行政院國家科學委員會補助專題研究計畫成果報告

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高速鐵路相關振動問題研究(3/3)

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計畫類別：☑個別型計畫　□整合型計畫
計畫編號：NSC 89－2211－E－002－113
執行期間：89 年 8 月 1 日至 90 年 7 月 31 日

計畫主持人：楊永斌　教授
共同主持人：

本成果報告包括以下應繳交之附件：
- □赴國外出差或研習心得報告一份
- □赴大陸地區出差或研習心得報告一份
- ☑出席國際學術會議心得報告及發表之論文各一份
- □國際合作研究計畫國外研究報告書一份

執行單位：國立台灣大學土木工程學系

中　華　民　國　90年　12月　1　日

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一、中文摘要

本文旨在敘述高速列車所引起之振動問題三年期研究計畫案中，第三年之研究成果。內容主要包含二個部分，其一為利用先前所發展的有限/無限元素，配合 2.5 維之分析模式，模擬列車通過所造成的土壤振動歷時反應，並利用分析結果評估三種不同振動阻隔方式之優缺點。其二是利用動態濃縮的方式，建立三維車-軌-橋互制(VRBI)元素及對應之分析流程，探討各種系統參數下之列車、軌道及橋梁的動態反應，另外，利用所建模型，考慮地表運動效應，配合相關準則，評估不同地震作用下列車行駛於橋梁上之穩定性。

關鍵詞：高速列車、橋梁、軌道、土壤振動、有限/無限元素、振動阻隔、車-軌-橋互制元素、穩定性

Abstract

In this report, the results obtained during the last year of the three-year project on vibration problems related to high speed railroads will be presented. There are two major components contained in this project. One is the application of the finite/infinite element developed previously, along with the 2.5D modeling method, to simulating the time-history response of the vibration of soils induced by moving trains. Using the results obtained, the three different vibration isolation countermeasures in reducing the train-induced soil vibrations were assessed. The other is the development of a 3D vehicle-rail-bridge interaction (VRBI) element and associated procedure by using the dynamic condensation technique. This procedure is then employed to investigate the dynamic responses of the train, track and bridge to different system parameters. Moreover, the procedure was augmented to include the effect of ground motions to evaluate the stability of trains moving over bridges under earthquakes.

Keywords: high speed train, bridge, track, soil vibration, finite/infinite element, vibration isolation countermeasure, vehicle-rail-bridge interaction element, stability

二、緣由與目的

我國目前正在積極興建的高速鐵路，由於土地取得不易，因此全線超過百分之七十的土建結構都採用高架橋的型式，以節省成本，在這種情形之下，列車高速行經橋梁所產生的振動效應便成為影響高鐵營運良窳一個重要的因素。以往相關的研究大都只侷限於二維的分析，無法有效反映出列車與橋梁實際的動態反應，對於高速鐵路「安全、快速與舒適」的營運要求並無法進行有效的探討與評估。另一方面，由於台灣位處地震帶，當地震發生時，軌道橋梁可能安全無虞，但行駛於其上之列車卻很可能因橋梁

振動過大而發生不穩定或脫軌、翻車的現象，嚴重影響列車乘客的安全，值得加以重視。有鑑於此，本研究採用三維的列車車廂和高鐵制式橋梁，考慮軌道效應與軌道不平整的影響，探討高速列車與軌道橋梁在常時與地震作用下之動力行為。

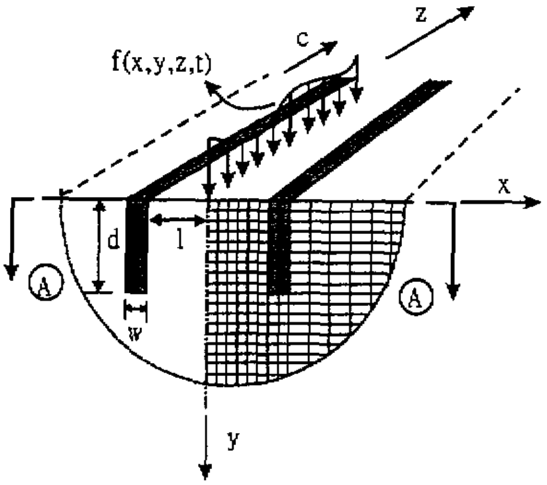
除橋梁的振動之外，高速列車行進時所產生的振動會經由橋柱和基礎或路堤經土壤傳遞至周圍地區，造成沿線居民不適或儀器設備過大的振動等環境衝擊，容易引起反彈與抗爭，這個問題隨著高鐵行經台南科學園區造成不利影響後而日漸浮上抬面，引起各界的高度關切；如何建立一套正確的分析模式，深入瞭解這種列車高速通過所產生的土壤振動，並且提出有效的阻隔對策，實為興建高速鐵路一個相當重要的課題。

三、研究內容

本研究計畫延續前二年的作法，分為土壤振動與橋梁振動兩大部分一前者包括土壤振動傳遞進一步的探討，以及評估不同的振動阻隔對策；後者則包括列車與軌道橋梁之三維互制分析，以及探討地震力作用下列車之行駛穩定性。 茲分述如下：

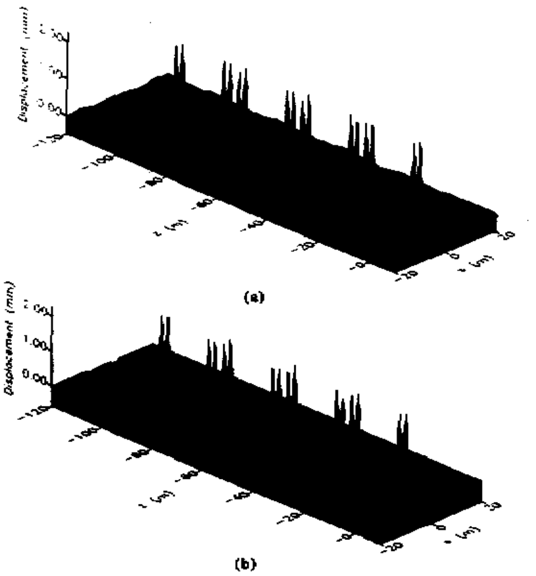
1.土壤振動方面

延續先前所建立的混合分析模式[1]，將近域以 Q8 有限元素來模擬，而遠域部分則利用自行發展以 Q8 元素為基礎之無限元素(infinite element)來模擬，如圖一所示，配合 2.5 維元素及富利葉轉換之技巧，來探討土壤的振動特性，以及三種不同振動阻絕設施(包括開放式溝渠(open trench)、充填式溝渠(in-filled trench)和振波阻隔塊(wave impeding block))之隔絕效果。這種分析模式的最大優點在於：(1)可以考慮近域土層或結構之不規則性，且可輕易分析溝渠在任何車速下(包括車速大於土壤雷利(Rayleigh)波速之情形)之隔振效果，為文獻上所僅見者；(2)可以只用平面元素而獲得三維之振動反應，大大降低分析模型所需之自由度個數，提昇電腦程式執行的效率。



圖一 列車引致土壤振動問題之分析模型

為簡化分析過程以及便於探討主要的影響因素，將列車簡化成序列移動荷重，考慮輪軸規則不均勻分佈和車廂振動的效應，得出一個可反映出列車行駛特性之動態作用力函數，利用此函數進行土壤波傳時間域分析，經比較驗證，上述分析方式及程序可有效模擬列車行駛所引致的土壤振動，詳細內容請參見文獻[2]。部分研究結果如圖二所示，為裝設開放式溝渠前後之土壤振動反應(中間明顯之波峰為輪軌接觸點下方之土壤反應)。



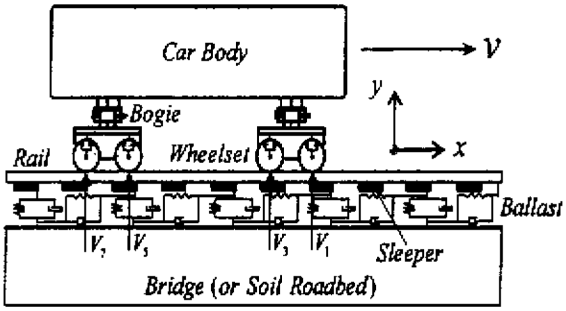
(a)

(b)

圖二 列車引致之周邊土壤振動：(a)無阻隔情形；(b)利用開放式溝渠阻隔情形

2.橋梁振動方面

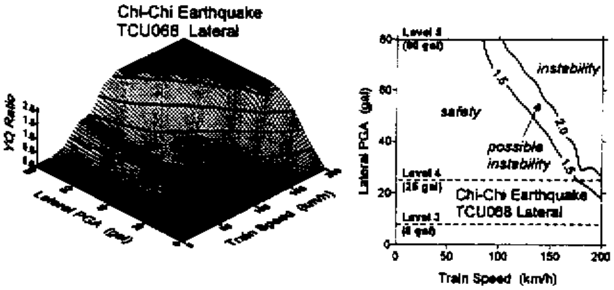
採用具有一個車體、二個台車及四個輪軸組的三維車輛模型來模擬列車車廂(如圖三所示)，可考慮俯仰(pitching)，翻滾(rolling)及搖擺(yawing)效應，橋梁則考慮鋪設雙軌的箱型簡支橋梁，以三維直梁元素加以模擬，其上之軌道則簡化成雙連續梁與均佈黏彈層之系統，同樣以三維直梁元素來模擬，如此可有效探討影響高速列車與橋梁動力行為之各種因素。



圖三 列車車廂與軌道橋梁分析模型

接著利用先前建立多元車-橋互制(VBI)元素[3]的方式，考量三維模型的特性及接觸點束制關係，建立車-軌-橋互制(vehicle-rail- bridge interaction，VRBI)元素，並提出一套動力歷時分析的程序，具有以下幾個優點：(1)理論構架之建立係採取廣義的方式，不限定車輛模型之自由度，因此可輕易分析各種車輛模型和支撐結構物間之互制問題；(2)推導過程清楚明確，便於了解及擴增修改；(3)程式之撰寫與偵錯較為容易。利用這套程序，探討鐵路橋梁在列車速度、煞車及加速、道渣勁度、軌道不平整度及雙向列車錯車等因素影響下的衝擊效應與列車反應，其中列車考慮新幹線(Shinkansen，SKS)系列 300 的型式[4]，同時列車、軌道及橋梁之性質均採用實測值，以有效模擬實際情形下之反應。

最後，根據所發展出的車-軌-橋模型，考慮地表運動的效應，建立可分析地震作用下之車-軌-橋分析模型與程序；不同於一般單純之橋梁地震反應，列車與軌道橋梁在地震下之互制行為，除地表加速度外，還與地表速度及地表位移有關。本研究利用求得之垂直(V)與水平(H)輪軌接觸力計算輪軸之脫軌係數(derailment ratio，YQ)，並利用相關準則[5]來評估列車在 El Centro、Northridge 及 921 地震-TAP003 及 TCU068 等地表運動下，靜止及行駛於軌道橋梁上之穩定性，其典型結果如圖四所示(TCU068)，其中(a)為列車速度(v)－脫軌係數(YQ)－地表水平尖峰加速度 (PGA)三相圖，(b)為列車行駛穩定性界限圖(v－PGA)。



(a) (b)

圖四 (a) 列車速度－脫軌係數－側向最大地表加速度三相圖, (b) 列車行駛穩定性界限圖

三、結論

根據研究的結果，可歸納得到一些結論，分土壤振動及橋梁振動兩方面敘述如下：

1. 土壤振動方面

(1)列車行駛所造成的土壤振動位移主要由列車靜重所導致，而速度及加速度反應則主要由列車動態效應所造成；(2)充填式溝渠阻隔列車靜重所造成的振動效果最佳；(3)開放式溝渠阻隔列車動態諧和荷重所造成的振動效果最佳；(4)對於阻隔列車靜重與動態荷重同時作用(一般情形)產生之振動，以開放式溝渠效果最好。(5)橋梁基樁深入岩盤雖可降低高架橋之振動，但卻會使擾動傳得更遠，同時也會降低開放式溝渠阻隔振動的效果，因此在設計時須審慎考慮是否將基樁埋至岩盤。

2. 橋梁振動方面

(1)列車與橋梁會發生垂直與側向共振行為，其中垂直主要共振速度(約 350km/h)與理論預測者[6]相吻合，而且其值落於高速鐵路實際營運範圍 (200~400 km/h)之內；(2)在具週期性之軌道不平整度的情況下，列車車廂會出現自身共振、台車同步或反同步運動等現象，而這些現象有可能與(1)所述之車一橋共振行為同時發生；(3)作用於軌道之煞車或加速力隨列車車廂數之增加而增加，但其作用範圍與大小有一上限值；(4)道渣勁度愈小(即愈鬆軟)，列車反應會愈大，傳遞至橋梁的水平力也愈大，反之亦然；(5)錯車時橋梁垂直反應加大，但側向及扭轉反應則可能放大或減少，視實際錯車情況而定，列車反應亦然；(6)軌道品質(即不平整度)愈差，列車反應愈大，且隨速度增加而更加明顯，但會逐漸趨於穩定值，說明列車在特定之軌道品質下將有一最大之反應。

地表運動之 PGA 愈大，避免列車失穩或脫軌(YQ 大於安全界限)之最大容許速度愈小，反之，列車速度愈快，所能抵抗而不發生脫軌情形之地表運動 PGA 愈小，結果非常合理。另外，在相同 PGA 之下，不同地震所容許之最大列車速度也不同，依實際分析結果，最大容許速度依序為 El Centro，Northridge，TCU068 及 TAP003，主要取決於地表運動之特性。垂直地表運動對列車穩定性具有明顯的影響，說明當列車在斷層附近之橋梁上行駛時須特別注意。

四、研究結果自評

本計畫執行的進度與成果符合當初擬定之目標，而研究所得之結果，也使我們對高速鐵路引起的土壤及橋梁之振動特性有深入的了解，同時對這些問題也提出了有效解決的辦法，可供未來高速鐵路及捷運等系統的參考。由於獲得不少有用的結果，因此也積極於國內外之期刊進行發表，如參考文獻[2]、[3]、[7]及[8]等所示。

五、參考文獻

1. Yang, Y. B., and Hung, H. H., “A parametric study of wave barriers for reduction of train-induced vibrations, ” *Int. J. for Numer. Meth. in Engrg.*, 40, 1997, 3729-3747.
2. Yang, Y. B., and Hung, H. H., “A 2.5D finite/infinite element approach for modelling visco-elastic body subjected to moving loads, ” *Int. J. for Numer. Meth. in Engrg.*, 50, 2001, 1317-1336.
3. Yang, Y. B., and Wu, Y. S., “A versatile element for analysing vehicle-bridge interaction response, ” *Engrg. Struct.*, 23, 2001, 452-469.
4. Wakui, H., Matsumoto, N., Matsuura, A., and Tanabe, M., “Dynamic interaction analysis for railway vehicles and structures,” *J. Structural Mech. and Earthquake Engrg.*, 513, 1995, 129-138. (in Japanese)
5. Elkins, J. A., and Carter, A., “Testing and analysis techniques for safety assessment of rail vehicles: the state-of-the-art,” *Vehicle System Dynamics*, 22, 1993, 185-208.
6. Yang, Y. B., Yau, J. D., and Hsu, L. C., “Vibration of simple beams due to trains moving at high speeds,” *Engrg. Struct.*, 19, 1997, 936-944.
7. Wu, Y. S., Yang, Y. B., and Yau, J. D., “Three-dimensional analysis of train - rail - bridge interaction problems,” *Vehicle System Dynamics*, 36, 2001, 1-35.
8. Yang, Y. B., and Wu, Y. S., “Dynamic stability of moving trains over bridg4es shaken by earthquakes,” submitted to *J. Sound and Vib.*, for publication. (2001)

行政院國家科學委員會補助國內專家學者出席國際學術會議報告

90 年 12 月 11 日

附件三

報告人姓名	楊 永 斌	服務機構及職稱	國立台灣大學土木系教授 兼院長
會議 時間 地點	2001 年 11 月 20-23 日 澳大利亞 雪梨市	本會核定補助文號	NSC90-2211-E-002-057
會議名稱	（中文）第一屆亞太計算力學大會 （英文）First Asian-Pacific Congress on Computational Mechanics (APCOM'01)		
發表論文題目	（中文）曲梁理論及相關計算考量 （英文）Curved beam theories and related computational aspects		

報告內容應包括下列各項：

一、 參加會議經過

亞太計算力學會議(Asian-Pacific Conference on Computational Mechanics)事實上並不是一個新的會議，最近曾經舉辦過此一會議的國家包括韓國與新加坡，此次是首次以 Congress 的名稱出現，因此重新排序，稱為「第一屆亞太計算力學大會」(First Asian-Pacific Congress on Computational Mechanics)，簡稱為 APCOM'01，藉此形成 Asian-Pacific Association for Computational Mechanics (APACM)，其目的是要與 International Association for Computational Mechanics (IACM)接軌，IACM 是近年來由美國 Atluri 教授等所推動成立的組織，參加的人士包括來自世界各地的計算力學專家，估計應在近千人左右，未來不論是 IACM 或 APCAM 均有可能更形具體化，而成為一具規模之世界性組織。

此一會議的主辦人為澳洲新南威爾斯大學的 S. Valliappan 教授，他是土壤動力學方面的專家，在世界上頗負名氣，本人長久以來與 Valliappan 教授常在世界上不同的會議上見面，與他建立了相當深厚的友誼，此次大會中，本人獲邀為地區組織委員會(Regional Organizing Committee)委員、邀請講員(Invited Speaker)，以及一個場次的主席(Session Chairman)，本人所主講的題目是:「曲梁理論及相關計算考量」(Curved beam theories and related computational aspects)，主要是在說明以有限元素法模擬曲梁挫曲所遭遇的幾個問題，以及正確的解決辦法。

會議於二〇〇一年十一月二十至二十三日在雪梨舉行，本人因學校行政工作忙碌之故，無法全程參與，出國日期前後計三天，從二〇〇一年十一月二十至二十二日。但在基本上仍然完成了論文報告，並主持了一個場次的論文發表，更重要的是與幾位

計算力學界的重要人士會了面，在許多方面的合作都作了有效的溝通，此行所見到的比較活躍的學者包括：香港大學的 Y. K. Cheung 教授（中國大陸中科院院士、前副校長），香港城市大學的 S. Kitipornchai 教授(去年剛由澳洲 Queensland 大學被延聘至香港擔任講座教授，Eng. Struct.期刊主編)，美國康乃爾大學的 Mukherjee 教授，新加坡大學的 C. M. Wang 教授，韓國 KAIST 的 C. K. Choi 教授，京都大學的 Adachi 教授；另外在名單上看到，而因場地分隔未曾見到的重要學者包括：有限元素專家 Owen 教授 (Univ. of Wales), IJNME 期刊主編 Lewis 教授(Univ. of Wales), 土壤動力專家 Wolf 教授(Swiss Federal Inst. of Tech.), 土壤動力期刊主編 Wriggers 教授 (Univ. of Hannover)，以及 Kanok-Nukulchai (AIT), Cescotto (Univ. Liege), Wunderlich (Tech. Univ. of Munchen), Waszczyszyn (Cracow Univ. of Tech.), Middleton (Univ. of Wales), Attard (Univ. of New South Wales)等教授，從上述參加的人員可以看出，本次會議是一次相當高水準的會議。

二、 與會心得

這是一個品質很高的國際會議，原來投稿的摘要有 400 篇，最後被接受的只有 270 篇，很可惜幾乎看不到其他從國內前來參加的同道。長久以來，在 APCOM 系列會議中，本人經常是唯一或少數幾個國內參與者之一，外國的朋友也經常在問：台灣做計算力學的的教授到那裡去了？我個人也覺尷尬萬分。

在開此一會議之前，會議主辦人 Valliappan 教授也曾來信說服本人爭取下一次大會 2004 年的主辦權，本人因考慮到已經爭取到 2003 年 IASS-APCS Symposium (IASS 年會)的主辦權，如果再爭取此一會議，則連續兩年每年舉辦一個國際會議，財務負擔實在太重；另方面，國內多年來參與此一會議的教授人數並不多，缺乏人力奧援，遂因此作罷。

後來極力爭取主辦權的，計有中國大陸大連理工大學和韓國 KAIST 兩個單位，開會討論結果由中國大陸獲得 2004 年的主辦權，大陸方面將此一會議視為計算力學界的奧運會，與 2008 年的奧運會相提並論。

三、 考察參觀活動(無是項活動者省略)

無。

四、 建議

建議國人對於已具組織雛形，而將成為正式世界性組織的會議背後主辦單位，應該積極投入參與，佔領一適當的位子，以備將來該單位正式成為世界性組織時，能夠順理成章，成為其原始會員。否則待中共先行進入，站穩腳步之後，則我國要申請入會，必然又會產生國籍的困擾，其程序一定困難重重。

五、　攜回資料名稱及內容

攜回會議論文集上下兩冊，名為 Computational Mechanics: New Frontiers in Computational Mechanics，編輯者為 S. Valliappan 和 N. Kahlili，總共收集了 270 篇論文，出版社為 Elsevier.

六、　其他

無。

CURVED BEAM THEORIES AND RELATED COMPUTATONAL ASPECTS

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ABSTRACT

The theories of buckling for horizontal curved beams presented by Timoshenko, Vlasov, Yoo, and Yang and Kuo are first reviewed, with their key features identified. The previous argument concerning the incapability of straight beam elements to predict the buckling loads of curved beams in incorrect, due to overlook of the conditions of equilibrium for structural joints connecting non-aligned members in the deformed position, as implied by conventional finite element approaches. If such conditions are duly taken into account, then the straight beam elements derived, which are referred to as the semitangential elements, can be used as a reliable tool for predicting the buckling loads of curved beams. Moreover, by simulating a curved beam in the limit as an infinite number of infinitesimal elements, the theory for straight beams can be manipulated through use of the concept of transfer matrix to yield the ones for curved beams for the cases of uniform bending and uniform compression. It is in this sense that the theories of straight beams and curved beams are unified.

KEYWORDS

Buckling, curved beam, curved beam element, joint equilibrium, stability, straight beam element

INTRODUCTION

The out-of-plane buckling of horizontally curved beams under in-plane loads has been a subject of intensive research. In this paper, some recent theories proposed by researchers (Yoo 1982; Yang and Kuo 1986. 1987) are reviewed, along with the classical theories of Timoshenko and Gere (1961) and Vlasov (1961). For a curved beam under the two special cases of uniform bending and uniform compression, most of the existing theories can be used to obtain closed-form solutions for the buckling loads, based on which comparison can be made among the various theories.

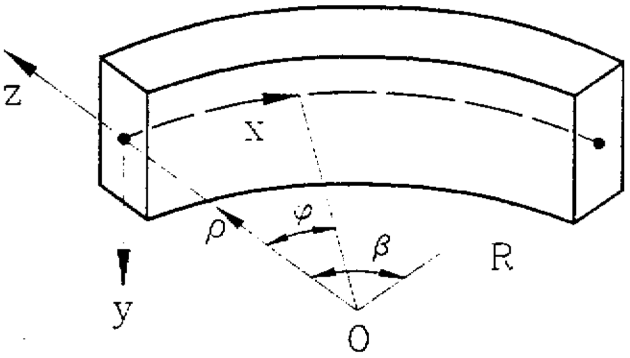

Figure 1 Curved Beam.

An old argument in the literature has been that straight-beam elements cannot be relied upon to predict the buckling behavior of curved beams (Bazant and El Nimeiri 1973; Rajasekaran and Ramm 1984; Yoo 1984). Historically, such an argument was made at the time when the lack of equilibrium of "angled joints" of space frames in the deformed position pointed out by Argyris et al. (1979) in his series of works was not made fully known to researchers. Moreover, very few researchers have realized that *simulation of curved beams by straight beam elements creates a number of "angled joints" that share the same nature as that of the problems encountered in the buckling analysis of space frames*. This implies that the approaches used to remedy the problem of angled joints in space frames can be adopted as well in the simulation of curved beams by straight-beam elements. Based on such an idea, a series of analytical and numerical studies were carried out by the author and co-workers (Yang and Kuo 1991a; Yang et al. 1991; Kuo and Yang 1991; Yang 1993). It has been demonstrated that if due consideration is taken of the equilibrium conditions of "angled joints" in the deformed position, which are created by representation of the curved beam by a number of piecewise straight elements, then the straight-beam element approach can be reliably used to produce accurate buckling solutions for curved beams.

Furthermore, it can be demonstrated that the buckling equations for curved beams under uniform bending or uniform compression can be derived from those for straight beams using the concept of transfer matrix, if a curved beam is approximated in the limit as an infinite number of infinitesimal chordwise straight elements. Also, by letting the radius of curvature of a curved beam approach infinity, the curved beam equations reduce to those for straight beams. In this sense, it is seen that the buckling equations for both the straight and curved beams can be unified.

REVIEW OF EXISTING BUCKLING THEORIES

The horizontally curved beam discussed in this paper is shown in Figure 1, for which R denotes the radius of curvature and L is the length of the curved beam. In order to concentrate on the fundamental aspects of instability for the curved beam, only ***solid, rectangular cross sections*** are considered. The effect of cross-sectional warping is omitted so as to simplify the mathematical complexities involved.

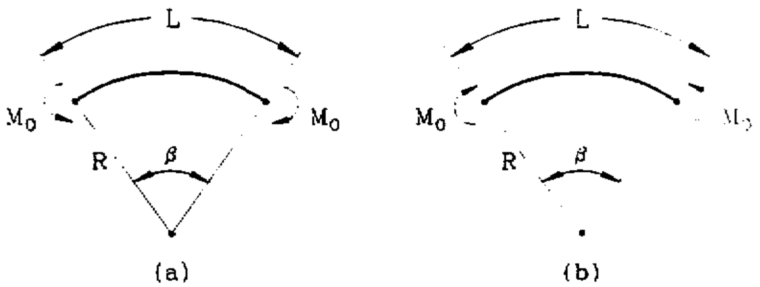

Figure 2 Uniform Bending.

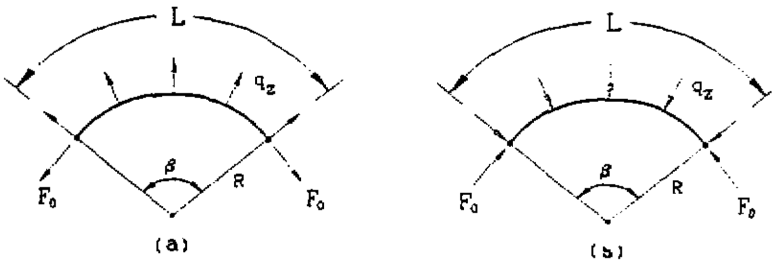

Figure 3 Uniform Compression.

Throughout the present study, two special loading cases, i.e., uniform bending and uniform compression, as shown in Figures 2 and 3, respectively, will be adopted as the test cases, since for these two cases, the member forces appear to be *constant* along the beam axis x, which allow the problems to be analytically solved. In particular, Figures 2(a) and (b) are referred to as positive and negative bending cases, respectively, and Figures 3(a) and (b) as uniform tension and compression cases. The uniform tension case in Figure 3(a) will just be omitted from here on, since no critical load exists for this case.

The following is a summary of the relevant existing theories on the buckling of curved beams:

Timoshenko and Gere's (1961) Theory

Based on an equilibrium approach, Timoshenko and Gere derived the differential equations for a solid curved beam. This theory is not general, since it works only for solid beams and for two types of loadings, i.e., uniform bending and uniform compression. However, for the special cases considered, the results have been quite good, compared with the numerical results.

Vlasov's (1961) Theory

Assuming the existence of an analogy between the generalized strains for straight and curved beams, Vlasov derived the differential equations for curved beams from the straight-beam equations by replacing the generalized strains for straight beams with those for curved beams. One question with this approach is that the existence of analogy is essentially questionable.

Yoo's (1982) Theory

Yoo adopted basically the same analogy approach, but worked on the potential energy instead. By replacing the generalized strains in the potential energy for straight beams with those for curved beams, he obtained the potential energy for curved beams. The differential equations for curved beams are then obtained as the Euler-Lagrange equations of the functional. The fact that the differential equations derived by Yoo are generally different from those of Vlasov's is indicative of the inherent lack of consistence with the analogy approach. In particular, Yoo's theory yields *two zero eigenvalues* for a hinged semi-circular arch under uniform bending, which is not physically justified since only a rigid body mode is allowed for this special case (Yang and Kuo 1986).

Yang and Kuo's (1987) Theory

By the principle of virtual work given below, the differential equations were derived for the curved beam:

$$\int_V S_{ij}\delta\varepsilon_{ij}dV = E.V.W. \tag{1}$$

where S_{ij} denotes the stresses, $\delta\varepsilon_{ij}$ the variation of strains, V the volume of the element, and $E.V.W.$ is the external virtual work. In the study by Yang and Kuo (1986), three components of stresses, i.e., the axial stress S_x and shear stresses S_{xy}, S_{xz}, are considered. From this study, the following conclusions can be drawn: (1) There is no one-to-one correspondence between the generalized strains for curved beams and those for straight beams. (2) Due to the effect of coupling between in-plane and out-of-plane deformations, the axial stress formula that is valid for the straight beam cannot be adopted for the curved beam. (3) For the hinged semicircular arch under uniform bending, only a *single zero eigenvalue* was obtained, which is physically sound. However, the critical loads predicted by this theory shows large deviations from those of Timoshenko and Vlasov for the case of uniform bending.

In the 1987 paper by Yang and Kuo, the fourth stress component, i.e., the radial stress S_z, which is of comparable order of magnitude as that of the axial stress S_x for curved beams, was included in formulating the potential energy. The critical loads obtained from this new set of differential equations for curved beams under uniform bending appear to be very close to those of Timoshenko and Vlasov.

STRAIGHT-BEAM ELEMENT APPROACHES

As for the straight-beam element, the virtual work equation is identical in form to the one given in Eqn. (1). Conventionally, only the axial stress S_x and two shear stresses S_{xy}, S_{xz} are considered. By the updated Lagrangian procedure, the virtual work equation can be recast in an incremental form. From the strain energy part, the elastic stiffness matrix $[k_e]$ can be derived, and from the potential energy part, the geometric stiffness matrix $[k_g]$ that accounts for the instability effects of member actions can be derived. Typically, the following is the equation of equilibrium for the element in incremental form:

$$\left(\left[k_e\right]+\left[k_g\right]\right)\{u\}=\left\{{}^2f\right\}-\left\{{}^1f\right\} \tag{2}$$

where $\{{}^2f\}$ and $\{{}^1f\}$ denote the external loads acting on the element nodes at the initial (C_1) configuration and the deformed or buckling (C_2) configuration, respectively, and $\{u\}$ represents the incremental or buckling displacements generated during the motion from C_1 to C_2. Here, it should be stressed that by Eqn. (2), the equilibrium is established only for *discrete elements*, but not for the joints.

The idea herein is to simulate a curved beam as a number of chordwise elements as shown in Figure 4. Undoubtedly, this will introduce a number of "angled joints", *A*, *B*, *C*, ..., as shown in Figure 4. A model like this is the result of natural thinking. However, there has been argument that the model shown in Figure 4(b) cannot be used to simulate the buckling of the curved beam shown in Figure 4(a). In the following, we'll explain what is wrong with this argument using the "conventional" finite elements.

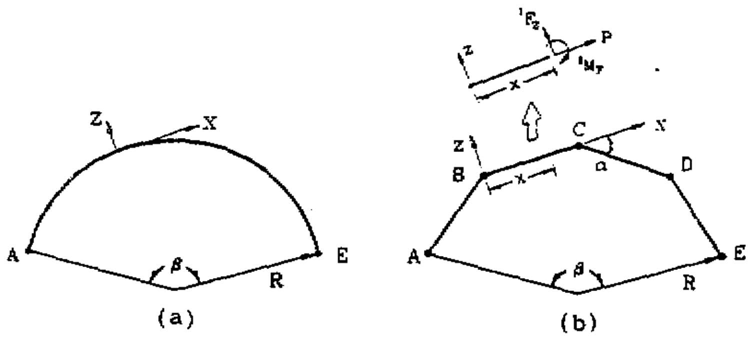

Figure 4 Straight Beam Model.

Conventional Buckling Approaches

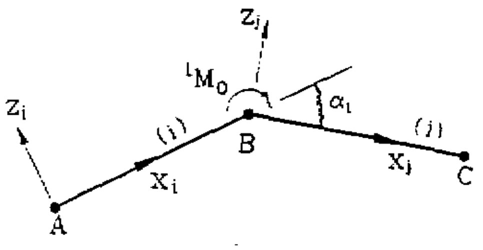

Figure 5 Angled Joint.

One essential step in the finite element analysis is to assemble all the element stiffness matrices to form the structure stiffness matrix. The key point here is the *configuration of reference*. If the element matrices are assembled with reference to the *initial* configuration, then all the equilibrium and compatibility conditions are established for the same initial configuration. Such an approach is good only for linear analysis, but not for buckling analysis, since it is a violation of the principle that *all the equilibrium and compatibility conditions of the structure should be expressed with reference to the deformed configuration in a buckling analysis.* The equilibrium conditions that are really affected in this regard are those involving moments and torques undergoing three-dimensional rotations. Consequently, the conventional approach of assembling both the $[k_e]$ and $[k_g]$ matrices withe reference to the initial configuration of the structural frame is *not* regarded as a valid one. One typical example is the work by Hasegawa et al. (1985).

Consistent Buckling Approach

Consider the angled joint B of part of a plane frame shown in Figure 5, which is subjected to an initial moment M_0. Here, we are interested in the out-of-plane buckling behavior of the frame. As was stated, all the equations of equilibrium should be established for the buckling configuration. In this regard, all the forces and moments should be expressed for the same configuration. In particular, the bending moment M_z and torque M_x should be written as

$$\begin{aligned} M_z &= EIv'' + M_y\theta \\ M_x &= GJ\theta' - M_y v' \end{aligned} \tag{3}$$

where θ is the angle of twist, v the lateral deflection, GJ the torsional rigidity, EI the flexural rigidity, $M_y = M_0$ is the initial bending moment, and ( )' denotes differentiation with respect to x.

By substituting the preceding equations into the equilibrium conditions for the angled joint B in the buckling position, a joint matrix $[k_j]$ can be derived (Yang and Kuo 1991a; Yang et al. 1991). If the joint matrices $[k_j]$ looped over all nodal points are combined with the $[k_e]$ and $[k_g]$ matrices looped over all the beam elements, then a structural stiffness matrix $[K]$ with full account of structural compatibility is obtained. By so doing, the inability of the conventional approach to cope with the deformed configuration is remedied. Such an approach is qualified for the buckling analysis of space frames, which in some cases has been referred to as the *semitangential (ST) approach.*

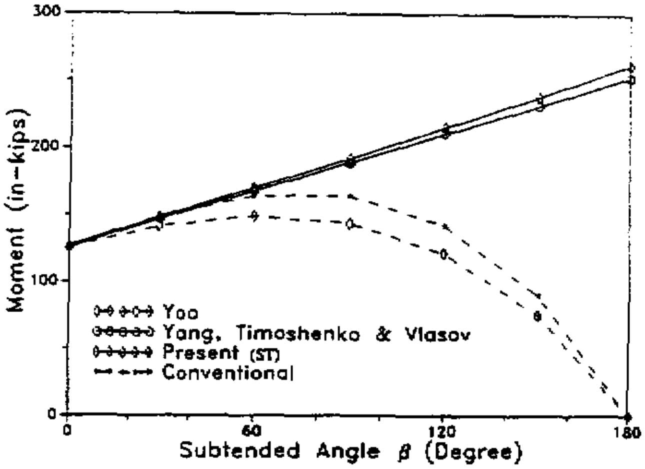

Figure 6 Positive Bending Case.

In fact, the conventional approach of assembling the $[k_e]$ and $[k_g]$ matrices with respect to the initial shape of the structure is equivalent to using the following relations:

$$M_z = EIv'' \qquad M_x = GJ\theta' \tag{4}$$

which are obviously not qualified for buckling analysis (Yang and Kuo 1991b).

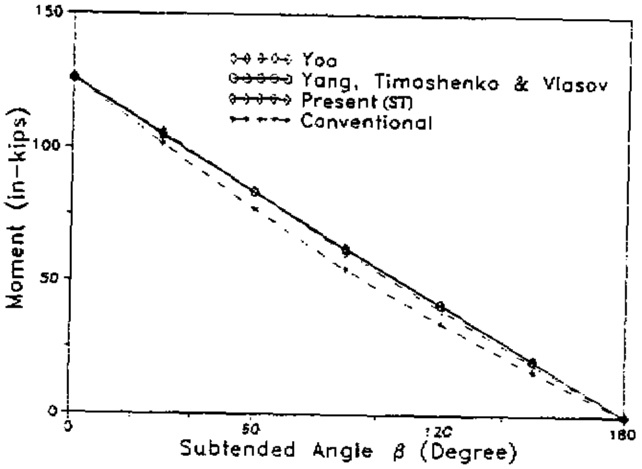

Figure 7 Negative Bending Case.

COMPARISON OF RESULTS

The results obtained by the theories of Timoshenko and Gere (1961), Vlasov (1961), Yoo (1982) and Yang and Kuo (1987) have been compared with the finite element solutions in Figures 6-8 for the three cases of positive bending, negative bending, and uniform compression. All the finite element solutions were obtained by approximating the curved beam by eight chordwise straight beam elements (Yang et al. 1991). In these figures, the solutions by Yang and Kuo (1987) have been marked as Yang.

Several conclusions can be drawn from the results plotted in Figures 6-8:

(1) For all the three cases considered, the solutions given by Yang and Kuo coincide with those by Timoshenko and Gere (1961).

(2) The "semitangential" (ST) straight beam element solutions are generally close to the analytical solutions of Yang and Timoshenko.

(3) The "conventional" straight beam solution appears to be close to Yoo's (1982), which yields two zero eigenvalues for the semicircular (180°) arch under uniform bending, which is unjustified.

(4) For the uniform compression case in Figure 8, no difference can be made between the "semitangential" (ST) and "conventional" finite element approach, due to the fact that *no* bending moments exist at the angled joints, which implies that no difference can be made between the joint equilibrium conditions at the deformed and initial configurations.

(5) For the uniform compression case, both the solutions given by Vlasov (1961) and Yoo deviate significantly from those of Yang, Timoshenko and the finite element solutions.

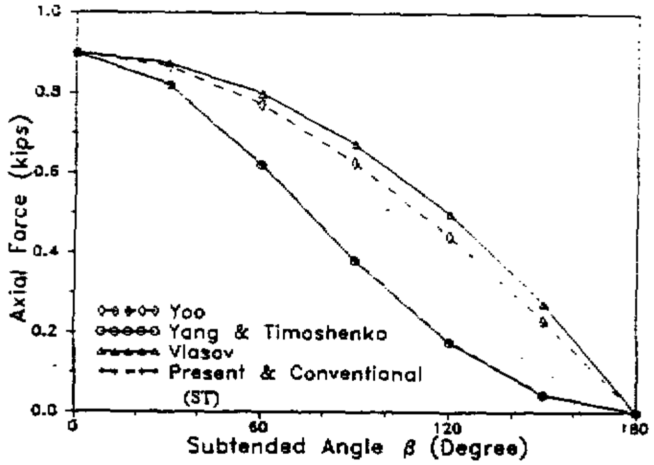

Figure 8 Uniform Compression Case.

CURVED BEAM EQUATIONS DERIVED FROM STRAIGHT BEAM EQUATIONS

A conceptual extension of the finite element approach is to treat a curved beam in the limit as the composition of an infinite number of infinitesimal elements, as shown in Figure 9, *the only kinematic* *assumption* made here, and to derive the curved beam equations directly from the classic straight beam equations. This has been the attempt made by Kuo and Yang (1991).

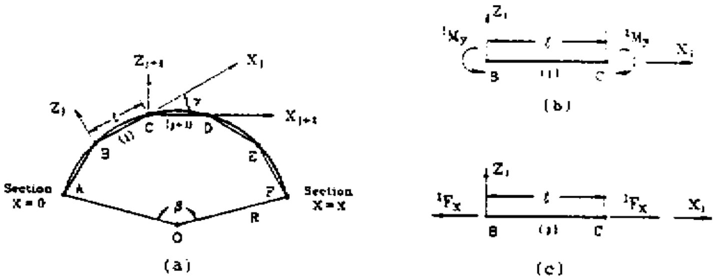

Figure 9 Curved Beam Modeling.

As shown in Figure 9, a curved beam is approximated by *n* straight elements. Based on the classic theory for straight beams (Vlasov 1961), the differential equations for the *j*th straight element under uniform bending are

$$\begin{aligned} EIv_j'''' + M\theta_j'' &= 0 \\ GJ\theta_j'' - Mv_j'' &= 0 \end{aligned} \qquad (5)$$

For this case, the member force expressions are exactly those given in Eqn. (3) considering the buckling displacements. By taking into account the conditions of compatibility and equilibrium at structural joints, letting $n \to \infty$, and by the concept of transfer matrix, the differential equations for straight beams can be manipulated to yield the differential equations for curved beams under uniform bending. The same procedure can be adopted to derive the differential equations for curved beams under uniform compression. The equations as derived appear to be essentially the same as those by Yang and Kuo (1987). One merit with such an approach is that the *fewest number of assumptions* have been adopted in the derivation, compared with the conventional approaches based on the virtual work or energy principles, in which assumptions have always to be made concerning the order of magnitudes of, say, the curvature $1/R$ and derivatives of displacements and rotations.

CONCLUDING REMARKS

In this paper, some major theories on the out-of-plane buckling of curved beams subjected to in-plane loads are briefly reviewed. The focus of this paper is to explain that the previous argument concerning the inability of straight beam elements in simulating the buckling of curved beams lacks a solid foundation, mainly due to overlook of the equilibrium conditions for angled joints in the deformed configuration, as implied by "conventional" finite element approached. If due account is taken of the equilibrium of angled joints in the deformed configuration of a structural frame, then the straight beam elements derived, known as the semitangential elements, can be reliably used to predict the buckling loads of curved beams. For the special cases of uniform bending and uniform compression, the classic differential equations for straight beams can be manipulated to yield the differential equations for curved beams, based merely on the assumption that a curved beam can be approximated in the limit as an infinite number of infinitesimal beam elements.

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56, NSC80-0410-E002-18, NSC80-0410-E002-59, and NSC81-0410-E002-05.

REFERENCES

Argyris J. H., Hilbert O., Malejannakis G. A., and Scharpf D. W. (1979), On the geometrical stiffness of a beam in space–a consistent v.w. approach. *Computer Methods in Applied Mechanics and Engineering*, **20**, 105-131.

Bazant Z. P. and El Nimeiri M. (1973), Large-deflection spatial buckling of thin-walled beams and frames. *Journal of Engineering Mechanics Division*, ASCE **99:6**, 1259-1281.

Hasegawa A., Liyanage K., Ikeda T., and Nishino F. (1985), A concise and explicit formulation of out-of-plane instability of thin-walled members. *Proc. Japanese Society of Civil Engineers, Struct. Eng./Earthquake Eng.*, **2**:1, 57-65.

Kuo S. R. and Yang Y. B. (1991), New theory on buckling of curved beams. *Journal of Engineering Mechanics*, ASCE **117:8**, 1698-1717.

Rajasekaran S. and Ramm E. (1984), Discussion of "Flexural-torsional stability of curved beams" by C. H. Yoo. *Journal of Engineering Mechanics*, ASCE **110:1**, 144-148.

Timoshenko S.P. and Gere J.M. (1961), *Theory of Elastic Stability*, 2nd ed., McGraw-Hill, New York, N.Y.

Vlasov V.Z. (1961), *Thin-Walled Elastic Beams*, 2nd ed., Israel Program for Scientific Translation, Jerusalem, Israel.

Yang Y. B. (1993), Recent researches on buckling of framed structures and curved beams. *Journal of Constructional Steel Research* **26**, 193-210.

Yang Y. B. and Kuo S. R. (1986), Static stability for curved thin-walled beams. *Journal of Engineering Mechanics*, ASCE **112:8**, 821-841.

Yang Y. B. and Kuo S. R. (1987), Effect of curvature on stability of curved beams. *Journal of Structural Engineering*, ASCE **113:6**, 1185-1201.

Yang Y. B. and Kuo S. R. (1991a), Consistent frame buckling analysis by finite element method. *Journal of Structural Engineering*, ASCE **117:4**, 1053-1069.

Yang, Y. B. and Kuo S. R. (1991b), Out-of-plane buckling of angled frames. *International Journal of Mechanical Science*, **33**:1, 55-67.

Yang Y. B., Kuo S. R. and Yau J. D. (1991), Use of straight beam approach to study buckling of curved beams. *Journal of Structural Engineering*, ASCE **117:7**, 1963-1978.

Yoo C. H. (1982), Flexural-torsional stability of curved beams. *Journal of Engineering Mechanics Division*, ASCE **108:6**, 1351-1369.

Yoo C. H. (1984), Closure to "Flexural-torsional stability of curved beams." *Journal of Engineering Mechanics*, ASCE **110:1**, 148-149.