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On the Generalized Sampling Theorem

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Being based on the generalized sampling theorem[1] presented by the authors, some series-expansions of functions are given as its special examples, including Kroll's expressions[2]. The discussions are also made on the conditions under which our generalized sampling theorem holds.

IN the previous paper[1], the authors presented a generalized sampling theorem with some of its examples, which includes Someya-Shannon's sampling theorem[3,4,5] as a special case. Recently the authors were informed that the sampling theorem was also generalized by Isomichi[10] in connection with the generalized frequency domain.

Here in the present paper, the authors wish to discuss the series-expansion of functions in an orthogonal set of functions, and to consider the conditions under which our generalized sampling theorem holds, and also to mention new formulae which were not given elsewhere.

Let f(t) belong to L_p, and let the following reciprocity relations hold:

$$F(s)=L_s\cdot f=\int_A K(s,t)f(t)dt, \quad \text{for } s\in B \tag{1}$$

and

$$f(t)=L_t^{-1}\cdot F=\int_B \tilde{K}(t,s)F(s)ds, \quad \text{for } t\in A \tag{2}$$

in the domains A and B, respectively.

From (1) and (2), we obtain at once:

$$\int_A K(s,t)\tilde{K}(t,\sigma)dt=\delta(s-\sigma), \tag{3}$$

(1) É. I. Takizawa, K. Kobayasi, and J.-L. Hwang: Chinese Journ. Phys. 5, 21 (1967).
(2) W. Kroll: Chinese Journ. Phys. 5, 86 (1967).
(3) I. Someya: *Transmission of Wave-Forms* (1949), (in Japanese) Syûkyôsya, Tôkyô. (染谷勲: 波形傳送 (1949), 修教社, 東京).
(4) C. E. Shannon: Hell System Techn. Journ. 27, 379; 623 (1948).
(5) C. E. Shannon and W. Weaver: *Mathematical Theory of Communication* (1949), Univ. of Illinois Press.

and

$$\int_B \tilde{K}(t,\ s)K(s,\ \tau)ds=\delta(t-\tau), \tag{3}$$

with delta function δ (2).

Further we assume that

$$F(s)=0,\ \text{for}\ \ s\notin D\subseteq B \tag{5}$$

in the outside of the domain D. Accordingly, we obtain from (2) and (5)

$$f(t)=L_t^{-1}\cdot F=\int_D \tilde{K}(t,s)F(s)ds. \tag{6}$$

Let $F(s)$ be expanded in a complete orthogonal set of functions:

$$\{\phi_n(s);\ \int_D \phi_m(s)\phi_n(s)ds=\gamma_m\cdot\delta_{m,n}\ (m,\ n=\text{integers})\}, \tag{7}$$

in the domain $D\subseteq B$; *i. e.*

$$F(s)=\sum_n a_n\ \phi_n(s),\quad \text{for s} \in D\subseteq B. \tag{8}$$

The expression (8), being multiplied by $\phi_m(s)$ and integrated over D, gives:

$$\int_D F(s)\phi_m(s)ds=\sum a_n\cdot\int_D \phi_m(s)\ \phi_n(s)\ \ ds=\sum_n a_n\cdot\gamma_m\cdot\delta_{m,n}=\gamma_m\cdot a_m. \tag{9}$$

From (6), (7), (8) and (9), we obtain:

$$\begin{aligned} f(t)&=L_t^{-1}\cdot F=\sum_n a_n\cdot L_t^{-1}\cdot\ \phi_n \\ &=\sum_n\Big(\frac{1}{\gamma_n}\int_D F(s)\ \phi_n(s)\ ds\Big)\bullet \int \tilde{K}(t,s)\ \phi_n(s)\,ds. \end{aligned} \tag{10}$$

If the functions $f(t)$, $\{\phi_n(s)\}$, and the integral kernels $K(s,t)$ and $\tilde{K}(t,s)$, are given, then we can construct a series-expansion of $f(t)$ by means of (10).

If we can take

$$\phi_n(s)=\tilde{K}(\lambda_n,\ s),\quad \text{for s} \in D \tag{11}$$

with constants λ_n, *i.* e. if the kernel $\tilde{K}(\lambda_n,s)$ can be put equal to $\phi_n(s)$, then the expression (10) is simplified into

$$\begin{aligned} f(t)&=\sum_n\Big(\frac{1}{\gamma_n}\int_D F(s)\ \tilde{K}(\lambda_n,\ s)\ ds\Big)\bullet \int \tilde{K}(t,\ s)\ \tilde{K}\ (\lambda_n,\ s)\ ds \\ &=\sum_n\frac{1}{\gamma_n}f(\lambda_n)\bullet \int \tilde{K}(t,\ s)\ \tilde{K}(\lambda_n,\ s)\ ds, \end{aligned} \tag{12}$$

by means of (6). The expression (12) gives a *generalized sampling theorem(')*. The points at the variable t:

$$t=\lambda_n,\quad (n=\text{integers}) \tag{13}$$

are called *sampling* points, and the function $g_n(t)$, defined by

$$g_n(t)=L_t^{-1}\cdot\phi_n=\int_D \tilde{K}(t,\ s)\ \phi_n(s)\ ds=\int_D \tilde{K}(t,\ s)\ \tilde{K}(\lambda_n,\ s)\ ds,$$

$$(n = \text{integers}) \tag{14}$$

is the sampling *function.*

If the set of functions $\{\tilde{K}(\lambda_n, s)$; n=integers$\}$ is not necessarily orthogonal, but complete and linearly independent, we can find normalized biorthogonal set $\{\psi_m(s)$; m=integers$\}$ by the method given by Iizima[11], in such a way that

$$\int_D \tilde{K}(\lambda_m,\ s)\ \psi_n(s)\ ds=\delta_{m,n}, \tag{15}$$

is to be satisfied. From the completeness of the set $\{\tilde{K}(\lambda_m, s)$; m=integers$\}$, we have:

$$\sum_n \tilde{K}(\lambda_n, s)\ \psi_n(\sigma)=\delta(s-\sigma). \tag{16}$$

Isomichi[10] expanded $F(s)$ in $\{\psi_n(s)$; n=integers$\}$, and obtained:

$$F(s)=\sum_n b_n\ \psi_n(s),\quad \text{for}\quad s\in D\subseteq B \tag{17}$$

with

$$b_n=\int_D \tilde{K}(\lambda_n,\ s)\ F(s)\ ds. \tag{18}$$

From (6) and (18), we obtain at once:

$$b_n=f(\lambda_n)=\int_D \tilde{K}(\lambda_n,\ s)\ F(s)\ \mathbf{ds}. \tag{19}$$

Operating L_t^{-1} on (17), we obtain, by means of (6) :

$$\begin{aligned} f(t)\ &=L_t^{-1}\cdot F=\sum_n b_n\cdot\int_D \tilde{K}(t,\ s)\ \psi_n(s)\ ds \\ &=\sum_n f(\lambda_n) \cdot \int_D \tilde{K}(t,s)\psi_n(s)\ ds. \end{aligned} \tag{20}$$

The expression (18) or (19) becomes to:

$$\begin{aligned} f(\lambda_n)&=\int_D ds\cdot\tilde{K}(\lambda_n,\ s)\ .\int_A dt\cdot K(s,\ t)\ f(t) \\ &=\int_A dt\cdot f(t)\cdot\int_D ds\cdot\tilde{K}(\lambda_n,\ s)\ K(s,\ t), \end{aligned} \tag{21}$$

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(10) Y. Isomichi: Research Group of Information Theory, Report IT 67-36 (磯道義典:情報理論研究會資料 IT 67-36) 電子通信學會 (1967), 1.

(11) T. Iizima: Method of Functional Analysis I, Theory of Biorthogonality (飯島泰藏:函數解析の手法 I, 陪直交性理論) (1966), 1. 電氣試驗所オートマトン研究室刊行.

by means of (1). The expresion (20) is the generalized sampling theorem obtained by Isomichi[10], and the expression

$$g_n(t)=\int_D \tilde{K}(t,s)\,\psi_n(s)\,ds,\quad (n=\text{integers}) \tag{22}$$

is the the sampling function, with sampling points $t=\lambda_n(n=\text{integers})$ and with

$$g_n(\lambda_m)=\int_D \tilde{K}(\lambda_m,s)\,\psi_n(s)\,ds=\delta_{m,n}.$$

The idea of the generalized sampling theorem was given by one of the present authors, Takizawa, in his book[6] of information theory.

If we take $\{\tilde{K}(\lambda_n, s)\}$ to be orthogonal, then we have at once

$$\{\psi_n(s)\}=\{\phi_n(s)\}=\{\tilde{K}(\lambda_n,s)\}, \tag{23}$$

as was given in (11), the expression (20) reduces to the sampling theorem (12), the dual vector $\{\psi_n(s)\}$ for $\{\tilde{K}(\lambda_n, s)\}$ reducing to $\{\tilde{K}(\lambda_n, s)\}$ itself.

The corresponding expressions of F(t) for (20) and (21) are as follows:

$$F(t)=\sum_n F(\mu_n)\cdot\int_E K(t,s)\,\psi_n(s)\,ds, \tag{24}$$

and

$$F(\mu_n)=\int_B dt\cdot F(t)\cdot\int ds\cdot K(\mu_n,s)\;\tilde{K}(s,t), \tag{25}$$

with the sampling function:

$$p_n(t)=\int_E K(t,s)\,\tilde{\psi}_n(s)\,ds, \tag{26}$$

and the sampling points $t=\mu_n$ (n=integers) , under the condition that for the complete set $\{K(\mu_n,t);\ \text{n=integers}\}$ we have a set of biorthogonal functions $\{\tilde{\psi}_n(s);\ \text{n=integers}\}$ in the domain $E\subseteq$ A, *i.* e.

$$\int_E K(\mu_m,t)\,\tilde{\psi}_n(t)\,dt=\delta_{m,n},\quad \text{for } t\in E\subseteq \text{A}. \tag{27}$$

From the completeness of the set $\{K(\mu_m, s);\ \text{m=integers}\}$, it follows that the expression:

$$\sum_n K(\mu_n,s)\,\tilde{\psi}_n(\sigma)=\delta(s-\sigma), \tag{28}$$

holds.

Now, let us take sampling functions $\{g_n(t);n=\text{integers}\}$, $\{h_n(t);n=\text{integers}\}$, $\{p_n(t);\ \text{n=integers}\}$, and $\{q_n(t);n=\text{integers}\}$, defined as follows:

(6) É. I. Takizawa: Information Theory and *its Exercises* (1966), (in Japancese). Hirokawasyoten, Tôkyô, p. 275. 瀧澤英一: 情報の理論と演習 (1966), 廣川書店, 東京.

$$g_n(t) \equiv \int_D \tilde{K}(t, s)\, \psi_n(s)\, ds, \tag{29}$$

$$h_n(t) \equiv \int_D K(s, t)\, \phi_n(s)\, ds = \int_D K(s, t)\, \tilde{K}(\lambda_n, s)\, ds, \tag{30}$$

$$p_n(t) \equiv \int_E K(t, s)\, \tilde{\psi}_n(s)\, ds, \tag{31}$$

and

$$q_n(t) \equiv \int_E \tilde{K}(s, t)\, K(\mu_n, s)\, ds. \tag{32}$$

From (3) and (4), we have

$$\int_A K(\sigma, t)\, g_n(t)\, dt = \int_D \psi_n(s) \cdot \delta(\sigma - s)\, ds = \begin{cases} \psi_n(\sigma), & \text{for } \sigma \in D \\ 0, & \text{for } \sigma \notin D \end{cases} \tag{33}$$

$$\int_A \tilde{K}(t, \sigma)\, h_n(t)\, dt = \int_D \phi_n(s) \bullet a(o\text{-}s)\, ds = \begin{cases} \tilde{K}(\lambda_n, \sigma), & \text{for } \sigma \in D \\ 0, & \text{for } \sigma \notin D \end{cases} \tag{34}$$

$$\int_B \tilde{K}(\sigma, t)\, p_n(t)\, dt = \int_E \tilde{\psi}_n(s) \cdot \delta(\sigma - s)\, ds = \begin{cases} \tilde{\psi}_n(\sigma), & \text{for } \sigma \in E \\ 0, & \text{for } \sigma \notin E \end{cases} \tag{35}$$

and

$$\int_B K(t, \sigma)\, q_n(t)\, dt = \int_E K(\mu_n, s) \bullet 6(\sigma - s)\, ds = \begin{cases} K(\mu_n, \sigma), & \text{for } \sigma \in E \\ 0, & \text{for } \sigma \notin E. \end{cases} \tag{36}$$

By means of (3) and (15), we obtain

$$\begin{aligned} \int_A g_m(t)\, h_n(t)\, dt &= \int_D ds \cdot \int_D d\sigma \cdot \psi_m(s)\, \phi_n(\sigma) \cdot \delta(\sigma - s) \\ &= \int_D \psi_m(s)\, \phi_n(s)\, ds = \int_D \tilde{K}(\lambda_n, s)\, \psi_m(s)\, ds = \delta_{m,n}, \end{aligned} \tag{37}$$

for any integers m and n. This expression (37) shows that any function of $\{g_n(t);\ n=\text{integers}\}$ and any function of $\{h_n(t);\ n=\text{integers}\}$ are *mutually orthogonal* in the domain D. Similarly, we obtain, from (4) and (27) :

$$\int_B p_m(t)\, q_n(t)\, dt = \int_E \tilde{\psi}_m(s)\, K(\mu_n, s)\, ds = \delta_{m,n}. \tag{38}$$

We shall return to the expressions (20), (21), (24), (25). They are written as follows :

$$f(t) = \sum_n f(\lambda_n)\, g_n(t), \tag{39}$$

$$f(\lambda_n) = \int_A f(t)\, h_n(t)\, dt, \tag{40}$$

$$F(t) = \sum_n F(\mu_n)\, p_n(t), \tag{41}$$

and

$$F(\mu_n) = \int_B F(t)\, q_n(t)\, dt, \tag{42}$$

in terms of (29) $\sim$ (32).

Let $\xi(t)$ be any function which can be expanded in $\{g_n(t); n = \text{integers}\}$, i.e.

$$\xi(t) = \sum_n c_n g_n c_n(t), \tag{43}$$

with constants c_n, then the integral operator:

$$\mathcal{G} = \int d\tau \cdot \sum_n g_n(t)\, h_n(\tau) = \int_A d\tau \cdot \delta(t-\tau), \tag{44}$$

applied to (43), is the ***identical** operator*, in the sense that

$$\mathcal{G} \cdot \xi(\tau) = \xi(t). \tag{45}$$

The proof of (45) is quite obvious, if we put (43) into (45) and refer (37).

Considering (4) and (44), we shall take

$$F^*(s) = \int_A \tilde{K}(t, s)\, f(t)\, dt, \tag{46}$$

with

$$f(t) = \int_B K(s, t)\, F^*(s)\, ds, \tag{47}$$

and

$$f^*(\lambda_n) = \int_A g_n(t)\, f(t)\, dt, \tag{48}$$

with

$$f(t) = \sum_n f^*(\lambda_n)\, h_n(t). \tag{49}$$

Then we obtain, from (2), (3), (47), (39), (49), and (37), the following expressions:

$$\begin{aligned} \|f\|^2 &= \int_A dt \cdot \int_B ds \cdot K(s, t)\; F^*(s) \cdot \int_B d\sigma \cdot \tilde{K}(t, \sigma)\; F(\sigma) \\ &= \int_B ds \cdot F^*(s) \quad \text{j-,}\, d\sigma \,.\, F(\sigma) \cdot \delta(s-\sigma) = \int_B F^*(s) F(s)\; ds, \end{aligned} \tag{50}$$

and

$$\begin{aligned} \|f\|^2 &= \int_A dt \cdot \sum_m \sum_n f^*(\lambda_m)\, h_m(t) \cdot f(\lambda_n)\, g_n(t) \\ &= \sum_m \sum_n f^*(\lambda_m) f(\lambda_n) \cdot \delta_{m,n} = \sum_n f^*(\lambda_n) f(\lambda_n), \end{aligned} \tag{51}$$

with the norm $\|f\|$ of function f(t) :

$$\|f\|^2=\int_A [f(t)]^2\, dt. \tag{52}$$

The expressions (37),(45), and the generalized Parseval equalities (50) and (51) are given by Isomichi[10].

Now we shall differentiate (39) with regard to t, then we have

$$f'(t)=\sum_n f(\lambda_n)\ g_n'(t). \tag{53}$$

On the other hand, we apply the generalized sampling theorem (39) to f'(t), then we obtain

$$f'(t)=\sum_n f'(\lambda_n)\ g_n(t), \tag{54}$$

with

$$f'(\lambda_n)=\left[\frac{df(t)}{dt}\right]_{t=\lambda_n}.$$

From (53) and (54), we obtain

$$\sum_n f'(\lambda_n)\ g_n(t)-\sum_n f(\lambda_n)\ g_n'(t)=0. \tag{55}$$

Accordingly, from the m-th derivative of f(t), we obtain:

$$\sum_n f^{(m)}(\lambda_n)\ g_n(t)-\sum_n f(\lambda_n)\ g_n^{(m)}(t)=0, \tag{56}$$

with

$$f^{(m)}(z)=\frac{d^m}{dz^m}f(z),\ \text{and}\ g_n^{(m)}(z)=\frac{d^m}{dz^m}g_n(z),\ (m=\text{integers}).$$

Here, we shall mention an expansion-formula, which contains both Taylor's and sampling formulae. If a function f(t) is expressed in (39), *i.e.*

$$f(t)=\sum_n f(\lambda_n)\ g_n(t), \tag{57}$$

and if we can expand $f(t+\xi)$ in a Taylor series, as

$$f(t+\xi)=\sum_{m=0}^{+\infty}\frac{1}{m!}\xi^m\cdot f^{(m)}(t), \tag{55}$$

then we have an expansion-formula:

$$f(t+\xi)=\sum_{m=0}^{+\infty}\sum_n\frac{1}{m!}\xi^m\cdot f^{(m)}(\lambda_n)\ g_n(t) \tag{59}$$

$$=\sum_{m=0}^{+\infty}\sum_n\frac{1}{m!}\xi^m\cdot f(\lambda_n)\ g_n^{(m)}(t). \tag{60}$$

For the special case of $\xi=0$, the expressions (59) and (60) reduce to (57). While, for the special case of $t=\lambda_k$ (with k fixed) in (59) and (60), we have

$$f(\lambda_k+\xi)=\sum_{m=0}^{+\infty}\sum_n \frac{1}{m!}\xi^m\cdot f^{(m)}(\lambda_n)\ g_n(\lambda_k)=\sum_m\sum_n\frac{1}{m!}\xi^m\cdot f^{(m)}(\lambda_n)\cdot\delta_{n,k}$$

$$=\sum_{m=0}^{+\infty}\frac{1}{m!}\xi^m\cdot f^{(m)}(\lambda_k), \tag{61}$$

and

$$f(\lambda_k+\xi)=\sum_{m=0}^{+\infty}\sum_n\frac{1}{m!}\xi^m\cdot f(\lambda_n)\,g_n^{(m)}(\lambda_k). \tag{62}$$

The expression (61) is nothing but the usual Taylor expansion of $f(\lambda_k+\xi)$, while the expression (62) gives $f(\lambda_k+\xi)$ in terms of the derivatives of sampling functions g_n at the sampling point λ_k.

Example 1

We shall take the Fourier transforms in (1) and (2), and take

$$K(s,t)=\frac{1}{2\pi}\exp[ist],\ A=(-\infty,+\infty)$$

$$\tilde{K}(t,s)=\exp[-its],\ B=(-\infty,+\infty),$$

$$\phi_n(s)=\tilde{K}(\lambda_n,s)=\exp[-i\lambda_n s],\ D=(-\beta,\beta),$$

$$\lambda_n=\frac{n-k}{\beta}\pi,\ n=\text{integers},\ k=\text{real number given arbitrarily},$$

$$\int_{-\beta}^{\beta}\exp[-i\lambda_m s]\cdot\exp[+i\lambda_n s]\,ds=\gamma_m\cdot\delta_{m,n},\qquad \gamma_n=2\beta,$$

then the sampling theorem (12) reads as follow:

$$f(t)=\frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}f\left(-\frac{n-k}{\beta}\pi\right)\cdot\frac{2\sin[\beta t+(n-k)\pi]}{t+\frac{x-k}{\beta}\pi}$$

$$=\sum_{o=-\infty}^{+\infty}f\left(\frac{n+k}{\beta}\pi\right)\cdot\frac{\sin[\beta t-(n+k)\pi]}{\beta t-(n-k)\pi}. \tag{63}$$

The expression is nothing but Someya's sampling [3] given in 1949. The expression (63) gives $f(t)$ in terms of the sampling function $\sin[\beta t-(n+k)\pi]/[\beta t-(n+k)\pi]$, with values of $f((n\pi+k\pi)/\beta)$ at sampling points $t=(n+k)\pi/\beta$ (n=integers). if we put $k=O$ in (63), we have Shannon's sampling theorem[4][5].

Example 2

We shall take the Fourier cosine transforms for (1) and (2), and take:

$$K(s, t) = \frac{2}{\pi}\cos(st),\ A=(0, +\infty),$$

$$\tilde{K}(t, s) = \cos(ts),\ B=(0, +\infty),$$

$$\phi_n(s) = \tilde{K}(\lambda_n, s) = \cos(\lambda_n s),\ D=(0, \beta),$$

$$r_n = \frac{\beta}{2}\{1+\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}\},$$

where $\lambda_n's$ are the roots (arranged in ascending order of magnitude) of the equation:

$$\lambda_n \tan(\lambda_n\beta) = M, \tag{64}$$

with a constant M. Then the expression (12) takes form:

$$\begin{aligned} f(t) &= \frac{2}{\beta}\sum_n \frac{1}{1+\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\left(\int_0^\beta F(s)\cos(\lambda s)ds\right) \cdot \int_0^\beta \cos(ts)\cos(\lambda_n s)ds \\ &= \frac{2}{\beta}\sum_n f(\lambda_n)\cdot\frac{\cos(\lambda_n\beta)}{1+\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\tan(\beta t)-M}{t^2-\lambda_n^2}\cos(\beta t), \end{aligned} \tag{65}$$

with sampling points $t=\lambda_n$ (n=integers), and the sampling function:

$$g_n(t) = \frac{t\tan(\beta t)-M}{t^2-\lambda_n^2}\cos(\beta t) \tag{66}$$

Example 3

We shall take the Fourier sine transforms (1) and (2), and take:

$$K(s,t) = \frac{2}{\pi}\sin(st),\ A=(0, +\infty),$$

$$\tilde{K}(t, s) = \sin(ts),\ B=(0, +\infty),$$

$$\phi_n(s) = \tilde{K}(\lambda_n, s) = \sin(\lambda_n s),\ D=(0, \beta),$$

$$r_n = \frac{\beta}{2}\{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}\},$$

where $\lambda_n's$ are the roots (arranged in ascending order of margnitude) of the equation :

$$\lambda_n \cot(\lambda_n\beta) = N, \tag{67}$$

with a constant N. Then the expression (12) becomes to:

$$f(t)=\frac{2}{\beta}\sum_{n}\frac{1}{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\left(\int_0^\beta F(s)\sin(\lambda_n s)ds\right)\cdot\int_0^\beta \sin(ts)\sin(\lambda_n s)ds$$

$$=-\frac{2}{\beta}\sum_{n}f(\lambda_n)\cdot\frac{\sin(\lambda_n\beta)}{1-\frac{\sin(2\lambda_n\beta)}{2\lambda_n\beta}}\cdot\frac{t\cot(\beta t)-N}{t^2-\lambda_n^2}\sin(\beta t), \tag{68}$$

with sampling points $t=\lambda_n$ (n=integers), and the sampling function:

$$g_n(t)=\frac{t\cot(\beta t)-N}{t^2-\lambda_n^2}\sin(\beta t). \tag{69}$$

The formulae (65) and (68) were obtained by Kroll[2] in connection with the solution of an integral equation.

Example 4

Let us take the Hankel transforms of order $\nu(\nu\geqq-1/2)$ for (1) and (2), and take the Fourier-Bessel series[7] for F(s) :

$$\sqrt{t}\ f(t)\in L_1\ (0,\ +\infty),$$

$$K(s,t)=t\ J_\nu(s\ t),\ A=(0,+\infty),$$

$$\tilde{K}(t,s)=s\ J_\nu(ts),\ B=(0,+\infty),$$

$$\phi_n(s)=J_\nu(j_n s),\ D=(0,\beta),$$

with the orthogonality relations :

$$\int_0^\beta s\ J_\nu(j_m s)\ J_\nu(j_n s)\ ds=r_m\cdot\delta_{m,n},$$

and

$$r_n=\frac{\beta^2}{2}J_{\nu+1}^2(j_n\beta),$$

where $j_n\beta$ (n=1, 2, 3,. ..) are the positive zeros of $J_\nu(z)$, being arranged in ascending order of magnitude, *i*. e.

$$J_\nu(j_n\beta)=0. \tag{70}$$

The expression (12) takes form :

$$f(t)=\frac{2}{\beta^2}\sum_{n=1}^{+\infty}\frac{1}{J_{\nu+1}^2(j_n\beta)}\left(\int_0^\beta sF(s)J_\nu(j_n s)ds\right)\bullet\int_0^\beta s\,J_\nu(ts)\,J_\nu(j_n s)\,ds$$

$$=-\frac{2}{\beta}\sum_{n=1}^{+\infty}f(j_n)\cdot\frac{j_n}{J_{\nu+1}(j_n\beta)}\cdot\frac{J_\nu(\beta t)}{t^2-j_n^2}, \tag{71}$$

(7) G. N. Watson: *Theory of Bessel Functions* (1944), Cambridge Univ. Press. p. 576.

with sampling points $t=j_n$ (n=positive integers), and the sampling function:

$$g_n(t)=\frac{J_\nu(\beta t)}{t^2-j_n^2}. \tag{72}$$

The formula (71) was given- by the present authors[1] in this journal.

Example 5

We shall take the Hankel transforms for (1) and (2) just as in Example 4. The function $F(s)$ is assumed to be expanded in the Deni expansion@) in the domain $D=(0, \beta)$. We shall take

$$K(s, t)=t\ J_\nu(st),\ A=(0, +\infty),$$

$$\tilde{K}(t, s)=s\ J_\nu(t\ s),\ B=(0,+\infty),$$

$$\phi_n(s)=J_\nu(\lambda_n s),\ D=(0, \beta),$$

with the orthogonality relation :

$$\int_0^\beta s\ J_\nu(\lambda_m s)\ J_\nu(\lambda_n s)\ ds=\gamma_m\cdot\delta_{m,\ n},$$

and

$$\gamma_n=\frac{1}{2\lambda_n^2}\{(\lambda_n^2+h^2)\beta^2-\nu\}J_\nu^2(\lambda_n\beta),$$

where λ_n's $(n=1,2,3,\ldots)$ are the positive roots (arranged in ascending order of magnitude) of the following equation:

$$\lambda_n J'_\nu(\lambda_n\beta)+hJ_\nu(\lambda_n\beta)=0, \tag{73}$$

with a constant h.

Then the expression (12) becomes:

$$f(t)=2\sum_{n=1}^{+\infty}\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2-\nu^2\}J_\nu^2(\lambda_n\beta)}\left(\int_0^\beta s\ F(s)J_\nu(\lambda_n s)ds\right)\cdot\int_0^\beta s\ J_\nu(ts)J_\nu(\lambda_n s)ds$$

$$=-2\beta\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2-\nu^2\}J_\nu(\lambda_n\beta)}\cdot\frac{t\ J'_\nu(\beta t)+hJ_\nu(\beta t)}{t^2-\lambda_n^2}, \tag{74}$$

with sampling points $t=\lambda_n$ (n=positive integers), and the sampling function:

$$g_n(t)=\frac{tJ'_\nu(\beta t)+hJ_\nu(\beta t)}{t^2-\lambda_n^2}, \tag{75}$$

The formula (74) was given by Kroll[2] in this journal.

Letting h tend to infinity in the expressions (73) and (74), we see that they reduce to the following expressions, respectively:

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(8) G. N. Watson: Theory of Bessel Functions (1944), Cambridge Univ. Press. p. 580.

$$J_\nu(\lambda_n\beta)=0, \tag{76}$$

and

$$f(t)=-\frac{2}{\beta}\sum_{n=1}^{+\infty}f(\lambda_n)\ \text{[illegible]}\cdot\frac{J_\nu(\beta t)}{t^2-\lambda_n^2}. \tag{77}$$

The expressions (76) and (77) are nothing but the expressions (70) and (71), respectively. Accordingly, the expression (71) is a limiting case of (74).

If we put $\nu=1/2$ in the expression (74), then we obtain

$$f(t)=-2\beta\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2\sqrt{\lambda_n}}{\{(\lambda_n^2+h^2)\beta^2-\frac{1}{4}\}\sin(\lambda_n\beta)}\times$$

$$\frac{t\cos(\beta t)+(h-\frac{1}{2\beta})\sin(\beta t)}{\sqrt{t}\,(t^2-\lambda_n^2)}, \tag{78}$$

where λ_n's are positive roots of the following equations, being arranged in ascending order of magnitude:

$$\lambda_n\cos(\lambda_n\beta)+(h-\frac{1}{2\beta})\sin(\lambda_n\beta)=0. \tag{79}$$

In the expression (78), if we replace f(t) and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)/\sqrt{\lambda_n}$ respectively, then we obtain :

$$f(t)=-2\beta\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2-\frac{1}{4}\}\sin(\lambda_n\beta)}\times$$

$$\frac{t\cot(\beta t)+(h-\frac{1}{2\beta})}{t^2-\lambda_n^2}\sin(\beta t). \tag{80}$$

Further, if we put

$$h-\frac{1}{2\beta}=-N, \tag{81}$$

in the expressions (79) and (80), then the expressions (79) and (80) reduce to (67) and (68), respectively.

If we put

$$h=\frac{1}{2\beta},\quad N=0, \tag{82}$$

in the expressions (80) and (79), then we obtain:

$$f(t)=-\frac{2}{\beta}\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{1}{\sin(\lambda_n\beta)}\cdot\frac{t\cos(\beta t)}{t^2-\lambda_n^2}, \tag{83}$$

with

$$\cos(\lambda_n\beta)=0, \tag{84}$$

i.e.

$$\lambda_n=\frac{2n-1}{2\beta}\pi, \quad (n=1,2,3,\ldots) \tag{85}$$

and

$$\sin(\lambda_n\beta) = (-1)^{n+1}.\ (n=1,2,3,\ldots) \tag{86}$$

Then the expression (83) reduces to :

$$f(t)=2\sum_{n=1}^{+\infty} f\left(\frac{2n-1}{2\beta}\pi\right).(-)^n\cdot\frac{\beta t\ \cos(\beta t)}{\beta^2t^2-\frac{(2n-1)^2\pi^2}{4}} \tag{87}$$

The expression (87) correponds to Someya ' s sampling theorem (with $k=\pm 1/2$) for an odd function f(t) .

It we put $\nu=-1/2$ in (74), we obtain :

$$f(t)=2\beta\sum_{n=1}^{+\infty} f(\lambda_n).\frac{\lambda_n^2\sqrt{\lambda_n}}{\{(\lambda_n^2+h^2)\beta^2-\frac{1}{4}\}\cos(\lambda_n\beta)}\times$$

$$\frac{t\sin(\beta t)-(h-\frac{1}{2\beta})\cos(\beta t)}{\sqrt{t}\,(t^2-\lambda_n^2)}, \tag{88}$$

with

$$\lambda_n\ \sin(\mathrm{M})\ -(h-\frac{1}{2\beta})\cos(\lambda_n\beta)\ =0. \tag{89}$$

In the expression (88), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/\sqrt{t}$ and $f(\lambda_n)\sqrt{\lambda_n}$ respectively, then we obtain

$$f(t)=2\beta\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{\lambda_n^2}{\{(\lambda_n^2+h^2)\beta^2-\frac{1}{4}\}\cos(\lambda_n\beta)}\times$$

$$\frac{t\ \tan(\beta t)\ -(\mathrm{h}--\mathrm{þ})}{t^2-\lambda_n^2}\cos(\beta t) \tag{90}$$

Further, if we put

$$h-\frac{1}{2\beta}=M, \tag{91}$$

in the expressions (89) and (90), then the expressions (89) and (90) reduce to (64) and (65), respectively.

If we put

$$h=\frac{1}{2\beta},\ M=0, \tag{92}$$

in the expressions (90) and (89), then we obtain the following expressions :

$$f(t)=\frac{2}{\beta}\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{1}{\cos(\lambda_n\beta)}\cdot\frac{t\sin(\beta t)}{t^2-\lambda_n^2}, \tag{93}$$

with

$$\sin(\lambda_n\beta)=0, \tag{94}$$

i. e.

$$\lambda_n=\frac{n\pi}{\beta},\ (n=1,2,3,\dots) \tag{95}$$

and

$$\cos(\lambda_n\beta)=(-1)^n,\qquad (n=1,2,3,\dots). \tag{96}$$

Then the expression (93) reduces to:

$$f(t)=2\sum_{n=1}^{+\infty}f(\frac{n\pi}{\beta})\cdot(-)^n\cdot\frac{\beta t\sin(\beta t)}{\beta^2t^2-n^2\pi^2}. \tag{97}$$

The expression (97) corresponds to Shannon's sampling theorem for an even function f(t).

Example 6

We shall take other Hankel transforms due to Weber[9] for (1) and (2), namely

$$t\ f(t)\in L_1(p,\ +\infty),\ a\geqslant p>0,$$

$$F(s)=\int_A K(s,t)f(t)dt,\ \text{for}\ \nu\geqslant-\frac{1}{2} \tag{98}$$

and

$$G_\nu\,(at)\ f(t)=\int_B\tilde{K}(t,s)F(s)ds,\quad \text{for}\ t\in A \tag{99}$$

with

$$K(s,t)=t\,T_\nu(ta,\ st),\ A=(p,+\infty),$$

$$\tilde{K}(t,\ s)=s\,T_\nu(ta,ts),B=(a,\ +\infty),$$

$$T_\mu(x,z)=Y_\nu(x)J_\mu(z)-J_\nu(x)Y_\mu(z), \tag{100}$$

and

$$G_\nu(z)=e^{i\nu\pi}.\ Y_\nu(z)\ Y_\nu(-z), \tag{101}$$

under the condition that the integral:

$$\int_p^{\infty+} t\ f(t)\ dt<+\infty,$$

is absolutely convergent.

We shall take another Fourier-Bessel expansion[9] for $F(s)$ of the form:

$$F(s)=\sum_{n=1}^{+\infty} a_n\ \phi_n(s),\quad \text{for } s\in D\subseteq B$$

with

$$\phi_n(s)=T_\nu(\lambda_n a, \lambda_n s),\quad D=(a, \beta),$$

$$\int_\beta^a s\ T_\nu(\lambda_m a, \lambda_m s)\ T_\nu(\lambda_n a, \lambda_n s)=\gamma_m \bullet \delta_{m,n}, \tag{102}$$

and

$$\gamma_n=\frac{\beta^2}{2}T^2_{\nu+1}(\lambda_n a, \lambda_n\beta)-\frac{a^2}{2}T^2_{\nu+1}(\lambda_n a, \lambda_n a)$$

$$=\frac{2}{\pi^2\lambda_n^2}\left[\frac{J_\nu^2(\lambda_n a)}{J_\nu^2(\lambda_n\beta)}-1\right], \tag{103}$$

where $\lambda_n's$ are the positive roots of the equation:

$$T_\nu(\lambda_n a, \lambda_n\beta)=0, \tag{104}$$

i. e.

$$Y_\nu(\lambda_n a)\ J_\nu(\lambda_n\beta)=J_\nu(\lambda_n a)\ Y_\nu(\lambda_n\beta).$$

From (99), the expression (12) becomes to:

$$f(t)=\frac{\pi^2}{2}\sum_{n=1}^{+\infty}\frac{\lambda_n^2 J_\nu^2(\lambda_n\beta)}{J_\nu^2(\lambda_n a)-J_\nu^2(\lambda_n\beta)}\cdot G_\nu(\lambda_n a)\cdot f(\lambda_n)\cdot\beta\frac{d}{d\beta}T_\nu(\lambda_n a, \lambda_n\beta)\cdot\frac{T_\nu(at, \beta t)}{(t^2-\lambda_\nu^2)\,G_\nu(at)}$$

$$=-\pi\sum_{n=1}^{+\infty} f(\lambda_n)\cdot\frac{\lambda_n^2\ G_\nu(\lambda_n a)\ J_\nu(\lambda_n a)\ J_\nu(\lambda_n\beta)}{J_\nu^2(\lambda_n a)-J_\nu^2(\lambda_n\beta)}\cdot\frac{T_\nu(at, \beta t)}{(t^2-\lambda_n^2)\,G_\nu(at)}, \tag{105}$$

with sampling points $t=\lambda_n(n=1, 2\ 3, \dots)$, and the sampling function:

$$g_n(t)=\frac{T_\nu(at, \beta t)}{t^2-\lambda_n^2}. \tag{106}$$

In the expression (105), if we replace $f(t)$ and $f(\lambda_n)$ by $f(t)/G_\nu(at)$ and $f(\lambda_n)/G_\nu(a\lambda_n)$ respectively, then we obtain

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(9) I. N. Sneddon: *Fourier Transforms* (1951), McGraw-Hill. p. 56.
G. N. Watson: *Theory of Bessel Functions* (1944), Cambridge Univ. Press. p. 470.
C. J. Tranter: *Integral Transforms in Mathematical Physics* (1956), Methuen, London, p. 89.

$$f(t) = -\pi \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2 J_\nu(\lambda_n \alpha) \; J_\nu(\lambda_n \beta)}{J_\nu^2(\lambda_n \alpha) - J_\nu^2(\lambda_n \beta)} \cdot \frac{T_\nu(\alpha t, \; \beta t)}{t^2 - \lambda_n^2} \cdot \tag{107}$$

with sampling points $t = \lambda_n$ and the sampling function:

$$g_n(t) = \frac{T_\nu(\alpha t, \; \beta t)}{t^2 - \lambda_n^2}. \tag{108}$$

The expressions (63), (65), (68), (71), (74), (78), (SO), (87), (88), (90), (97), (105) and (107), are examples of the sampling formulae derived from our generalized sampling theorem (12).