Statistical Analysis and Empirical Approaches for Importance Sampling and Monte Carlo Techniques in Digital Transmission System Simulation

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Abstract

Computer simulation is currently the most powerful approach in evaluating the error rate performance of a digital transmission system. Classical Monte Carlo (MC) method is the fundamental method that requires a very large sample size, while importance sampling (IS) technique is a variance-reduction method to reduce the required sample size. In the previous literature the analysis of the statistics of MC and IS estimators is not very complete. In this paper, a detailed analysis of the statistical behavior of the classical MC and several IS estimators is presented. This analysis enables us to give a fair comparison for these estimators, and find the confidence intervals of simulation results. A new empirical approach to estimate the IS estimator variance is further proposed in the paper. Numerical results are also included to support the discussions.

1 Introduction

Evaluation of the error rate performance of a digital transmission system is the most basic problem in communication system design. Simulation is the most powerful method for obtaining the accurate estimate. The fundamental method of estimating the error rate from a simulation is the classical Monte Carlo (MC) method [1], which consists of basically counting errors. The more advanced techniques for simulation include the "variance-reduction" methods [1], in which importance sampling (IS) is one of the most attractive methods to be used in simulating digital transmission systems. In the last decade, substantial progress has been achieved in the simulation of bit error rate (BER) and symbol error rate (SER) of binary as well as multilevel systems using IS techniques [2, 3, 4, 5].

In the analysis and comparison of simulation techniques in the literature [2, 3, 4, 5], it was first assumed that all the estimators are Gaussian distributed, and second that the estimator variances can be exactly obtained. The first assumption enables us to compare these techniques simply by looking at their variances, while the second assumption enables us to write down the confidence intervals of simulated results based on the first assumption. Unfortunately, the second assumption obviously does not hold in an actual simulation because the estimator outcomes are themselves random. And the first assumption requires a close inspection. In this paper, after a brief review of classical MC and IS methods in Section 2, we will first investigate the distribution of classical MC estimator and IS estimators in Section 3 and 4. The results enable us to present empirical methods for evaluating the efficiency of IS estimators and obtain confidence intervals of simulation results in Section 5. Numerical results are given in Section 6, and finally the conclusion is given in Section 7.

2 A Brief Review

A discrete-time vector channel model [6, 7] for M-ary pulse amplitude modulation (PAM) is depicted in Figure 1. Let $\{X_k\}$ be a sequence of random symbols such that the outcome $x_k \in X = \{x | x = \pm1, \pm3, \ldots, \pm(M-1)\}$, and let $g = \{g_0, g_1, \ldots, g_\nu\}$ be the normalized channel impulse response such that $\sum_{i=0}^{\nu} g_i^2 = 1$, then the channel output sequence is $\{Z_k\} = \{X_k\} * g + \{N_k\}$, where '$*$' stands for convolution and N_k is additive white Gaussian noise (AWGN) with variance σ^2. The system is intersymbol-interference (ISI) free if $\nu = 0$. The system has small to moderate ISI if $\nu \geq 1$ and g_0 dominates over all $g_i, 1 \leq i \leq \nu$, where ν is called the ISI *span*. The appropriate detector here is now the symbol by symbol dectector. Let $\mathcal{D}(x)$ be the decision region of $x \in X$, the detector decides that $\hat{X}_k = x$ if $Z_k \in \mathcal{D}(x)$. The decision regions for M = 2,4, and 8 are plotted in Figure 2.

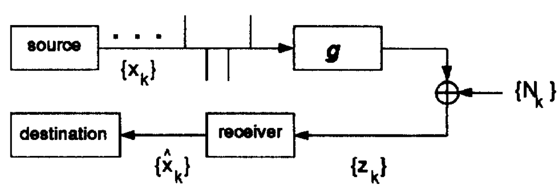

Figure 1: Discrete model for a PAM system.

The performance measures of interest are both BER and SER for all M. Because BER is obtained from SER [5], we shall review only the SER:

$$P_s = P[\hat{X}_k \neq X_k] \quad (1)$$

where $\hat{X}_k$ is the estimated symbol at the detector output.

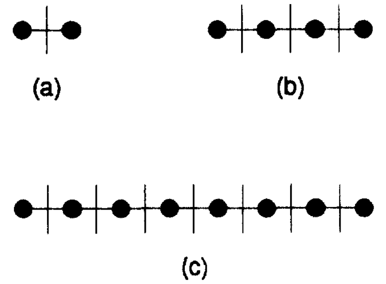

Figure 2: Decision regions for (a) 2PAM, (b) 4PAM, and (c) 8PAM.

Define the symbol error indicator function

$$I_E(x_k, \hat{x}_k) = \begin{cases} 1 & \hat{x}_k \neq x_k \\ 0 & \hat{x}_k = x_k \end{cases} \quad (2)$$

and then we have

$$P_s = E[I_E(x_k, \hat{x}_k)] \quad (3)$$

the expectation of the indicator function. The crude MC estimator of SER is

$$\hat{P}_s = \frac{1}{N} \sum_{k=0}^{N-1} I_E(x_k, \hat{x}_k) \quad (4)$$

which is simply the relative frequency of symbol errors. To apply IS technique, we generate the noise N_k by a probability density function (p.d.f.) $f^*_{N_k}(n_k)$ different from $f_{N_k}(n_k)$. Let

$$w^*_{N_k}(n_k) = f_{N_k}(n_k)/f^*_{N_k}(n_k) \quad (5)$$

be the "weight", or "likelihood ratio" of $f_{N_k}(n_k)$ with respect to $f^*_{N_k}(n_k)$. Then the IS estimator of SER is

$$\hat{P}^*_s = \frac{1}{N^*} \sum_{k=0}^{N^*-1} I_E(x_k, \hat{x}_k) w^*_{N_k}(n_k). \quad (6)$$

Two IS methods used in the literature will be used in this paper to illustrate our analysis. One is the method proposed by Shanmugan[2, 5], referred to as the conventional IS (CIS) technique here in this paper, which generates the zero mean Gaussian noise with a new variance $\sigma_*^2 > \sigma^2$:

$$f^*_{N_k}(n_k) = \frac{1}{\sqrt{2\pi}\sigma_*} \exp(-\frac{n_k^2}{2\sigma_*^2}) \quad (7)$$

where the optimal value of σ_* is

$$\sigma_{*,\mathrm{opt}} \approx \sqrt{1 + \sigma^2 + \frac{9}{16}\sigma^4}. \quad (8)$$

The other is the new technique proposed recently [5] , referred to as the new improved importance sampling (NIIS) technique here in this paper, which generate a new noise with p.d.f.:

$$f^{**}_{N_k}(n_k) = \frac{1}{2} f_{N_k}(n_k + c) + \frac{1}{2} f_{N_k}(n_k - c) \quad (9)$$

where c is a parameter to be optimized, and the optimum value of c is

$$c_{\mathrm{opt}} \approx \sqrt{1 + \sigma^2}. \quad (10)$$

3 Statistics of the Classical Monte Carlo Estimator

Here we consider only ISI free systems for simplicity, in which the error indicators are mutually independent. The results can be assumed to be also essentially applicable to systems with small to moderate ISI.

Because $I_E(x_k, \hat{x}_k)$ is simply a Bernoulli trial with parameter P_s, $N\hat{P}_s$ is a binomial distribution $\mathcal{B}(N, P_s)$. By central limit theorem [8] $\hat{P}_s$ will converge in law to Gaussian distribution $\mathcal{N}(P_s, P_s(1-P_s)/N)$ as N approaches infinity. This is the reason why most discussions [2, 3, 4, 5] use Gaussian approximation to obtain the confidence interval for classical MC simulation. However, because P_s is extremely smaller than $1 - P_s$, $I_E(x_k, \hat{x}_k)$ has very bad coefficients of *skewness* and *kurtosis* such that $\hat{P}_s$ converges too slow to justify the use of Gaussian approximation for reasonable values of N. This can be found very clearly by the *Edgeworth expansion* [9].

Let $F_N(x)$ denote the distribution of $T_N = \sqrt{N}(\hat{P}_s - P_s)/\sqrt{P_s(1-P_s)}$, which is simply the shifted and normalized classical MC estimator, and let γ_{1N} and γ_{2N} denote the coefficients of skewness and kurtosis of T_N. Then the Edgeworth expansion of $F_N(x)$ is

$$\begin{aligned} F_N(x) &= \Phi(x) - \phi(x)[\frac{1}{6}\gamma_{1N} H_2(x) + \frac{1}{24}\gamma_{2N} H_3(x) \\ &\quad + \frac{1}{72}\gamma_{1N}^2 H_5(x)] + r_N \end{aligned} \quad (11)$$

where r_N is a residue that tends to zero at a rate faster than $1/N$, and $H_2(x), H_3(x)$, and $H_5(x)$ are *Hermite polynomials*

$$H_2(x) = x^2 - 1,\ H_3(x) = x^3 - 3x,\ H_5(x) = x^5 - 10x^3 + 15x \quad (12)$$

and

$$\gamma_{1N} = \frac{\mu_3(T_N)}{[\mathrm{Var}(T_N)]^{3/2}},\ \gamma_{2N} = \frac{\mu_4(T_N)}{[\mathrm{Var}(T_N)]^2} - 3 \quad (13)$$

in which $\mathrm{Var}(\cdot), \mu_3(\cdot)$, and $\mu_4(\cdot)$ are the second, third, and fourth central moments of the enclosed variable, and $\Phi(x)$ and $\phi(x)$ are the cumultive distribution and p.d.f. of $\mathcal{N}(0,1)$, respectively. Let γ_1 and γ_2 denote the coefficients of skewness and kurtosis of $I_E(x_k, \hat{x}_k)$, then the coefficients of skewness and kurtosis of T_N and $I_E(x_k, \hat{x}_k)$ have very simple relation $\gamma_{1N} = \gamma_1/\sqrt{N}$ and $\gamma_{2N} = \gamma_2/N$, which will greatly simplify the calculation. It is then easy to show

$$\gamma_{1N} \approx \frac{1}{\sqrt{NP_s}},\ \gamma_{2N} \approx \frac{1}{NP_s} \quad (14)$$

and $F_N(x)$ can be well approximated by $\Phi(x)$ only when N is well over the number of required simulation samples, say $N > 100/P_s$. This suggests that using Gaussian approximation to obtain the confidence interval for classical MC might not be very accurate.

Instead of using Gaussian approximation, we shall use Poisson approximation for $N\hat{P}_s$, since $\mathcal{B}(N, P_s)$ is well approximated by Poisson distribution $\mathcal{P}(NP_s)$ for small P_s

and moderate NP_s [9]. Because the distribution of $N\hat{P}_s$ under Poisson distribution depends on NP_s, the method of pivot [9] is not applicable, in order to find the confidence interval we have to plot the set C such that $P[(N\hat{P}_s, NP_s) \in C] = 0.95$ [9], where $(N\hat{P}_s, NP_s)$ is a 2-tuple. In Figure 3 this set C is obtained by plotting the vertical sections C_v such that $P[N\hat{P}_s \in C_v] = 0.95$ for every value of NP_s. The 95 percent confidence interval for NP_s is then simply the horizontal section $C_h(N\hat{P}_s)$ of C. From Figure 3, it can be seen that when $N\hat{P}_s = 5$ the 95 percent confidence interval for NP_s is

$$(N\hat{P}_s - 2.7, N\hat{P}_s + 6.5) = (2.3, 11.5) \tag{15}$$

which corresponds to the 95 percent confidence interval of P_s $(\hat{P}_s - 0.54\hat{P}_s, \hat{P}_s + 1.3\hat{P}_s)$. Likewise, when $N\hat{P}_s = 10$ the 95 percent confidence interval for P_s is $(\hat{P}_s - 0.46\hat{P}_s, \hat{P}_s + 0.8\hat{P}_s)$. Comparing with the Gaussian approximation, which give a 95 percent confidence interval $(\hat{P}_s - 0.62P_s, \hat{P}_s + 0.62P_s)$ when $NP_s = 10$, the confidence interval obtained from the Poisson approximation is skewed on one side instead of being symmetric, but the width of the confidence interval is approximately the same. Hence Gaussian approximation is not too bad, yet Poisson approximation is certainly better. A different form of Gaussian approximation has also been used [3] to produce skewed confidence interval for P_s, which gave an interval $(\hat{P}_s - 0.5\hat{P}_s, \hat{P}_s + \hat{P}_s)$ of 95 percent confidence when $NP_s = 10$. But the width of the confidence interval given by this approximation is slightly larger.

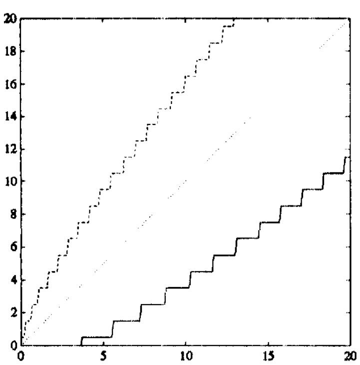

Figure 3: Confidence region for classical MC estimator.

4 Importance Sampling Estimators

We will consider the distribution of CIS estimator $\hat{P}_s^*$ and NIIS estimator $\hat{P}_s^{***}$ with the help of Edgeworth expansion. Only ISI free binary system is considered, but the results can be assumed to be applicable in principle for systems with small to moderate ISI and multilevel systems, and for other appropriately designed IS techniques.

For CIS, let $U = I_B(x_k, \hat{x}_k) w_{N_k}^*(n_k)$, the second moment of U is

$$E(U^2) = \frac{\sigma_*}{\sigma d} Q(\frac{d}{\sigma}) \tag{16}$$

where

$$d = \sqrt{2 - \frac{\sigma^2}{\sigma_*^2}} \tag{17}$$

and the third and fourth moments of U are

$$\begin{aligned} E(U^3) &= \int\int I_B \frac{\sigma_*^2}{\sqrt{2\pi}\sigma^3} \exp\left(\frac{-n_k^2}{2\sigma^2 d_3^{-2}}\right) f_{X_k}(x_k)\, dx_k dn_k \\ &= \frac{\sigma_*^2}{\sigma^2 d_2} Q(\frac{d_2}{\sigma}) \end{aligned} \tag{18}$$

$$\begin{aligned} E(U^4) &= \int\int I_B \frac{\sigma_*^3}{\sqrt{2\pi}\sigma^4} \exp\left(\frac{-n_k^2}{2\sigma^2 d_3^{-2}}\right) f_{X_k}(x_k)\, dx_k dn_k \\ &= \frac{\sigma_*^3}{\sigma^3 d_3} Q(\frac{d_3}{\sigma}) \end{aligned} \tag{19}$$

where

$$d_2 = \sqrt{3 - \frac{2\sigma^2}{\sigma_*^2}},\ d_3 = \sqrt{4 - \frac{3\sigma^2}{\sigma_*^2}}. \tag{20}$$

Thus the second, third, and fourth central moments of U can be easily calculated, and the coefficients of skewness and kurtosis of $U_{N^*} = \sqrt{N^*}(\hat{P}_s^* - P_s)/[\mathrm{Var}(U)]^{1/2}$ can be obtaind by the relations

$$\gamma_{1N^*} = \frac{\mu_3(U)}{[\mathrm{Var}(U)]^{3/2}\sqrt{N^*}},\ \gamma_{2N^*} = \left[\frac{\mu_4(U)}{[\mathrm{Var}(U)]^2} - 3\right] \Big/ N^*. \tag{21}$$

For $\sigma_* = \sigma_{*,\mathrm{opt}}, \sigma = 1/4$, we have

$$\gamma_{1N^*} = \frac{7.6}{\sqrt{N^*}},\ \gamma_{2N^*} = \frac{64.0}{N^*} \tag{22}$$

and for $\sigma_* = \sigma_{*,\mathrm{opt}}, \sigma = 1/5$, we have

$$\gamma_{1N^*} = \frac{-0.4}{\sqrt{N^*}},\ \gamma_{2N^*} = \frac{-2.9}{N^*} \tag{23}$$

all small enough to justify Gaussian approximation even for very modest sample sise N^*. Thus we conclude that this CIS estimator is approximately Gaussian distributed.

For the NIIS technique, define $V = I_B(x_k, \hat{x}_k) w_{N_k}^{***}(n_k)$, then the second moment of V is

$$E(V^2) \approx 2\exp(\frac{c^2}{\sigma^2}) Q(\frac{1+c}{\sigma}) \tag{24}$$

and the third and fourth moments are

$$E(V^3) \approx 4\exp(\frac{c^2}{\sigma^2}) Q(\frac{1+2c}{\sigma}) \tag{25}$$

$$E(V^4) \approx 8\exp(\frac{c^2}{\sigma^2}) Q(\frac{1+3c}{\sigma}). \tag{26}$$

Following the same procedure as in the preceding paragraph, the coefficients of skewness and kurtosis can be easily calculated. For $c = c_{\mathrm{opt}}, \sigma = 1/4$, we have

$$\gamma_{1N^{***}} = \frac{-1.0}{\sqrt{N^{***}}},\ \gamma_{2N^{***}} = \frac{-2.3}{N^{***}} \tag{27}$$

small enough to justify Gaussian approximation even for very modest sample sise N^{***}. Thus we conclude that the NIIS estimator is also approximately Gaussian distributed.

5 The Empirical Approaches

Estimating the variance of an IS estimator is very important in an actual simulation. A new method for estimating IS estimator variance is discussed here. This method is basically applicable for any well-designed IS technique. It is illustrated using the CIS technique as an example. The confidence intervals of simulation results can also be obtained by this method.

5.1 Conventional Method

According to Section 4, the CIS estimator is approximately Gaussian distributed even for very modest N^*. An elaborate approach to estimate σ^2_{CIS} is discussed by Shanmugan [2], which requires l independent simulation runs with N^* samples in each run. Let $A_i, i = 1, 2, \ldots, l$, be the outcomes of $\hat{P}_s^*$ for the l simulation runs, respectively. Let

$$\bar{A} = \frac{1}{l}\sum_{i=1}^{l} A_i,\ s_l^2 = \frac{1}{l-1}\sum_{i=1}^{l}(A_i - \bar{A})^2 \tag{28}$$

be the estimates of P_s and σ^2_{CIS}, respectively, then $(l-1)s_l^2/\sigma^2_{\text{CIS}}$ has a *chi square distribution with $l-1$ degrees of freedom* χ^2_{l-1}, and it is easy to find a level $(1-\alpha)$ confidence interval for σ^2_{CIS} by the method of pivot [9]. For example, if $l = 10$, then the 95 percent confidence interval for σ^2_{CIS} is $(0.47s_l^2, 3.33s_l^2)$, or equivalently the 95 percent confidence interval for σ_{CIS} is $(0.69s_l, 1.82s_l)$. We can also obtain the confidence interval for P_s using this approach, since $\sqrt{l}(\bar{A} - P_s)/s_l$ has a *t distribution with $l-1$ degrees of freedom* $\mathcal{T}_{l-1}$. For example, if $l = 10$, then the 95 percent confidence interval for P_s is $(\bar{A} - 2.26s_l/\sqrt{10}, \bar{A} + 2.26s_l/\sqrt{10})$, a very narrow confidence interval.

5.2 A New Approach

The above method has the drawback of having to perform l simulation runs to obtain an estimate of σ^2_{CIS} and a confidence interval of P_s. We shall present here a new method to obtain an estimate of σ^2_{CIS} and a confidence interval of P_s using only a single simulation run with N^* samples. We divide N^* into n, say 10 for example, equal sections. It is not difficult to adjust N^* such that N^* divide n, and for this method to work, n should not be too large, and N^* should not be too small. Let

$$B_j = \frac{1}{N^*/n}\sum_{k=(j-1)N^*/n}^{jN^*/n-1} I_B(x_k, \hat{x}_k)w^*_{N_k}(n_k),\ j = 1, 2, \ldots, n \tag{29}$$

then

$$\bar{B} = \frac{1}{n}\sum_{j=1}^{n} B_j = \hat{P}_s^* \tag{30}$$

and $B_j, j = 1, 2, \ldots, n$ are approximately identically Gaussian distributed. They are also independent for ISI free systems, and are approximately independent for systems with small to moderate ISI if $\nu \ll N^*/n$. Thus

$$\sigma^2_{\text{CIS}} = \frac{1}{n}\sigma^2(B_j) \tag{31}$$

and

$$s_n^2 = \frac{1}{n(n-1)}\sum_{j=1}^{n}(B_j - \bar{B})^2 \tag{32}$$

is a good estimate of σ^2_{CIS}, and $(n-1)s_n^2/\sigma^2_{\text{CIS}}$ is χ^2_{n-1} distributed and $(\bar{B} - P_s)/s_n$ is $\mathcal{T}_{n-1}$ distributed. It is thus easy to find level $(1-\alpha)$ confidence intervals for σ^2_{CIS} and P_s. For example, if $n = 10$, $(0.47s_n^2, 3.33s_n^2)$ is a 95 percent confidence interval for σ^2_{CIS}, or equivalently $(0.69s_n, 1.82s_n)$ is a 95 percent confidence interval for σ_{CIS}, and $(\bar{B} - 2.26s_n, \bar{B} + 2.26s_n)$ is a 95 percent confidence interval for P_s.

This new approach are better than the conventional approach in comparing the efficiency of IS techniques. In order to know which IS technique is the best, right now we only need one simulation run instead of, say 10 simulation runs. So a saving of 10 times is achieved.

6 Numerical Results

In the numerical simulation to be discussed here, all random variables are derived from the uniform $(0,1)$ distribution with suitable transformations. The uniform $(0,1)$ distribution is generated by a maximum length linear congrunential pseudo-random generator [10] defined by [11]

$$f(z) = 16807z \bmod 2^{31} - 1. \tag{33}$$

The Gaussian distribution is derived using the Kinderman-Ramage algorithm [12]. The distribution specified by Equation (9) is derived by adding cB to a Gaussian variable, where c is a constant and B is a Bernoulli trial with equally probable outcomes ± 1.

First we shall consider the ISI free 8PAM. Tables 1 (a) lists the simulation results of SER using the CIS technique, whereas Tables 1 (b) lists the results of SER using the NIIS technique. In Table 1, when simulation samples per run is not very large we use 10 simulation runs to obtain an estimate of the estimator variances and estimates of P_s, when simulation samples are large we use single simulation run only and the samples are equally divided into 10 sections as discussed in Section 5. It is easy to understand this numerical data. For example, in the first row of Table 1 (a), the simulated system is 8PAM with noise standard deviation $\sigma = 0.2702$ (corresponds to a SNR $= E(|x_k|^2)/\sigma^2$ of 22.42 dB), 10 independent runs are performed with 300 samples per run, and the 95 percent confidence interval for $P_s = 1.869E-4$ is $(\hat{P}_s^* - 2.26s_l/\sqrt{10}, \hat{P}_s^* + 2.26s_l/\sqrt{10}) = (1.731E-4, 2.451E-4)$, while in the last row of Table 1 (a), a single run is performed with 8000 samples divided into 10 sections. It can be found that NIIS achieves approximately the same estimator variance as CIS using several times less samples for 8PAM. Thus NIIS outperformes CIS for PAM.

Next we consider $g = \{0.9933, 0.0993, 0.0591\}$ 2PAM. Table 2 (a) lists the simulation results of SER using the CIS technique, whereas Table 2 (b) lists the results of SER using the NIIS technique. By comparing Tables 2 (a) and (b) with Tables 1 (a) and (b), it can be found that both CIS and NIIS for systems with ISI are less efficient than CIS and NIIS for ISI free systems, since CIS and NIIS for 2PAM with ISI achieve approximately the same estimator variance as CIS and NIIS for ISI free 8PAM using 4 to 8 times more samples. Furthermore, NIIS outperforms CIS for 2PAM with ISI, since it achieves approximately the same extimator variance as CIS with 3 to 5 times less samples.

Table 1: SER simulation results for 8PAM (a) using the CIS technique,(b) using the NIIS technique.

(a)

σ	N^*	l/n	$\hat{P}^*_s$	s_l/s_n	P_s
0.2702	300	10(l)	2.02E-4	5.04E-5(s_l)	1.87E-4
0.2344	550	10(l)	1.69E-5	3.61E-6(s_l)	1.74E-5
0.2104	2000	10(n)	1.80E-6	1.90E-7(s_n)	1.76E-6
0.1925	3000	10(n)	1.76E-7	2.46E-8(s_n)	1.79E-7
0.1677	5000	10(n)	2.18E-9	2.09E-10(s_n)	2.17E-9

(b)

σ	N^{***}	l/n	$\hat{P}^{***}_s$	s_l/s_n	P_s
0.2702	100	10(l)	1.75E-4	5.44E-5(s_l)	1.87E-4
0.2344	180	10(l)	1.82E-5	3.32E-6(s_l)	1.74E-5
0.2104	500	10(l)	1.77E-6	2.60E-7(s_l)	1.76E-6
0.1925	800	10(n)	1.81E-7	1.55E-8(s_n)	1.79E-7
0.1677	1000	10(n)	2.24E-9	3.29E-10(s_n)	2.17E-9

Table 2: SER simulation results for 2PAM with ISI (a) using the CIS technique,(b) using the NIIS technique.

(a)

σ	N^*	l/n	$\hat{P}^*_s$	s_l/s_n	P_s
0.2702	1200	10(l)	3.00E-4	6.02E-5(s_l)	3.23E-4
0.2344	2200	10(l)	5.05E-5	8.74E-6(s_l)	5.38E-5
0.2104	5000	10(n)	9.12E-6	1.17E-6(s_n)	1.00E-5
0.1925	9000	10(n)	1.77E-6	2.64E-7(s_n)	1.93E-6
0.1677	20000	10(n)	8.90E-8	1.01E-8(s_n)	8.34E-8

(b)

σ	N^{***}	l/n	$\hat{P}^{***}_s$	s_l/s_n	P_s
0.2702	400	10(l)	3.03E-4	5.24E-5(s_l)	3.23E-4
0.2344	700	10(l)	5.22E-5	9.24E-6(s_l)	5.38E-5
0.2104	1200	10(n)	1.11E-5	1.43E-6(s_n)	1.00E-5
0.1925	2200	10(n)	1.88E-6	2.70E-7(s_n)	1.93E-6
0.1677	4000	10(n)	8.54E-8	8.00E-9(s_n)	8.34E-8

7 Conclusion

The classical MC estimator is found to be better approximated by Poisson distribution, while the IS estimator is found to be well approximatedby Gaussian distribution. A reliable method is also presented to obtain the confidence intervals for the error rate and IS estimator variance using only a signle simulation run. Our arguments are strongly supported by the numerical results.

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