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Negatively Curved Sets on Surfaces of Constant Mean Curvature in $\mathbb{R}^3$ are Large

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Abstract

It is proved that the negatively curved set M_- on a nonparametric surface M of constant mean curvature in $\mathbb{R}^3$ must extend to the boundary ∂M, if M_- is nonempty. For M parametric, if M_- is compactly included in the interior of M, then M_- is at least as large as an extremal domain. The results imply certain convexity results on elliptic partial differential equations. Second-order calculus of variation is employed.

In this paper we show that on a nonparametric surface M of constant mean curvature in $\mathbb{R}^3$ the negatively curved set M_-, if non-empty, must extend to the boundary ∂M of M [Theorem 3]. By the *negatively curved set* we mean the set of all saddle points, i.e., the points of M at which the Gaussian curvature is negative. Geometric intuition suggests that if a surface of constant mean curvature has saddle points in the interior, then the set of such points cannot be confined to a small region. Theorem 3 proves this assertion in the sense that the set must extend to the boundary. More generally, if M is parametrically immersed in $\mathbb{R}^3$ and if M_- is compactly included in the interior of M, we show [Theorem 2] that it must contain a relatively large area called the *extremal domain* [Definition 2.1].

Our results relate to the convexity problem for solutions of nonlinear elliptic differential equations, such as the capillarity equation. In the case of nonzero contact angle γ on the boundary walls, FINN [F] and KOREVAAR [K] found nonconvex capillary surfaces defined on certain convex domains of $\mathbb{R}^2$, while CHEN & HUANG [C-H1] and KOREVAAR [K] showed that for $\gamma = 0$, capillary surfaces over convex domains are necessarily convex. In the examples of FINN and KOREVAAR the negatively curved sets extend to the boundary, consistent with our result. A similar behavior prevails for the convexity problem with constant Dirichlet boundary condition. That is, we may have a liquid drop resting in zero gravity on a plane domain, such that the wetted area on the plane is convex and simply covered by the drop, but the drop itself is not convex and the negatively curved set on it extends to the boundary [F2].

Although in both the Dirichlet and capillary cases, convexity of the surface cannot be expected, we may still ask about the level curves. WANG [WA] has given an example of nonconvex level curves, with the solution surface defined over a ring domain between two convex curves, while FINN [F2] constructed a nonconvex solution surface defined over a simply connected domain with the level curves all convex.

The main results of this paper may provide a clue toward re-examination of our intuition on convexity. For a nonparametric solution surface $u = u(x)$ over a domain $\Omega \in \mathbb{R}^2$, Theorem 3 is equivalent to

$$D^2 u \geqq 0 \ \text{ on } \ \partial\Omega \quad \Rightarrow \quad D^2 u \geqq 0 \ \text{ in } \ \Omega, \tag{$*$}$$

with $D^2 u \geqq 0$ denoting the nonnegative or semidefinite Hessian. In geometrical terms,

$$K(p) \geqq 0 \quad \forall\, p \in \partial\Omega \quad \Rightarrow \quad K(p) \geqq 0 \quad \forall\, p \in \Omega,$$

where $K(p)$ denotes the Gaussian curvature of the solution surface at p. In [H3] HUANG expressed the viewpoint that it is not the zeroth or first boundary condition, such as constant Dirichlet data or constant contact angle γ, which in conjunction with convexity of Ω makes the surface convex; rather, some properties of the second derivatives on the boundary play the crucial role. He examined the known convexity theorems of CHEN & HUANG, FINN and KOREVAAR, as well as those of BRASCAMP & LIEB [B-L] and CAFFARELLI & FRIEDMAN [C-F]. It is remarked in [H3] that the possibility of concluding convexity depends on whether or not the given zeroth or first boundary condition has as a consequence that

$$D^2 u \geqq 0 \ \text{ on } \ \partial\Omega.$$

With regard to the level curves, the second-order boundary condition must be replaced by $D^2 g(u) \geqq 0$ on $\partial\Omega$ for a suitable function $g(u)$, with the property that the convexity of $g(u)$ implies that of the level curves of u.

Our Theorem 3 supports this viewpoint. It asserts that a solution surface $w = w(x)$ is convex over Ω if and only if $D^2 w \geqq 0$ on $\partial\Omega$, where $w = u$ or $g(u)$. Hence the convertibility of the given boundary conditions into $D^2 w \geqq 0$ on $\partial\Omega$ is *decisive* for w to be convex. This seems to express the intrinsic nature of the convexity problem. As an example, the conditions of convex Ω and $\gamma = 0$ imply by a simple geometric reasoning that $K(p) \geqq 0$ for $p \in \partial\Omega$. Therefore $D^2 u \geqq 0$ on $\partial\Omega$ and by ($*$), u is convex. So Theorem 3 yields a simple intrinsic proof of the theorem of CHEN & HUANG. On the other hand, when $\gamma \not\equiv 0$ convexity of Ω does not yield control over $D^2 u$ on $\partial\Omega$. That is why nonconvex solution surfaces exist as shown by FINN.

Also in the case of parametric surfaces, the original expectations on convexity conditions need to be adjusted. The many initial attempts to prove HOPF's conjecture were based on a feeling that closed immersed surfaces in $\mathbb{R}^3$ of constant mean curvature were necessarily convex. The conjecture would in fact be a consequence of such a result. WENTE's counterexample [W] opened up new fields of investigation, among them that of classifying the abundant examples of such immersions. Thus the general intuition turned out to be unfounded. But it seems to us that a related geometrical insight still makes sense. For example, it is correct to us that the negatively curved set must be relatively large, as is shown by the main result of this paper. For the nonconvex closed immersed surfaces of constant mean curvature, we show the geometrical property that the negatively curved set *on one side* of each such surface must extend continuously to its *other side*. More generally, the negatively curved set must be at least as large as an extremal domain on the surface.

The concept of *extremal domain* that we introduce in Section 2 admits an empirical interpretation, which has an independent interest. We blow up a spherical balloon from the inside while holding the part outside a domain M fixed. If M is contained properly in a hemisphere, then the mean curvature H (chosen positive) increases with the interior volume V of the balloon. That is, $dH/dV > 0$, or equivalently, the interior pressure increases with the interior volume. When M is a hemisphere, the mean curvature is initially stationary, i.e., $dH/dV = 0$. However, when M properly includes a hemisphere, we have $dH/dV < 0$. In Theorem 1, we establish this phenomenon for general stable surfaces of constant mean curvature in $\mathbb{R}^n$. The critical domain determined by the criterion $dH/dV = 0$ is the extremal domain, which basically generalizes the notion of the "extremal domain" defined by GIUSTI [G] for nonparametric surfaces. As a corollary of Theorems 1 and 2, we see that the negatively curved set on a stable surface of constant mean curvature has its mean curvature decreasing as it blows up. This behavior provides geometric insight as to how large a negatively curved set must be.

1. Preliminaries

Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be an immersion of an n-dimensional connected hypersurface into an Euclidean space $\mathbb{R}^{n+1}$. We consider a smooth variation $X(t) \equiv X + t\varphi N$, where N is a unit normal vector of M^n in $\mathbb{R}^{n+1}$, φ is the variation function with $\varphi|_{\partial M} = 0$ and $t \in (-\delta, \delta)$. For convenience, the notation $F(t)$ in this paper does not necessarily mean a function of a single variable t. It may also depend on $p \in M^n$.

Let $A(t) \equiv \int_M dM_t$ be the area functional and

$$V(t) \equiv \frac{1}{n+1} \int_M \langle X(t), N(t) \rangle \, dM_t$$

be the volume of the cone with base $M_t \equiv X(t)(M^n)$ and vertex $O \equiv$ the origin of $\mathbb{R}^{n+1}$. Here $V(t)$ counts the multiplicity, and $N(t)$ denotes the unit normal vector of M_t in $\mathbb{R}^{n+1}$, chosen continuously in $p \in M^n$ as well as in $t \in (-\delta, \delta)$. It is well known that

$$\frac{dA}{dt}(0) = -\int_M nH\varphi \, dM\,, \quad \frac{dV}{dt}(0) = \int_M \varphi \, dM\,,$$

where H is the mean curvature with its sign determined by

$$\vec{H} = HN,$$

with $\vec{H}$ the mean curvature vector defined by $\Delta_M X = n\,\vec{H}$. Here Δ_M denotes the Beltrami-Laplacian operator of M^n relative to the Riemannian metric induced by X, i.e., $\Delta_M X = \sum_{i=1}^n X_{ii}$, where X_{ij}'s are the second-order covariant derivatives relative to an orthonormal frame at every point M^n. We denote

$$\mathscr{F} \equiv \Big\{\varphi \in C^\infty(M)\,;\ \varphi\,|_{\partial M} = 0\Big\},$$
$$\mathscr{G} \equiv \Big\{\varphi \in C^\infty(M)\,;\ \varphi\,|_{\partial M} = 0 \ \text{ and } \ \int_M \varphi\, dM = 0\Big\} \subset \mathscr{F}.$$

For the constrained variational problem for the area functional A subject to the volume constraint that V be constant in t, we obtain that

$$0 = \delta_\varphi(A - \lambda V) = \int_M (-nH - \lambda)\varphi\, dM$$

for any $\varphi \in \mathscr{F}$, λ being the Lagrange multiplier. We have that

$$\lambda = -nH,$$

and therefore the mean curvature of M in $\mathbb{R}^{n+1}$ is constant. We hereafter assume that H = a constant H_0 and consider the functional

$$J(t) \equiv A(t) + nH_0 V(t).$$

It is evident that

$$\frac{dJ}{dt} = \int_M \big(-nH(t) + nH_0\big)\Big\langle \frac{\partial X}{\partial t}, N\Big\rangle\, dM_t,$$

where dM_t is the area element induced by $X(t)$ and $H(t)$ is the mean curvature of M_t in $\mathbb{R}^{n+1}$. We have the well-known second variation formula for J:

$$\begin{aligned} J''(0) \equiv \frac{d^2J}{d^2t}(0) &= \int_M -n\frac{\partial H}{\partial t}(0)\cdot \varphi\, dM \\ &= \int_M \big(-\varphi\Delta_M\varphi - \|B\|^2\varphi^2\big)\, dM, \end{aligned}$$

where the first equality follows by $-nH(t) + nH_0 = 0$ when $t = 0$, and the second equality is valid based on a straightforward computation (cf. [B-C]). Here Δ_M denotes the Beltrami-Laplace operator of M and $\|B\|$ is the length of the second fundamental form (h_{ij}) of M^n in $\mathbb{R}^{n+1}$, i.e.,

$$\|B\|^2 = \sum_{i,j=1}^n h_{ij}^2 = \alpha_1^2 + \cdots + \alpha_n^2,$$

where $\alpha_1, \ldots, \alpha_n$ are the principal curvatures of M^n in $\mathbb{R}^{n+1}$, and $h_{ij} = \langle X_{ij}, N\rangle$.

Definition 1.1. Let $X : M^n \hookrightarrow \mathbb{R}^{n+1}$ be such that the mean curvature is constant. We call M *stable* if

$$A''(0) \geqq 0$$

for any volume preserving variation $X(t)$ that is fixed on ∂M.

Remark 1.1. It is known, from an example from [B-C], that M is stable if and only if $J''(0) \geqq 0$ for any variation $X(t) = X + t\varphi N$ with $\varphi \in \mathscr{G}$.

2. Extremal Domains

Let M^n have constant mean curvature H_0. Let

$$L\varphi = -\Delta_M \varphi - \|B\|^2\varphi \quad \text{for} \quad \varphi \in \mathscr{F}.$$

Then we have

$$J''(0) = \int_M L\varphi \cdot \varphi \, dM.$$

Let φ_i be an i-th eigenfunction of L on $\mathscr{F}$ with i-th eigenvalue $\lambda_i = \lambda_i(M)$. It is clear that $\lambda_1(M) \geqq 0$ implies the stability of M. Yet the converse is not true in general.

Definition 2.1. M is called an *extremal domain* if $\lambda_1(M) = 0$.

The reason to name it "extremal" is motivated by nonparametric surfaces. Assume that M^n is nonparametrically defined over a bounded open set Ω in $\mathbb{R}^n$, i.e., it is defined by $u = u(x),\ x \in \Omega \subset \mathbb{R}^n$. If

$$|Du(x)| \to \infty \quad \text{as} \quad x \to \partial\Omega,$$

then M^n is extremal, i.e., $\lambda_1(M^n) = 0$. In fact, let $\zeta \equiv \langle N, A\rangle$, where N is the upward unit normal of M^n in $\mathbb{R}^{n+1}$ and $A = (0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. It is easy to see that ζ is the first eigenfunction of L on $\mathscr{F}$ with eigenvalue $\lambda_1 = 0$. To show this, we present a quick computation using the method of moving frames as follows. Let $e_i,\ i = 1, 2, \ldots, n$, be a local orthonormal frame on M^n; then

$$\begin{aligned} \Delta_M N &= N_{ii} = (-h_{ij}e_j)_i = -h_{iji}e_j - h_{ij}h_{ji}N \\ &= -h_{iij}e_j - h_{ij}^2 N = -\|B\|^2 N, \end{aligned} \tag{A}$$

where h_{ijk} is symmetric in any of its two indices in space forms, and therefore $h_{iji} = h_{iij} = H_j = 0$, since H is constant. Hence we have

$$L\zeta = 0.$$

But $\zeta|_{\partial M} = 0$ and ζ has no zero point in M^n. Therefore $\lambda_1(M^n) = 0$.

On the other hand, M^n is called a *first boundary-conjugate domain* if ∂M is the first conjugate boundary with respect to J on $\mathscr{G}$, i.e., if there exist a Jacobi field $Y = gN$ with $g \in \mathscr{G} = \mathscr{G}(M)$, but no such a field exists on any smaller domain $M' \subsetneqq M$. Let $\widehat{M}$ be a first boundary-conjugate domain. The Morse index theory shows that M is stable if $M \subset \widehat{M}$ and unstable otherwise.

Since $\mathscr{G} \subset \mathscr{F}$ but $\mathscr{G} \nsupseteq \mathscr{F}$, an extremal domain M in general is smaller than a first boundary-conjugate domain $\widehat{M}$ and therefore is stable.

Remark 2.1. We remark that $M_a \subset M_b \Longrightarrow \lambda_1(M_a) \geqq \lambda_1(M_b)$. Let a domain be enlarged gradually along a given surface of constant mean curvature, starting with a domain so small that it contains no extremal domain and is stable. As it is enlarged enough, it becomes extremal and is still stable. Further enlargement converts the domain into a first boundary-conjugate domain. After this stage, the domain becomes unstable.

Now we perturb M in a one-parameter family $\{M_t : t \in (-\varepsilon, \varepsilon)\}$, each M_t having constant mean curvature H_t with $M_0 = M$ and $\partial M_t = \partial M_0$. For example, we hold a soap bubble fixed except at a north cap, i.e., a geodesic disk centered at the north pole. Let M be the north cap. As we blow up the soap bubble, M deforms into M_t of constant mean curvature H_t. We may reparametrize the family $\{M_t\}$ by the enclosed volume V in place of t. In order to compare the general surface of constant mean curvature with the soap bubble, we have to readjust the signs of H and V as follows. The normal N is chosen opposite to $\vec{H}$, i.e., $\vec{H} = H \cdot (-N)$ with $H > 0$. Also the coordinate origin O is put so that $V = \frac{1}{n+1} \int_M \langle X, N \rangle \, dM > 0$. This makes V increase as the surface M perturbed along N.

Theorem 1. *Let M be deformed in a one-parameter family $\{M_V ; V \in (V_0 - \delta, V_0 + \delta)\}$, each M_V having constant mean curvature, i.e.,*

$$H(V) = \text{ constant } H_V .$$

If $M \equiv M_{V_0}$ is stable, then

(i) $\frac{dH}{dV} > 0$ *if and only if M is properly contained in an extremal domain,*

(ii) $\frac{dH}{dV} = 0$ *if and only if M is extremal,*

(iii) $\frac{dH}{dV} < 0$ *if and only if M properly contains an extremal domain.*

Proof. Consider the bilinear form I defined by J'' on $\mathscr{F}$, i.e.,

$$I(f, g) \equiv \int_M Lf \cdot g \, dM \quad \forall \ f, g \in \mathscr{F} .$$

Let φ = the deformation function $\frac{d}{dV}$, i.e., the position vector $X(V)$ of M_V is $X(V) = X + V\varphi N$. Then for all $g \in \mathscr{G} \subset \mathscr{F}$, we have

$$I(\varphi, g) = \int_M L\varphi \cdot g \, dM = \int_M n \frac{dH}{dV} \cdot g \, dM = n \frac{dH}{dV} \cdot \int_M g \, dM = 0$$

since dH/dV is constant on M and $g \in \mathscr{G}$. This perpendicularity of φ to $\mathscr{G}$ with respect to I is the key point of the following proof.

Lemma 1. *"M is stable and $dH/dV \geqq 0$" if and only if $\lambda_1(M) \geqq 0$.*

Proof of Lemma 1. To show the necessity, let φ_1 be a first eigenfunction of L on $\mathscr{F}$. Then $\varphi_1 = \alpha\varphi + \beta g$ for some $\alpha, \beta \in \mathbb{R}$ and some $g \in \mathscr{G}$. It is seen that

$$\begin{aligned} I(\varphi_1, \varphi_1) &= \alpha^2 I(\varphi, \varphi) + 2\alpha\beta I(\varphi, g) + \beta^2 I(g, g) \\ &\geqq n \frac{dH}{dV} \alpha^2 \int_M \varphi \qquad \text{(since } I(\varphi, g) = 0 \text{ and } I(g, g) \geqq 0\text{)} \\ &= n \frac{dH}{dV} \alpha^2 \qquad \text{(since } \textstyle\int_M \varphi = \frac{dV}{dV} = 1\text{)}. \end{aligned}$$

But

$$I(\varphi_1, \varphi_1) = \lambda_1 \int_M \varphi_1^2.$$

Hence $dH/dV \geqq 0$ implies $\lambda_1 \geqq 0$. Conversely, if $\lambda_1 \geqq 0$ then $I(f, f) \geqq 0\ \forall f \in \mathscr{F}$. Thus $I(g, g) \geqq 0$, $\forall g \in \mathscr{G}$ and M is stable. Also $I(\varphi, \varphi) \geqq 0$. But $I(\varphi, \varphi) = ndH/dV$. Hence $dH/dV \geqq 0$.

Lemma 2. *Let* $\lambda_1(M) < 0 \leqq \lambda_2(M)$, *then*

(a) $\frac{dH}{dV} < 0$ *if* M *is stable, and*

(b) $\frac{dH}{dV} > 0$ *if* M *is unstable.*

Proof of Lemma 2. The statement (a) is trivial by Lemma 1. For the statement (b), we consider in $\mathscr{F}$ the cone K defined by $\{f \in \mathscr{F} \mid I(f, f) < 0\}$. Suppose M is unstable but $\frac{dH}{dV} \leqq 0$. Let $f \in \mathscr{G}$ be such that $I(f, f) < 0$; it is evident that for all $h \in$ the linear span of φ and f, i.e., $h = \alpha\varphi + \beta f$, we have $I(h, h) < 0$. In fact,

$$I(h, h) = \alpha^2 \frac{dH}{dV} \int_M \varphi + \beta^2 I(f, f) < 0,$$

since $dH/dV \leqq 0$, $\int_M \varphi = 1$ and $I(f, f) < 0$. But $\lambda_2 \geqq 0$ and the space of the first eigenfunctions of L on $\mathscr{F}$ is only one-dimensional. This leads to a contraction.

By combining Lemma 1 and Lemma 2, we complete the proof of Theorem 1.

Remark 2.2. For $\lambda_2 < 0$, it is clear that M is unstable.

Example 2.1. Evidently, on a sphere S^n, a hemisphere M_h is an extremal domain. If we consider a domain M_a properly contained in M_h, holding $S^n - M_a$ fixed while blowing up M_a from the inside of the sphere, we would find that the mean curvature of M_a increases by Theorem 1. On the other hand, if we consider $M_b \subset S^n$ properly containing M_h, then the mean curvature of M_b decreases while it blows up.

Example 2.2. Consider a Delaunay surface obtained by rolling an ellipse. A domain M generated by the segment between a pair of adjacent maximum and minimum points of the generating curve is extremal. This is easily seen by taking $\varphi_1 = \langle N, B\rangle$ where B is a fixed vector along the axis of the Delaunay surface. The application of Theorem 1 yields a phenomenon like that of the last example.

Remark 2.3. RUCHERT proved that if

$$\frac{1}{2}\int_M \|B\|^2 dM < 2\pi,$$

then M is "stable" in the sense that he defined in [R]. Using our terminology here, this can be restated that the last inequality implies that M is still contained in an extremal domain, and therefore $dH/dV > 0$.

Remark 2.4. Theorem 1 may be explained empirically thus: Blowing up a small piece of a soap bubble or soap film from inside would increase the inner pressure. But if the piece is large enough to contain an extremal domain, then the inner pressure decreases instead.

Remark 2.5. Stability domains on Delaunay surfaces in the capillary and the Dirichlet sense had been considered in [VT, CS, HC].

3. Negatively Curved Sets

Theorem 2. *Let M be a surface of constant mean curvature immersed in $\mathbb{R}^3$, with or without boundary ∂M. Let the negatively curved set M_- be compactly included in M, i.e., the closure $cl\, M_-$ of M_- in the interior* $\operatorname{int} M$ *of M be compact. Then M_- must contain an extremal domain, i.e., $\lambda_1(M_-) \leqq 0$.*

Proof. Let $\zeta \equiv K$ be the deformation function on M_-. We have $\zeta|_{\partial M_-} = 0$. Suppose that $\lambda_1(M_-) > 0$. Then $I_{M_-}(\varphi,\varphi) \equiv \int_{M_-} L\varphi \cdot \varphi dM > 0,\ \forall\ \varphi \in \mathscr{F}$. But we shall show that

$$I_{M_-}(\zeta,\zeta) \leqq 0,$$

which would yield a contradiction. In fact, for surfaces of constant mean curvature, we have from [H2] that

$$\Delta_M K = -4(H^2 - K)K - \frac{|\nabla K|^2}{H^2 - K}, \tag{B}$$

$$\begin{aligned} I(\zeta,\zeta) &= -\int_{M_-} \left(K\Delta_M K + \|B\|^2 K^2\right) dM \\ &= -\int_{M_-} K\Big(-4(H^2-K)K - \frac{|\nabla K|^2}{H^2-K}\Big) - \int_{M_-} (4H^2K^2 - 2K^3) \\ &= -2\int_{M_-} K^3 + \int_{M_-} \frac{K|\nabla K|^2}{H^2-K} \end{aligned}$$

(when we have omitted the "dM" under the integral sign for convenience). Let the second integral in the last formula be denoted by S; then

$$S = \int_{M_-} \sum_{i=1}^{2} \frac{K}{H^2 - K} \cdot K_i^2 = \int_{M_-} \Big(\frac{K}{H^2 - K} K_i\Big) K_i$$

(by using the summation convention)

$$= \int_{M_-} \Big(\frac{K^2}{H^2 - K} K_i\Big)_i - \int_{M_-} \Big(\frac{KK_i}{H^2 - K}\Big)_i K \qquad \text{(by integration by parts)}$$

$$= \int_{\partial M_-} \frac{K^2}{H^2 - K} \cdot \frac{\partial K}{\partial n} - \int_{M_-} \Big(\frac{K^2|\nabla K|^2}{(H^2 - K)^2} + \frac{K|\nabla K|^2}{H^2 - K} + \frac{K^2}{H^2 - K} \Delta_M K\Big)$$

$$= -S + 4 \int_{M_-} K^3 \qquad \text{(by again using (B)).}$$

Hence

$$S = 2 \int_{M_-} K^3,$$

and therefore $I(\zeta, \zeta) = 0$. This completes the proof.

Remark 3.1. If we take $\zeta = K^{1+\gamma}$, $\gamma > \frac{1}{2}$, we find that

$$I(\zeta, \zeta) = \frac{4\gamma^2}{1 + 2\gamma} \int_{M_-} H^2(-K)^{2+2\gamma} \geqq 0,$$

and the equality holds only when $\gamma = 0$.

Theorem 3. *Let M be a nonparametric surface of constant mean curvature H_0 defined over an open set Ω in $\mathbb{R}^2$, i.e., M is defined by $u = u(x)$, $x = (x^1, x^2) \in \Omega$, satisfying*

$$D_i\left(\frac{D_i u}{\sqrt{1 + |Du|^2}}\right) = 2H_0 \quad \text{in} \quad \Omega\,.$$

Then each component M_-^α of the negatively curved set M_- must extend continuously to the boundary ∂M or to infinity, i.e., M_-^α is not compactly included in M.

Proof. We first need a lemma.

Lemma 3.1. *For M_* compactly included in M, $\lambda_1(M_*) > 0$.*

Proof of Lemma 3.1. Supposing the contrary, we may assume that $\lambda_1(M_*) = 0$ by shrinking M_*. Let φ_1 be the first eigenfunction of

$$L = -\Delta_M - \|B\|^2$$

on M_* with zero boundary value. Then

$$\begin{aligned} \Delta_M \varphi_1 &= -\|B\|^2 \varphi_1 \quad \text{in } M_*, \\ \varphi_1 &\geqq 0 \qquad\qquad \text{in } M_*, \\ \varphi_1 \big|_{\partial M_*} &= 0. \end{aligned}$$

Let $\zeta \equiv \langle N, A\rangle$ where N is the unit surface normal $\left(\frac{-u_x}{\sqrt{1+|Du|^2}}, \frac{-u_y}{\sqrt{1+|Du|^2}}, \frac{1}{\sqrt{1+|Du|^2}}\right)$ of M in $\mathbb{R}^3$ and let $A = (0, 0, 1)$ be perpendicular to Ω. Then

$$\begin{aligned} \Delta_M \zeta &= -\|B\|^2\zeta \quad \text{in } M_*, \\ \zeta &> 0 \qquad\quad\ \ \text{in } M_*, \\ \zeta|_{\partial M_*} &> 0. \end{aligned}$$

Consider $\zeta - \alpha\varphi_1,\ \alpha > 0$. For α small, we have $\zeta - \alpha\varphi_1$ positive. Now enlarge α until $\zeta - \alpha\varphi_1$ has a zero point P_0. It is clear that $P_0 \notin \partial M_*$. This would imply that $\zeta - \alpha\varphi_1$ has a zero interior minimum. But

$$\Delta_M(\zeta - \alpha\varphi_1) = -\|B\|^2(\zeta - \alpha\varphi_1) \leqq 0,$$

contradicting the Hopf maximum principle. Lemma 3.1 is proved.

For the proof of Theorem 3, suppose that there is a component M_-^α of M_- compactly included in M; then by Theorem 2, $\lambda_1(M_-^\alpha) \leqq 0$. Hence there exists $M_* \subset M_-^\alpha$ with $\lambda_1(M_*) = 0$, contradicting Lemma 3. This completes the proof of Theorem 3.

Corollary 3.1. *The results in Theorems 2 and 3 are still valid if the surface M has mean curvature H satisfying a weaker condition:*

$$D^2H \leqq 0,$$

i.e., the Hessian of H on M is nonpositive-definite. In fact, we have

$$\Delta_M K \leqq -4(H^2 - K)K - \frac{|\nabla K|^2}{H^2 - K},$$

and the proofs are similar to those of Theorems 2 and 3. The last inequality can be seen in [H2].

Corollary 3.2. *On a stable surface M of constant mean curvature immersed in $\mathbb{R}^3$, if a connected negatively curved set is compactly contained in M, then in the procedure of blowing up the set as defined in Section 2, we have*

$$\frac{dH}{dV} \leqq 0.$$

Remark 3.2. If the boundary segment $\Gamma \equiv \partial M_-^\alpha \cap \partial M$ is compact and M can be extended across Γ as a parametric surface of constant mean curvature, then on the interior segment $\operatorname{int}\Gamma$ of Γ the Gaussian curvature is still negative. This is easily seen from the argument in the proof of Theorem 3.

Remark 3.3. Lemma 3.1 also gives a simple proof of the known fact that a nonparametric surface of constant mean curvature is stable. We, however, do not know the existence of a proof in literature.

Finally, we raise some related open problems:

Problem 1. Are the theorems about the negatively curved sets true for general n-dimensional surfaces of constant mean curvature immersed in $\mathbb{R}^{n+1}$? Let K_ν denote the ν-th mean curvature of the surfaces in $\mathbb{R}^{n+1}$ for $2 \leqq \nu \leqq n$. In [H3], $\Delta \log K_\nu$ has been computed and estimated.

Problem 2. For M simply connected, is it possible to estimate the size of an extremal domain quantitatively? More precisely, does each extremal domain contain a geodesic ball with the radius independent of where the extremal domain is sitting? What is the largest radius? The assumption of simply connectedness is essential, since there exists a Delaunay surface with constant mean curvature 1 on which the area of extremal domain is arbitrarily small.

Problem 3. Are the theorems, especially Theorem 3, valid for some other kinds of elliptic solution surfaces, e.g., $\Delta u = f(u, \nabla u)$?

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