. A scaling model of a storm hyetograph. Water Resources Research, 29(7): 2345-2361.
La Barbera, P. and Rosso, R., 1989. On the fractal dimension of stream networks. Water Resources Research, 25(4): 735-741.
Olsson, J., Niemcznowicz, J., and Berndtsson, R., 1993. Fractal analysis of high-resolution rainfall time series. Journal of Geophysical Research, 98(D12): 23265-23274.
Pilgrim, D.H. and Cordery, I., 1975. Rainfall temporal patterns for design floods. Journal of the Hydraulics Division, ASCE, 101(HY1): 81-95.
Soil Conservation Service (SCS), 1986. Urban Hydrology for Small Watersheds, Technical Release 55. U.S. Soil Conservation Service, Washington, D.C.
Turcotte, D.L., 1989. Fractals in geology and geophysics. Pure and Applied Geophysics, 131:171-196.
Veltri, M., Veltri, P., and Maiolo, M., 1996. On the fractal description of natural channel networks. Journal of Hydrology, 187: 137-144.
Woolhiser, D.A. and Osborn, H.B., 1985. A stochastic model of dimensionless thunderstorm rainfall. Water Resources Research, 21(4):511-522.
Yen, B.C. and Chow, V.T., 1980. Design hyetographs for small drainage structures. Journal of the Hydraulics Division, ASCE, 106(HY6): 1055-1076.

Table 1. Beginning dates of some annual maximum rainfall events in Taiwan.

1965-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	9/5	8/18	8/18	8/18	8/18	8/18	8/18	8/18	8/18	8/18
Wutuh	6/11	8/21	9/6	9/6	9/6	9/6	8/18	8/18	8/18	8/18
1969-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	5/14	5/14	5/14	5/14	9/26	9/26	9/9	9/9	9/9	9/9
Wutuh	9/21	9/8	9/8	9/8	9/8	9/8	9/8	9/8	9/8	9/8
1974-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	9/27	9/27	10/11	10/11	10/11	10/11	10/11	10/11	10/11	10/11
Wutuh	9/15	9/15	10/11	10/11	9/15	10/11	10/11	10/11	10/11	10/11
1983-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	5/23	5/23	5/23	5/23	5/23	10/10	10/10	10/10	10/10	10/10
Wutuh	10/1	10/1	6/3	6/3	6/3	10/1	10/1	10/1	10/1	10/1
1987-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	7/26	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22
Wutuh	10/22	7/26	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22
1994-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	6/18	6/18	6/18	6/18	6/18	10/9	10/9	10/9	10/9	10/9
Wutuh	6/18	6/18	6/18	9/12	9/12	9/12	9/12	9/12	9/12	9/12

1. Hosoliau locates approximately 20 km southeast of Wutuh.

2. Entries in the table represent month/day of a rainfall event.

*Design duration.

Table 2. Statistics for selected annual maximum events at two raingauges in Taiwan.

Station	Number of events	Event duration (hrs)			
		minimum	maximum	average	standard deviation
Hosoliau	65	12	144	55	32.5
Wutuh	49	12	144	37.8	26.6

Table 3. Parameters for the distributions of normalized rainfalls for two raingauges in Taiwan.

Station / i	Hosoliau			Wutuh		
	μ_i	σ_i	$\rho_1(i)$	μ_i	σ_i	$\rho_1(i)$
1	2.02	2.74	0.00	2.90	4.15	0.00
2	2.87	4.34	0.26	3.52	4.01	0.65
3	4.04	4.53	0.62	3.48	3.86	0.67
4	3.79	4.04	0.74	3.03	3.59	0.78
5	2.98	2.91	0.38	2.80	3.56	0.49
6	3.38	3.22	0.57	3.21	3.02	0.40
7	3.68	3.81	0.57	3.43	3.16	0.72
8	4.17	3.76	0.56	4.67	5.06	0.67
9	3.97	3.42	0.76	4.40	3.63	0.26
10	4.67	4.48	0.62	5.21	4.81	0.71
11	5.30	4.39	0.61	4.47	3.95	0.79
12	4.68	3.07	0.31	5.05	4.93	0.49
13	5.09	4.42	0.67	5.99	5.36	0.57
14	5.52	4.28	0.53	5.19	4.58	0.65
15	6.01	4.87	0.68	6.20	4.72	0.67
16	6.54	4.51	0.68	6.20	3.90	0.52
17	5.79	4.28	0.54	5.65	4.91	0.63
18	6.29	5.02	0.57	4.52	4.29	0.39
19	5.27	4.70	0.62	5.00	4.86	0.46
20	3.86	4.06	0.72	4.55	4.79	0.45
21	3.38	3.45	0.48	3.78	3.76	0.71
22	2.87	3.71	0.38	2.90	2.90	0.51
23	2.45	3.26	0.72	2.37	3.33	0.77
24	1.36	1.55	0.45	1.51	2.22	0.48

Table 4. Results of the normality check for normalized rainfalls at two raingauges in Taiwan.

Time *i*	Hosoliau	Wutuh	Time *i*	Hosoliau	Wutuh
1	**X**	**X**	13	○	○
2	○	**X**	14	○	○
3	○	○	15	○	○
4	○	○	16	○	○
5	○	○	17	○	○
6	○	○	18	○	○
7	○	○	19	○	○
8	○	○	20	○	○
9	○	○	21	○	○
10	○	○	22	○	**X**
11	○	○	23	○	○
12	○	○	24	**X**	**X**

○: Hypothesis not rejected at $\alpha = 0.05$.

X: Hypothesis rejected at $\alpha = 0.05$.

Table 5. Parameters of the IDF curves.

Station	a	m	c	R^2
Hosoliau	375.49	0.1635	0.4513	0.9928
Wutuh	330.36	0.1823	0.4380	0.9883

$\bar{i}_T(D) = \dfrac{aT^m}{D^c}$, where $\bar{i}_T(D)$: rainfall intensity in mm/hr,

D: duration in minutes, and T: recurrence interval in years.

Table 6. Comparison of the peak rainfall depths (mm) from design storm hyetographs and the corresponding rainfall depths from IDF curves (recurrence interval = 100 yr).

Stations	Design Storm Hyetograph[1]		Design Storm Hyetograph[2]		IDF Curves	
	D=2 hrs Δ= 5 min	D=24hrs Δ= 1 hr	D=2 hrs Δ= 5 min	D=24hrs Δ= 1 hr	D=5 min	D=1 hr
Hosoliau	28	110	12	47	32	126
Wutuh	30	123	12	47	32	127

[1]Design storm hyetograph with the constraint on peak rainfall characteristics.

[2]Design storm hyetograph without the constraint on peak rainfall characteristics.

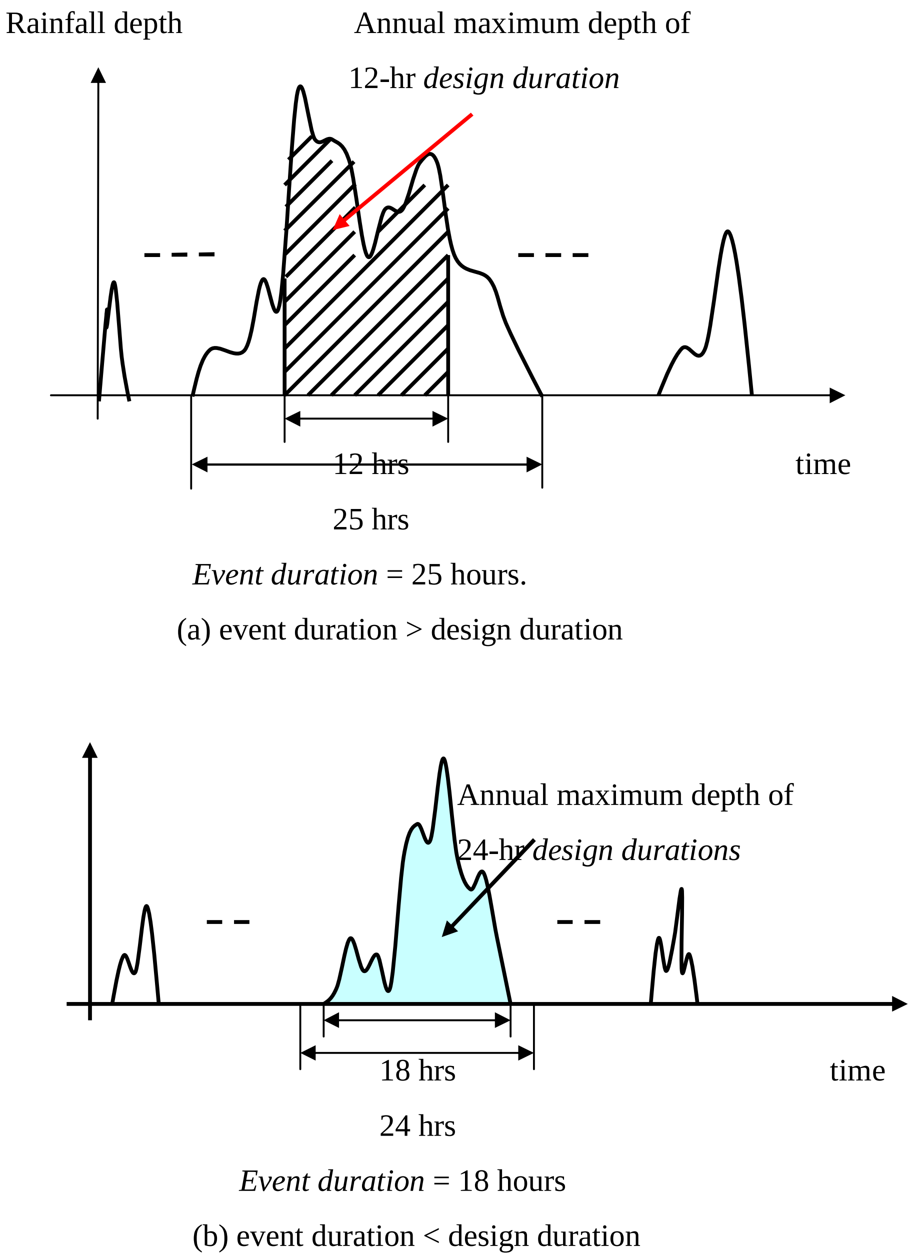
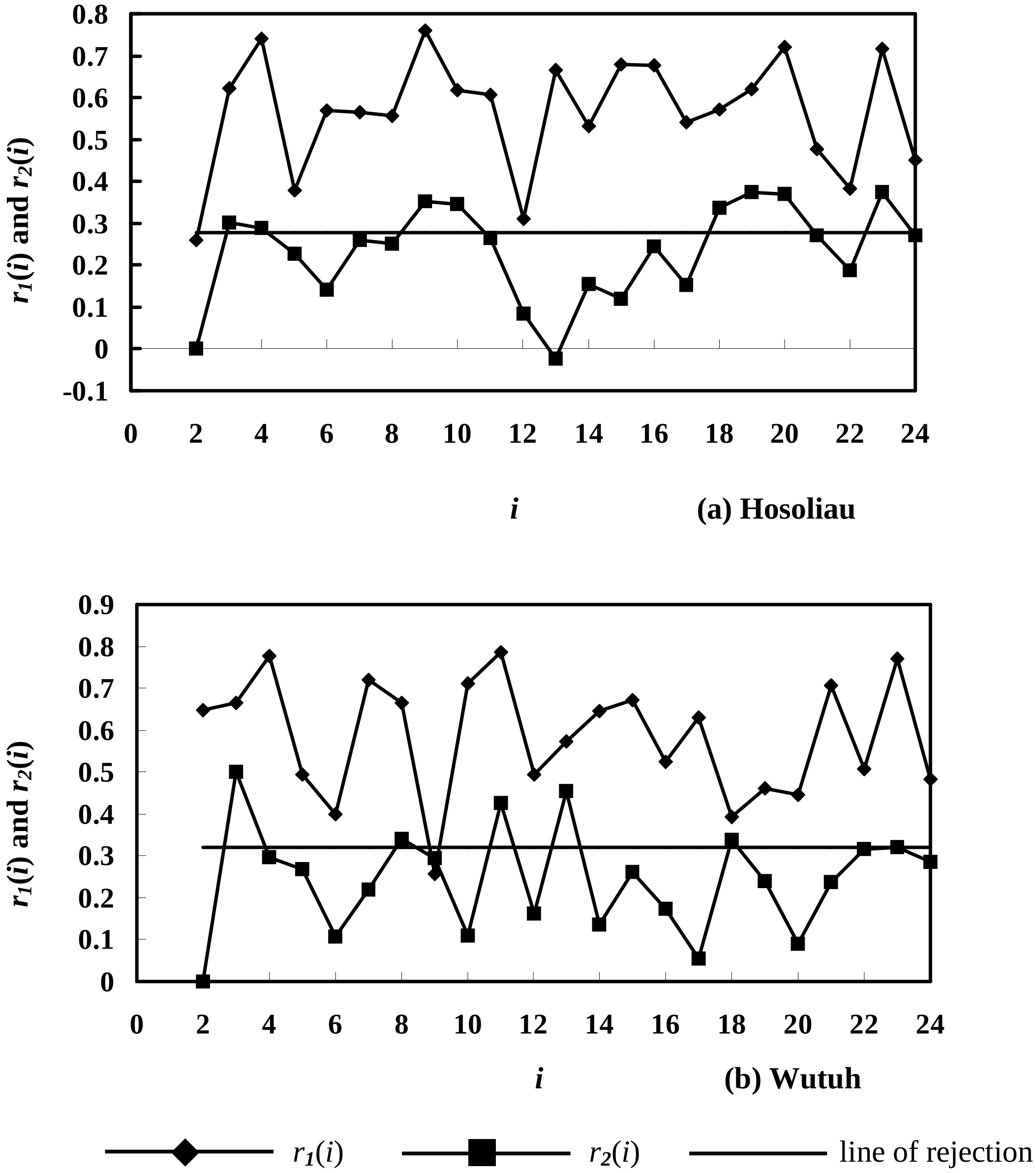


Figure 1. Design duration and event duration of an annual maximum event.



Null hypothesis H_o: $\rho_k(i) = 0$ is rejected if $r_1(i)$ or $r_2(i)$ falls below the line of rejection.

Figure 2. Lag-1 and lag-2 correlation coefficients of normalized rainfalls for two stations in Taiwan.

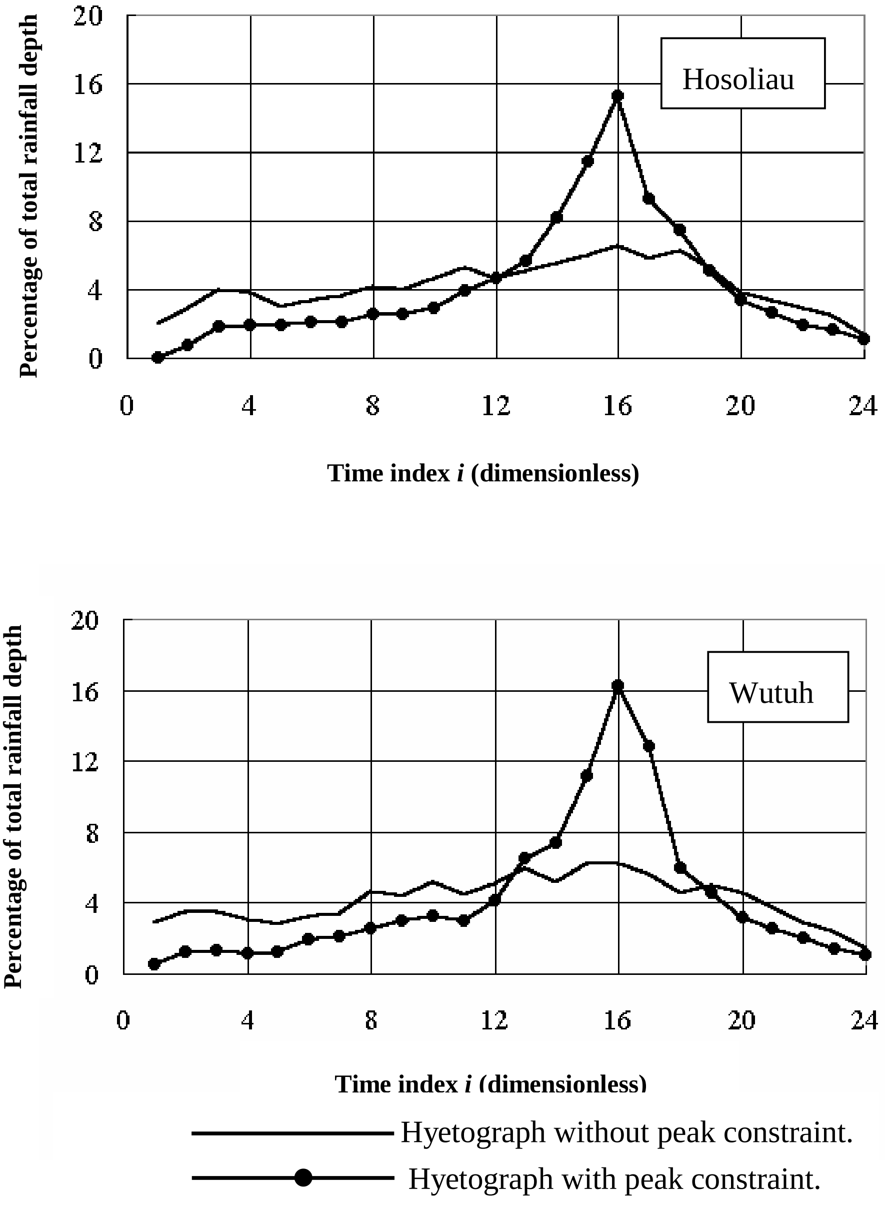


Figure 3. Dimensionless hyetographs for two raingauges in Taiwan.