VARIETIES FIBERED BY GOOD MINIMAL MODELS

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Abstract. Let $f : X \to Y$ be an algebraic fiber space such that the general fiber has a good minimal model. We show that if f is the Iitaka fibration or if f is the Albanese map with relative dimension no more than three, then X has a good minimal model.

Two of the main conjectures in higher dimensional birational geometry are:

- **Existence of minimal models (Mori's program)**: Its aim is to provide a nice representative, *a minimal model*, in the birational class of a given variety X. A minimal model is required to have a nef canonical divisor and hence it is the "simplest" one in its birational class.
- **Abundance conjecture**: Any minimal model has semiample canonical system so that the m-th canonical system is base point free for some $m > 0$.

If for a variety we can find a minimal model such that abundance holds, then we say this variety has a *good minimal model*. Existence of a good minimal model has been established in several cases. Amongst these

- $\dim(X) \le 3$ by S. Mori, Y. Kawamata, and others,
- varieties of general type by [BCHM06], and
- maximal Albanese dimensional varieties by [Fuj09].

In this paper we prove the existence of good minimal models in the following cases:

- The general fiber of the Iitaka fibration of a smooth projective variety X has a good minimal model (Theorem 4.4).
- The general fiber of the Albanese morphism of a smooth projective variety X has dimension no more than three (Theorem 4.5).

The original motivation for this paper is to study **Ueno's Conjecture C** ([Uen75, §11]).

Conjecture 0.1. *If $f : X^n \to Y^m$ is an algebraic fiber space of smooth projective varieties with general fiber F, then we have*

- $C_{n,m} : \kappa(X) \ge \kappa(F) + \kappa(Y)$, *and*
- $C^+_{n,m} : \kappa(X) \ge \kappa(F) + \mathrm{Max}\{\mathrm{Var}(f), \kappa(Y)\}$ *if* $\kappa(Y) \ge 0$, *where* $\mathrm{Var}(f)$ *is the variation of* f *(cf.* [Mor85, §6 and §7]).

Conjecture C has also been established in many cases. For example,

- $C^+_{n,m}$ holds if the general fiber F of f has a good minimal model by [Kaw85], and
- $C_{n,m}$ holds if the general fiber F of f is of maximal Albanese dimension by [Fuj02a].

(The reader can find a more complete list of the known results in [Mor85, §6 and §7] which we do not repeat here.) A related conjecture, **Viehweg's Question Q(f)** (cf. [Mor85, §7]) asks : Let $f : X \to Y$ be an algebraic fiber space with $\mathrm{Var}(f) = \dim(Y)$, then is $f_*(\omega^k_{X/Y})$ big for some positive integer k? It is known that a positive answer to $Q(f)$ implies $C^+_{n,m}$.

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Kawamata proved in [Kaw85] that $Q(f)$ holds when the general fiber F has a good minimal model. A question of Mori in [Mor85, Remark 7.7] then asks if $Q(f)$ holds by assuming that the general fiber of the Iitaka fibration of F has a good minimal model. Hence a corollary of the above mentioned results gives a positive answer to Mori's question:

Corollary 0.2. *Let $f : X \to Y$ be an algebraic fiber space of normal projective varieties with general fiber F. Suppose that the general fiber of the Iitaka fibration of F has a good minimal model. Then Ueno's conjecture C holds on f.*

This paper is organized as follow: We recall some definitions in section 1. In section 2 we establish the necessary ingredients for constructing good minimal models. In section 3 we prove a nonvanishing theorem by using generic vanishing results. Section 4 is the heart of this paper where we construct our good minimal models in Theorems 4.2, 4.4, and 4.5.

Remark 0.3. After the completion of this paper, we were informed that Professor Yum-Tong Siu has announced a proof of the abundance conjecture in [Siu09] which in particular would imply many of the results in this paper.

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1. Preliminaries

We work over the complex number field $\mathbb{C}$. We refer the readers to [KM98] and [BCHM06] for the standard terminology on singularities and the minimal model program, and to [Laz04b] and [BCHM06] for definition of a multiplier ideal sheaf and the related asymptotic constructions.

In this paper, *a pair* (X, Δ) *over* U consists of a $\mathbb{Q}$-factorial normal projective variety X with an effective $\mathbb{R}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and a projective morphism $X \to U$ to a quasi-projective variety U. We recall the definition of a minimal model here.

Definition 1.1. For a log canonical pair (X, Δ) over U, we say that a birational map $\phi : (X, \Delta) \dashrightarrow (X', \Delta' = \phi_*\Delta)$ over U is a *minimal model* if

1. X' is normal and $\mathbb{Q}$-factorial,
2. ϕ extracts no divisors,
3. $K_{X'} + \Delta'$ is nef over U, and
4. ϕ is $(K_X + \Delta)$-negative, i.e. $a(F, X, \Delta) < a(F, X', \Delta')$ for each ϕ-exceptional divisor F.

Moreover, we say that *abundance* holds on (X', Δ') if $K_{X'} + \Delta'$ is semiample over U, i.e. $K_{X'} + \Delta'$ is an $\mathbb{R}$-linear sum of $\mathbb{Q}$-Cartier semiample over U divisors. A *good minimal model* of a pair (X, Δ) over U is a minimal model such that abundance holds.

Remark 1.2. A minimal model in this paper is a log terminal model as defined in [BCHM06].

Remark 1.3. Let $X \to U$ and $Y \to U$ be two projective morphisms of normal quasi-projective varieties. Let $\phi : X \dashrightarrow Y$ be a birational contraction over U. Let D and D' be $\mathbb{R}$-Cartier divisors such that $D' = \phi_*D$ is nef over U. Then ϕ is D-negative if given a common resolution $p : W \to X$ and $q : W \to Y$, we may write

$$p^*D = q^*D' + E,$$

where $p_*E \geq 0$ and the support of p_*E contains the union of all ϕ-exceptional divisors (cf. [BCHM06, Lemma 3.6.3]).

A proper morphism $f : X \to Y$ of normal varieties is *an algebraic fiber space* if it is surjective with connected fibers. For an effective divisor Γ on X, we write $\Gamma = \Gamma_{\text{hor}} + \Gamma_{\text{ver}}$ where Γ_{hor} and Γ_{ver} are effective without common components such that Γ_{hor} dominates Y and $\operatorname{codim}(\operatorname{Supp}(f(\Gamma_{\text{ver}}))) \geq 1$ on Y respectively.

For general results on Fourier-Mukai transforms, we refer to [Muk81]. We recall the definition of certain cohomological support loci which will be used in the proof of Theorem 3.1.

Definition 1.4. Let $\mathcal{F}$ be a coherent sheaf on an abelian variety A. Then we define for each $i = 0, ..., \dim(A)$ the subset

$$V^i(\mathcal{F}) := \{P \in \hat{A} |\ h^i(\mathcal{F} \otimes P) > 0\}.$$

These subsets are studied in [GL1], [GL2], and [Hac04].

Definition 1.5. Let L be a line bundle on a smooth projective variety X. For each non-negative integer m, we define

$$V_m(L) := \{P \in \operatorname{Pic}^0(X) |\ h^0(X, L^{\otimes m} \otimes P) > 0\}.$$

These subsets are studied in [CH].

2. Preparation

2.1. **Good Minimal Models.**

Lemma 2.1. *Let (X_i, Δ_i), $i = 1, 2$, be two klt pairs over U and $\alpha : (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2)$ be a birational map over U with $\alpha_* \Delta_1 = \Delta_2$. Suppose that α is $(K_{X_1} + \Delta_1)$-negative and extracts no divisors, then (X_1, Δ_1) has a good minimal model over U if (X_2, Δ_2) does.*

Proof. This is [BCHM06, Lemma 3.6.9]. □

Lemma 2.2. *Let (X, Δ) be a terminal pair over U. Then for any resolution $\mu : (X', \Delta') \to (X, \Delta)$ with $\Delta' := \mu_*^{-1} \Delta$, a good minimal model of (X', Δ') is also a good minimal model of (X, Δ).*

Proof. Note that if we write $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + E$, then E is effective and its support equals to the set of all μ-exceptional divisors. Hence the same argument as in [BCHM06, Lemma 3.6.10] applies (without adding extra μ-exceptional divisors). □

Theorem 2.3. *Let $\phi_i : (X, \Delta) \dashrightarrow (X_i, \Delta_i)$, i=1,2, be two minimal models of a klt pair (X, Δ) over U with $\Delta_i = (\phi_i)_* \Delta$. Then the natural birational map $\psi : (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2)$ over U can be decomposed into a sequence of $(K_{X_1} + \Delta_1)$-flops over U.*

Proof. By [KM98, Theorem 3.52], (X_i, Δ_i) are isomorphic in codimension one, and hence the argument in [Kaw07] applies. □

Proposition 2.4. *Let (X, Δ) be a klt pair over U. If (X, Δ) has a good minimal model over U, then any other minimal model of (X, Δ) over U is also good.*

Proof. Suppose (X_g, Δ_g) is a good minimal model of (X, Δ) over U and $(\tilde{X}, \tilde{\Delta})$ is another minimal model of (X, Δ) over U. From Theorem 2.3, the birational map $\alpha : (X_g, \Delta_g) \dashrightarrow (\tilde{X}, \tilde{\Delta})$ over U may be decomposed into a sequence of flops over U. If an intermediate step is $X_i \dashrightarrow X_{i+1}$ with X_i a good minimal model of (X, Δ) over U, then the morphism $X_i \to Z := \mathbf{Proj}_U(K_{X_i} + \Delta_i)$ factors through the contraction morphism $g_i : X_i \to Z_i$ by $\psi : Z_i \to Z$. Hence for the corresponding flop $g_{i+1} : X_{i+1} \to Z_i$, there exists a divisor H on Z ample over U such that $K_{X_{i+1}} + \Delta_{i+1} = g_{i+1}^* \psi^*(H)$. In particular, $K_{X_{i+1}} + \Delta_{i+1}$ is semiample over U and X_{i+1} is also a good minimal model of (X, Δ) over U. □

Proposition 2.5. *If a klt pair (X, Δ) over U has a good minimal model over U, then any $(K_X + \Delta)$ minimal model program with scaling of an ample divisor A over U terminates.*

Proof. Let $\phi : (X, \Delta) \dashrightarrow (X_g, \Delta_g)$ with $\Delta_g = \phi_*\Delta$ be a good minimal model of (X, Δ) over U and $f : X_g \to Z = \mathbf{Proj}_U(K_{X_g} + \Delta_g)$ the corresponding morphism over U. Note that ϕ contracts exactly the divisorial part of $\mathbf{B}(K_X + \Delta/U)$ (cf. [BCHM06, Lemma 3.6.3]).

Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0A_g)$ with $A_g = \phi_*A$ is klt and an ample divisor H on X_g. By [BCHM06], the outcome of running a $(K_{X_g} + \Delta_g + t_0A_g)$-minimal model program with scaling of H over Z exists and is a minimal model $\psi : X_g \dashrightarrow X'$ of $(X_g, \Delta_g + t_0A_g)$ over Z. As $K_{X_g} + \Delta_g \equiv_Z 0$, we have $K_{X'} + \Delta' \equiv_Z 0$ where $\Delta' = \psi_*\Delta_g$. Hence those curves contracted in each step of this minimal model program over Z have trivial intersection with $K_{X_g} + \Delta_g$ and negative intersection with A_g. In particular, this shows that X' is a minimal model of $(X_g, \Delta_g + tA_g)$ over Z for all $t \in (0, t_0]$. Since $\Delta' + t_0A'$ with $A' = \psi_*A_g$ is big over U, there exists only finitely many $(K_{X'} + \Delta' + t_0A')$-negative extremal rays in $\overline{\mathrm{NE}}(X'/U)$ by [BCHM06, Corollary 3.8.2]. Hence by considering smaller $t_0 > 0$, we can assume that X' is a minimal model of $(X_g, \Delta_g + tA_g)$ over U for all $t \in (0, t_0]$. As a map being negative is an open condition, we may choose $t_0 > 0$ sufficiently small such that $\psi \circ \phi$ is $(K_X + \Delta + tA)$-negative for all $t \in (0, t_0]$, and hence X' is a minimal model of $(X, \Delta + tA)$ over U for all $t \in (0, t_0]$. This implies that $\psi \circ \phi$ contracts exactly the divisorial part of $\mathbf{B}(K_X + \Delta + t_0A/U)$ which is contained in $\mathbf{B}(K_X + \Delta/U)$ and is contracted by ϕ. Hence ψ contracts no divisors, and in particular $\psi \circ \phi$ is $(K_X + \Delta + tA)$-negative for all $t \in [0, t_0]$. This implies that X' is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [0, t_0]$. Note that then $\mathbf{B}(K_X + \Delta + tA/U)$ has the same divisorial components for all $t \in [0, t_0]$.

Now choose $0 < t_1 < t_0$ such that $(X, \Delta + t_1A)$ is klt and run a minimal model program of $(X, \Delta + t_1A)$ with scaling of A over U. By [BCHM06], the outcome $\phi : X \dashrightarrow \tilde{X}$ exists and is a minimal model of $(X, \Delta + t_1A)$ over U. Since being $(K_X + \Delta + tA)$-negative is an open condition and $K_{\tilde{X}} + \tilde{\Delta} + t\tilde{A} := \phi_*(K_X + \Delta + tA)$ is nef over U for $t \in [t_1, t_0]$, by picking $t_0 > 0$ smaller if necessary we can assume that $\tilde{X}$ is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [t_1, t_0]$. Since $\mathbf{B}(K_X + \Delta + tA/U)$ has the same divisorial components for all $t \in [0, t_0]$, X' and $\tilde{X}$ are isomorphic in codimension one. **For each** $t \in [t_1, t_0]$, by Theorem 2.3 we may decompose the birational map $X' \dashrightarrow \tilde{X}$ over U into *possibly different* sequences S_t of $(K_{X'} + \Delta' + tA')$-flops over U as X' and $\tilde{X}$ are both minimal models of $(X, \Delta + tA)$ over U. Since $\Delta' + tA'$ is big over U for any $t \in [t_1, t_0]$ and each outcome of a $(K_{X'} + \Delta' + tA')$-flop over U is also a minimal model of $(X, \Delta + tA)$ over U, by finiteness of models in [BCHM06] we can only have finitely many $(K_{X'} + \Delta' + tA')$-flop over U as t ranges in $[t_1, t_0]$. In particular, we can find an uncountable subset $T_1 \subseteq [t_1, t_0]$ such that for all $t \in T_1$, the first $(K_{X'} + \Delta' + tA')$-flops over U of the corresponding sequences S_t's are all the same. Note that those curves contracted by this flop then have trivial intersection with A' and hence this flop is a $(K_{X'} + \Delta')$-flop over U. As each sequence S_t is finite, inductively we can find a $t^* \in [t_1, t_0]$ such that all the steps of the sequence S_{t^*} connecting X' and $\tilde{X}$ are $(K_{X'} + \Delta')$-flops over U. Since X' is a minimal model of (X, Δ) over U, we then also have that $\tilde{X}$ is a minimal model of (X, Δ) over U. In particular, this shows that the minimal model program of (X, Δ) with scaling of A over U terminates. □

Corollary 2.6. *Let (X, Δ) be a klt pair over U. Suppose that (X, Δ) has a good minimal model over U, then there exists a $t_0 > 0$ such that: if $\tilde{X}$ is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [\alpha, \beta]$ for some $0 \le \alpha < \beta \le t_0$, then $\tilde{X}$ is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [0, t_0]$. In particular, the set of all such minimal models $\tilde{X}$ is finite.*

Proof. By Proposition 2.5, there exists a $t_0 > 0$ and a birational map $X \dashrightarrow X'$ over U such that X' is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [0, t_0]$. By the proof of Proposition 2.5, there is a finite sequence of $(K_{X'} + \Delta')$-flops over U connecting $X' \dashrightarrow \tilde{X}$ which are also A'-trivial and hence $(K_{X'} + \Delta' + tA')$-flops over U for all $t \in [0, t_0]$, where Δ' and A' are the proper transforms of Δ and A on X'. Therefore the corollary follows. Note that X' and the varieties given by $(K_{X'} + \Delta')$-flops over U appearing in the proof are all minimal models of the big pair $(X, \Delta + t_0 A)$ over U and hence by [BCHM06] there can only be finitely many of these. □

Proposition 2.7. *Let $f : X \to Y$ be an algebraic fiber space of normal quasi-projective varieties such that X is $\mathbb{Q}$-factorial with klt singularities and projective over Y. Suppose that the general fiber F of f has a good minimal model, then X is birational to some X' over Y such that the general fiber of $f' : X' \to Y$ is a good minimal model.*

Proof. Pick an ample divisor H on X and run a minimal model program of X with scaling of H over Y. Suppose that $\mathrm{cont}_R : X \to W$ is the contraction morphism corresponding to an extremal ray $R \in \overline{\mathrm{NE}}(X/Y)$. If R doesn't give an extremal contraction of F, then $\mathrm{cont}_R|_F = \mathrm{id}_F$. Otherwise it's easy to see that cont_R and $\mathrm{cont}_R|_F$ must be of the same type (divisorial or small). Suppose that we have a sequence of infinitely many flips whcih are nontrivial on the general fiber F with $t_i > t_{i+1} > 0$ such that $K_{F_i} + tH_i|_{F_i}$ is nef for all $t \in [t_{i+1}, t_i]$. Since F has a good minimal model, by Corollary 2.6 the set of such F_i's is finite (modulo isomorphisms) and each F_i is a good minimal model of F. Then we get a contradiction by the same argument as in the last step of the proof of [BCHM06, Lemma 4.2]. Hence after finitely many steps, we may assume that all flips are trivial on the general fiber, and so we get an algebraic fiber space $f' : X' \to Y$ such that the general fiber is a good minimal model. □

2.2. **Degenerate Divisors.** This part concerns the negativity property of a "degenerate" divisor. The following definition is taken from [Tak08].

Definition 2.8. Let $f : X \to Y$ be a proper surjective morphism of normal varieties and $D \in \mathrm{WDiv}_{\mathbb{R}}(X)$ be an effective Weil divisor. Then

- D is f-exceptional if $\mathrm{codim}(\mathrm{Supp}(f(D))) \geq 2$.
- D is of insufficient fiber type if $\mathrm{codim}(\mathrm{Supp}(f(D))) = 1$ and there exists a prime divisor $\Gamma \nsubseteq \mathrm{Supp}(D)$ such that $f(\Gamma) \subseteq \mathrm{Supp}(f(D))$ has codimension one in Y.

In either of the above cases, we say that D is *degenerate.* In particular, a degenerate divisor is always assumed to be effective.

Lemma 2.9. *Let $f : X \to Y$ be an algebraic fiber space of normal projective varieties such that X is $\mathbb{Q}$-factorial. Then for a degenerate Weil divisor D on X, we can always find a component $F \subseteq$ $\mathrm{Supp}(D)$ which is covered by curves contracted by f and intersecting D negatively. In particular, we have $F \subseteq \mathbf{B}_-(D/Y)$, the diminished base locus of D over Y.*

Proof. Write $D = \Sigma r_i D_i$ with $r_i > 0$ and $D_i \in \mathrm{Div}(X)$ prime.

Case 1: Suppose D is f-exceptional, and hence $\dim Y \geq 2$. Cutting by general hyperplanes, we reduce to a birational morphism of surfaces with $E = \Sigma r_j \tilde{E}_j$, where $\tilde{E}_j = D_j \cap H_1 \cap ... \cap H_n$ may be nonreduced and reducible and $E = D \cap H_1 \cap ... \cap H_n$. Note that we may assume $P := f(E)$ is a point. By Hodge index theorem (cf. [Bǎd01, Corollary 2.7]), the intersection matrix of irreducible components of $f^{-1}(P)$ is negative-definite. Then $(E)^2 < 0$, and hence $(\tilde{E}_j.D) = (\tilde{E}_j.E) < 0$ for some j. In particular, $(\tilde{E}_j.D_j) < 0$ and D_j is covered by curves intersecting D negatively.

Case 2: Suppose D is of insufficient fiber type. Cutting by general hyperplanes, we reduce to a morphism from a surface to a curve with $E = \Sigma r_j \tilde{E}_j$ supported on fibers, where $E_j = D_j \cap H_1 \cap ... \cap H_n$ may be nonreduced and reducible and $E = D \cap H_1 \cap ... \cap H_n$. Then by [Bǎd01, Corollary 2.6], we have $(E)^2 \leq 0$. But $\mathrm{Supp}(E)$ can not be the whole fiber, hence we can find Γ an effective divisor having no common components with E such that $\mathrm{Supp}(E + \Gamma) = f^{-1}(f(E))$. For $F := f^*(f_*(E))$, then we can find a and b two positive real numbers such that $aF \leq E + \Gamma \leq bF$. If $(E)^2 = 0$, then E is nef and hence $E.F = 0$ implies $E.(E + \Gamma) = 0$. But we have $E.\Gamma > 0$ which implies $(E)^2 < 0$, a contradiction. Hence $(E)^2 < 0$ and the same argument as in case 1 applies.

To prove that $D_j \subseteq \mathbf{B}_-(D/Y)$, we pick an ample divisor A on X and $\epsilon > 0$ a small rational number such that $\tilde{E}_j.(D + \epsilon A) < 0$. Note that we then also have $\tilde{E}_j.(D + \epsilon A + f^*R) < 0$ for any $\mathbb{R}$-Cartier divisor R on Y. In particular, this shows that $\tilde{E}_j \subseteq \mathbf{B}(D+\epsilon A/Y)$. As $\tilde{E}_j$ passes through a general point of D_j, we have $D_j \subseteq \mathbf{B}(D+\epsilon A/Y) \subseteq \mathbf{B}_-(D/Y)$. □

3. Nonvanishing Theorems

Theorem 3.1. *Let X be a smooth projective irregular variety with $\alpha := alb_X : X \to A := Alb(X)$ the Albanese morphism and $\alpha' : X \to Y$ with general fiber F be the Stein factorization of $\alpha : X \to \alpha(X) \subseteq A$. Suppose $\kappa(F) \geq 0$, then $\kappa(X) \geq 0$.*

Lemma 3.2. *Assumptions as in Theorem 3.1, then K_X is pseudo-effective.*

Proof. We have $\alpha'_*\omega^N_{X/Y} \neq 0$ and is weakly positive by [Vie83]. Hence for any $\epsilon > 0$ and H ample on Y, $\alpha'_*\omega^N_{X/Y} \otimes (\epsilon H)$ is big. As Y is finite over $\alpha(X)$, a subvariety in A, we have $\kappa(Y) \geq 0$ and hence $\alpha'_*\omega^N_X \otimes (\epsilon H)$ is also big. In particular $\kappa(K_X + \frac{\epsilon}{N}(\alpha')^*H) \geq 0$ for any $\epsilon > 0$, and hence K_X is pseudo-effective. □

Lemma 3.3. *Let X be a smooth projective variety. Suppose $\{D_k\}$ is a collection of effective $\mathbb{Q}$-divisors with $k \in \mathbf{N}$ such that the corresponding multiplier ideal sheaves $\mathcal{J}_k := \mathcal{J}(D_k)$ satisfy $\mathcal{J}_k \subseteq \mathcal{J}_{k'}$ whenever $k \leq k'$. If there exists a line bundle L such that $L - D_k$ is nef and big for all $k > 0$, then $\bigcap_{i>0} \mathcal{J}_i = \mathcal{J}_k$ for k sufficiently large.*

Proof. The proof is taken from [Hac04, Proposition 5.1]. We reproduce the proof here for the convenience of the reader. Take a sufficiently ample divisor H on X and consider the line bundle $M = L + (n+1)H$ for $n = \dim(X)$, then

$$M - D_k - (iH) \equiv L - D_k + (n-i+1)H$$

is nef and big for all $k > 0$ and $1 \leq i \leq n$. Hence we have $H^i(X, \mathcal{O}_X(K_X + M - iH) \otimes \mathcal{J}_k) = 0$ for all $i > 0$ by Nadel vanishing, and then $\mathcal{O}_X(K_X + M) \otimes \mathcal{J}_k$ is generated by global sections by Mumford regularity. In particular, if $\mathcal{J}_k \neq \mathcal{J}_{k'}$ for $k \leq k'$, then we get a strict inclusion $H^0(X, \mathcal{O}_X(K_X+M) \otimes \mathcal{J}_k) \subseteq H^0(X, \mathcal{O}_X(K_X+M) \otimes \mathcal{J}_{k'})$ of $\mathbb{C}$ vector spaces. But this can not happen for infinitely many times, hence the lemma follows. □

Lemma 3.4. *The same setting as in Theorem 3.1. Then for H an ample divisor on A and a non-negative integer m, $\mathcal{J}(\|mK_X + \epsilon\alpha^*H\|)$ is independent of $\epsilon \in \mathbb{Q}$ for any $\epsilon > 0$ sufficiently small. Hence we can define the sheaf*

$$\mathcal{F}_m := \alpha_*(\omega^m_X \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^*H\|))$$

on A for $\epsilon > 0$ a sufficiently small rational number. Then for L any sufficiently ample line bundle on the dual abelian variety $\hat{A}$ with $\hat{L}$ the Fourier-Mukai transform of L on A, we have $H^i(A, \mathcal{F}_m \otimes \hat{L}^\vee) = 0$ for all $i > 0$. From [Hac04, Corollary 3.2], *we then have for any non-negative integer m the inclusions:*

$$V^0(\mathcal{F}_m) \supseteq V^1(\mathcal{F}_m) \supseteq ... \supseteq V^n(\mathcal{F}_m).$$

In particular, $V^0(\mathcal{F}_m) = \phi$ implies $\mathcal{F}_m$=0.

Proof. The first statement follows from Lemma 3.3 by taking L to be $mK_X + \alpha^*H$ on X. The vanishing of cohomologies follows from [Hac04, Theorem 4.1] with a slight modification and hence we reproduce the argument here. Consider the isogeny $\phi_L : \hat{A} \to A$ defined by L, $\hat{\alpha} : \hat{X} \to \hat{A}$, and $f : \hat{X} = X \times_A \hat{A} \to X$. Then as $\phi^*_L \hat{L}^\vee = \oplus_{h^0(L)} L$, we have

$$\begin{aligned} H^i(A, \mathcal{F}_m \otimes \hat{L}^\vee) &\subseteq H^i(A, \mathcal{F}_m \otimes \hat{L}^\vee \otimes \phi_{L*}\mathcal{O}_{\hat{A}}) \\ &= H^i(\hat{A}, \phi^*_L\mathcal{F}_m \otimes \phi^*_L\hat{L}^\vee) \\ &= \oplus H^i(\hat{A}, \hat{\alpha}_* f^*(\omega^m_X \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^*H\|)) \otimes L) \\ &= \oplus H^i(\hat{A}, \hat{\alpha}_*(\omega^m_{\hat{X}} \otimes \mathcal{J}(\|(m-1)K_{\hat{X}} + \epsilon\hat{\alpha}^*\phi^*_L H\|)) \otimes L), \end{aligned}$$

where the last equality is the étale base change of multiplier ideal sheaves in [Laz04b, Theorem 11.2.16]. For $i > 0$, the cohomological groups above vanish by Nadel vanishing on $\hat{X}$, or by Kawamata-Viehweg vanishing theorem on a log resolution $\pi : Y \to \hat{X}$. The final statement follows from [Muk81, Theorem 2.2]. □

Proof. (of Theorem 3.1) For general point $z \in Y$ and m sufficiently divisible, we have for the sheaves defined by $\mathcal{F}'_m := \alpha'_*(\omega_X^m \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|))$ on Y:

$$\begin{aligned}(\mathcal{F}'_m)_z &= H^0(F, \omega_F^m \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|)|_F)\\ &\supseteq H^0(F, \omega_F^m \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|_F))\\ &= H^0(F, \omega_F^m \otimes \mathcal{J}(\|(m-1)K_F\|))\\ &\supseteq H^0(F, \omega_F^m \otimes \mathcal{J}(\|mK_F\|))\\ &= H^0(F, \omega_F^m) > 0.\end{aligned}$$

The first inclusion follows from the properties of the restriction of multiplier ideal sheaves in [Laz04b, Theorem 11.2.1], the second equality from the explanation of semipositivity in [Kol93, Proposition 10.2], and the last inequality from $\kappa(F) \geq 0$. Hence $\mathcal{F}'_m$ is non-trivial. In particular, $\mathcal{F}_m$ is also non-trivial for m sufficiently divisible.

For m sufficiently divisible, $\mathcal{F}_m \neq 0$ and hence $V^0(\mathcal{F}_m) \neq \phi$ by Lemma 3.4. This shows that we can find an element $P \in \mathrm{Pic}^0(X)$ with $H^0(X, \omega_X^m \otimes P) \neq 0$. Following the argument of [CH, Theorem 3.2] (cf. Theorem 3.5), $V_m(K_X)$ is a union of torsion translates of subvarieties in $\mathrm{Pic}^0(X)$ for $m \geq 1$ and in particular we can find an element $P' \in \mathrm{Pic}^0(X)_{\mathrm{tor}}$ with $H^0(X, \omega_X^m \otimes P') \neq 0$. Then $H^0(X, \omega_X^{md}) \neq 0$ for $d = \mathrm{ord}(P')$ in $\mathrm{Pic}^0(X)$ and hence $\kappa(X) \geq 0$. □

Theorem 3.5. *Let X be a smooth projective variety. Then the cohomological loci*

$$V_m(K_X) := \{P \in \mathrm{Pic}^0(X) |\ h^0(X, \omega_X^{\otimes m} \otimes P) > 0\}$$

for m a positive integer, if non-empty, is a finite union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$.

Proof. If $m = 1$, then by a result of Simpson in [Sim93] the loci $V_1(K_X)$ is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$. In general, let $\tilde{P} \in V_m(K_X)$ and write $\tilde{P} = mP$ for some $P \in \mathrm{Pic}^0(X)$. Let $\mu : X' \to X$ be a log resolution of $|m(K_X + P)|$, and $D \in \mu^*|m(K_X + P)|$ be a divisor with simple normal crossing support. Consider the line bundle $N := \mu^*\mathcal{O}_X((m-1)(K_X + P)) \otimes \mathcal{O}_{X'}(-\lfloor \frac{m-1}{m} D \rfloor)$, then it follows from [CL-H, Theorem 8.3] and [Sim93] that the cohomological loci

$$V^0(\omega_{X'} \otimes N) := \{R \in \mathrm{Pic}^0(X') | h^0(\omega_{X'} \otimes R) > 0\}$$

is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X')$. Note that $\mathrm{Pic}^0(X') \cong \mathrm{Pic}^0(X)$ as X is smooth, and hence we may identify the elements in these two groups (via pulling back by μ). It is easy to see that $P \in V^0(\omega_{X'} \otimes N)$, and hence there exists an abelian subvariety $T \subseteq \mathrm{Pic}^0(X)$ and a torsion element $Q \in \mathrm{Pic}^0(X)_{\mathrm{tor}}$ such that

$$P \in T + Q \subseteq V^0(\omega_{X'} \otimes N).$$

By pushing forward, it is also easy to see that

$$T + Q + (m-1)P \subseteq V_m(K_X).$$

Now since $rP \in rT$ for some positive integer r and rT is a group, we have that $r(m-1)P \in rT$ and hence $(m-1)P \in T + Q'$ for some torsion element $Q' \in \mathrm{Pic}^0(X)_{\mathrm{tor}}$. In particular, we have

$$\tilde{P} = mP \in T + Q + (m-1)P = T + Q + Q' \subseteq V_m(K_X),$$

and hence $V_m(K_X)$ is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$.

Let V be an irreducible component of $V_m(K_X)$ and denote $\mathrm{Pic}^0(X)$ by A. Note that for any general point of V, there is a torsion translate of an abelian subvariety of A contained in V passing through it. It is well-known that if V is of general type, then there are no nontrivial abelian subvarieties of A contained in V passing through general points of V. In this case, a general point of V must be torsion and hence $\dim V$ can only be zero since there are only countably many torsion points in A. It follows that V is a torsion point. If V is not of general type, then by [Uen75, Theorem 10.9] there is an algebraic fiber space $f : V \to B$ with general fiber A_1 induced by $\pi : A \to A/A_1$, where A_1 is an abelian subvariety of A and $B \subseteq A/A_1$ is a subvariety of general type. Since there are also torsion translate of abelian subvarieties of A/A_1 contained in B passing through general points of B, B is a torsion point and then V is a torsion translate of an abelian subvariety of A. Hence we conclude that the algebraic set $V_m(K_X)$, if non-empty, is a *finite* union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$. □

4. Main Theorems

We will now establish the existence of good minimal models in several different situations.

4.1. Kodaira Dimension $\kappa(X) = 0$.

Lemma 4.1. *Let X be a smooth projective variety with $\kappa(X) = 0$ and $\alpha : X \to A := Alb(X)$ be the Albanese morphism. Suppose that $|mK_X| \neq \phi$ for some $m > 0$ and F is the unique effective divisor in $|mK_X|$, then* $\mathrm{Supp}(F)$ *contains all α-exceptional divisors.*

Proof. Suppose that we have $P_1(X) = P_2(X) = 1$, where $P_n(X) := h^0(X, \omega_X^n)$ is the n-th plurigenus. Then by [EL97, Proposition 2.1], the origin of $\mathrm{Pic}^0(X)$ is an isolated point of the cohomological support loci

$$V_0(X) := \{y \in \mathrm{Pic}^0(X) | h^0(X, \omega_X \otimes P_y) \neq 0\}.$$

By [EL97, Corollary 1.5], this implies that for every non-zero $\eta \in H^0(X, \Omega^1_X)$, the map

$$\phi_\eta : H^0(X, \Omega^{n-1}_X) \overset{\wedge\eta}{\to} H^0(X, \Omega^n_X)$$

determined by wedging with η is surjective. Now for a given α-exceptional divisor E, the differential $d\alpha : T_eX \to T_{\alpha(e)}A$ is not of full rank at a general point $e \in E$. In particular, if η_e is the non-zero 1-form given by pulling back a flat 1-form on A which is in $\ker[(d\alpha)^\vee : T^*_{\alpha(e)}A \to T^*_eX]$, then the surjectivity of ϕ_{η_e} shows that F must pass through e (cf. [EL97, Proposition 2.2]). Since $e \in E$ is a general point, we have $E \subset F$.

For the general case, let $\mu : X' \to X$ be a log resolution of (X, F). Then we may write $mK_{X'} = \mu^*(mK_X) + E \sim \mu^*F + E =: F' \geq 0$ where E is effective and consists of μ-exceptional divisors, $\mathrm{Supp}(F')$ is a simple normal crossing divisor, and $\mu_*F' = F$. We take a cyclic cover $f : Y \to X'$ defined by $F' \in |mK_{X'}|$ followed by a resolution $d : Y' \to Y$. It is well-known that Y is normal with only quotient singularities and for $f' := f \circ d$ we have

$$f'_*\mathcal{O}_{Y'} = \bigoplus_{i=0}^{m-1} (\omega^i_{X'}(-\lfloor \frac{i}{m} F' \rfloor))^\vee$$

and

$$f'_*\omega_{Y'} = \omega_{X'} \otimes (\bigoplus_{i=0}^{m-1} \omega^i_{X'}(-\lfloor \frac{i}{m} F' \rfloor)).$$

An easy computation then shows that $\kappa(Y') = 0$ and $P_1(Y') = P_2(Y') = 1$. As the generic vanishing result still holds for the induced map $\alpha' : Y' \to A$ (cf. [EL97, Remark 1.6]), the argument above then shows that for $m > 0$ sufficiently divisible and the unique effective divisor $\Gamma \in |mK_{Y'}|$, Supp(Γ) contains all α'-exceptional divisors. It is then easy to see that $\mathrm{Supp}(F') = \mathrm{Supp}(f'(\Gamma))$ and hence the lemma now follows by pushing forward Γ to X. $\square$

Theorem 4.2. *Let X be a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities and $\kappa(X) = 0$. Suppose the general fiber F of the Albanese morphism has a good minimal model, then X has a good minimal model.*

Proof. By Lemma 2.2, we may assume that X is smooth. By [Kaw81], the Albanese map $\alpha := alb_X : X \to A := Alb(X)$ is an algebraic fiber space. Moreover we have $\kappa(F) = 0$ as $C_{n,m}$ holds by [Kaw85].

By Proposition 2.7, after running a minimal model program of X with scaling of an ample divisor over A, we have a birational map $X \dashrightarrow X'$ over A such that the general fiber F' of $\alpha' : X' \to A$ is a good minimal model. Moreover we may assume that $\mathbf{B}_-(K_{X'}/A)$ contains no divisorial components. Note that then $\kappa(F') = 0$ implies $K_{F'} \sim_{\mathbb{Q}} 0$. For $K_{X'} \sim_{\mathbb{Q}} \Gamma$ with Γ effective, we write $\Gamma = \Gamma_{\mathrm{hor}} + \Gamma_{\mathrm{ver}}$. Then as $\Gamma_{\mathrm{hor}}|_{F'} = \Gamma|_{F'} \sim_{\mathbb{Q}} K_{X'}|_{F'} \sim K_{F'} \sim_{\mathbb{Q}} 0$, we have $\Gamma_{\mathrm{hor}} = 0$. Suppose there exists an effective divisor $E \leq \Gamma$ with $P := \alpha'_*(E)_{\mathrm{red}}$ a codimension one point and E contains all divisors on X' dominating P. Then we have $\mathrm{Supp}((\alpha')^{-1}(P)) \subseteq \mathrm{Supp}(\Gamma)$ (note that $(\alpha')^{-1}(P)$ may have some exceptional divisorial components which are automatically contained in Supp(Γ) by Lemma 4.1). This implies that

$$0 = \kappa(X) = \kappa(X') \geq \kappa(\mathcal{O}_{X'}(\Gamma)) \geq \kappa(\mathcal{O}_A(P)) > 0,$$

a contradiction. Hence Γ is of insufficient fiber type. By Lemma 2.9, we can find a component D of Γ such that $D \subseteq \mathbf{B}_-(K_{X'}/A)$. But this is impossible as $\mathbf{B}_-(K_{X'}/A)$ contains no divisorial components. Hence $K_{X'} \sim_{\mathbb{Q}} 0$ and X' is a good minimal model of X. $\square$

Corollary 4.3. *Let X be a projective variety with terminal singularities and $\kappa(X) = 0$. Let V be a smooth projective variety of maximal Albanese dimension and $\alpha : X \to V$ be an algebraic fiber space. If the general fiber F of α has a good minimal model, then X has a good minimal model.*

Proof. By [Kaw85], we have $\kappa(V) = 0$ and hence V is birational to its Albanese variety $A := A(V)$. We may then replace V by A. By Proposition 2.2, we may assume that X is smooth. Then we have $\alpha : X \to A$ an algebraic fiber space such that the general fiber F has a good minimal model. As noted in the proof of Lemma 4.1, the argument in Lemma 4.1 and Theorem 4.2 still works in this general case. Hence the corollary follows. $\square$

4.2. **Iitaka Fibration.**

Theorem 4.4. *Let X be a $\mathbb{Q}$-factorial normal projective variety with non-negative Kodaira dimension and at most terminal singularities. Suppose the general fiber F of the Iitaka fibration has a good minimal model, then X has a good minimal model.*

Proof. The theorem is certainly true for the case $\kappa(X) = 0$. For varieties of general type, the theorem follows from [BCHM06] and base point free theorem in [KM98]. Hence we may assume $0 < \kappa(X) < \dim(X)$.

By [BCHM06], $R(K_X)$ is a finitely generated $\mathbb{C}$-algebra and hence there is an integer d such that the truncated ring $R^{[d]}(K_X)$ is generated in degree 1. Take a resolution $\mu : X' \to X$ of X and $|dK_X|$, then

- $\mu^*|mdK_X| = |mM| + mF$ with $|mM|$ base point free and $mF \geq 0$ the fixed divisor for all $m > 0$,
- $f := \phi_{|M|} : X' \to Y$ is birationally equivalent to the Iitaka fibration,
- $K_{X'} = \mu^* K_X + E$ with E effective and μ-exceptional,
- $dK_{X'} \sim M + F + dE$ with $F + dE$ effective and $F + dE \subseteq \mathbf{B}(K_{X'})$.

Write $\Gamma := F + dE = \Gamma_{\text{hor}} + \Gamma_{\text{ver}}$ with respect to f. By Proposition 2.7, after running a minimal model program of X' with scaling of an ample divisor over Y, we may assume that the general fiber of f is a good minimal model. As the general fiber F of f has Kodaira dimension zero, we have $\Gamma_{\text{hor}}|_F = (M + F + dE)|_F \sim (dK_{X'})|_F \sim dK_F \sim_{\mathbb{Q}} 0$ and hence $\Gamma_{\text{hor}} = 0$. In particular, we may assume $F + dE$ consists of only vertical divisors. Note that the condition $F + dE \subseteq \mathbf{B}(K_{X'})$ still holds after steps of a minimal model program.

Consider T an effective divisor with $\mathrm{Supp}(T) \subseteq \mathrm{Supp}(F + dE)$ and the exact sequences

$$0 \to f_*\mathcal{O}_{X'}((j-1)T) \to f_*\mathcal{O}_{X'}(jT) \to Q_j \to 0$$

on Y with $j \geq 1$ and Q_j the quotient. After tensoring with $\mathcal{O}_Y(k)$ for k sufficiently large, we have $Q_j(k)$ is generated by global sections and $H^1(Y, f_*\mathcal{O}_{X'}((j-1)T) \otimes \mathcal{O}_Y(k)) = 0$. As $T \subseteq \mathbf{B}(K_{X'})$, we have

$$H^0(Y, f_*\mathcal{O}_{X'}(jT) \otimes \mathcal{O}_Y(k)) = H^0(X', \mathcal{O}_{X'}(kM + jT)) = H^0(X', \mathcal{O}_{X'}(kM)) = H^0(Y, \mathcal{O}_Y(k))$$

for any $j \geq 0$ as $\mathcal{O}_{X'}(M) = f^*\mathcal{O}_Y(1)$. Hence the exact sequence of cohomological groups shows that $H^0(Y, Q_j(k)) = 0$ and then $Q_j = 0$. In particular, $f_*\mathcal{O}_{X'}(jT) = \mathcal{O}_Y$ for any $j \geq 0$. Suppose that $f_*(T)_{\text{red}} = P$ is a codimension one point on Y such that $\mathrm{Supp}(T)$ contains all divisors in X' dominating P. Note that we can find a big open subset $U \subseteq Y$ such that the image of the exceptional divisors contained in $f^*(P)$ is disjoint from U as it has codimension greater or equal to two. Then there is a positive integer j such that $f_*\mathcal{O}_{X'}(jT)|_U \supseteq \mathcal{O}_Y(P)|_U$. Since $f_*\mathcal{O}_{X'}(jT) = \mathcal{O}_Y$ and $\mathcal{O}_Y(P)$ are reflexive, we have an inclusion $\mathcal{O}_Y(P) \subseteq \mathcal{O}_Y$ which is impossible. In particular, this shows that $F + dE$ is of insufficient fiber type over Y.

By Lemma 2.9, we can find a component of $F+dE$ which is contained in $\mathbf{B}_-(K_{X'}/Y)$. The same argument as in Theorem 4.2 then says that this is impossible. Then $dK_{\tilde{X}} \sim M$ with $\mathcal{O}_{X'}(M) = f^*\mathcal{O}_Y(1)$ is base point free and hence X' is a good minimal model of X by Lemma 2.2 (as μ is a resolution of a terminal variety). □

4.3. **Albanese morphism.**

Theorem 4.5. *Let X be a smooth projective variety with Albanese map $\alpha : X \to A := Alb(X)$. If the general fiber F of α has dimension no more than three with $\kappa(F) \geq 0$, then X has a good minimal model.*

Proof. By Theorem 3.1, we have $\kappa(X) \geq 0$. Let $X \dashrightarrow Z$ be the Iitaka fibration and $X' \to X$ a resolution such that $X' \to Z$ is a morphism. By [CH, Lemma 2.6], the image of the general fiber X'_z of $X' \to Z$ over a general point $z \in Z$ is a translation of a fixed abelian subvariety K in A and $0 \to K \to A \to Alb(Z) \to 0$ is exact. Consider the Albanese map $\alpha_z : X'_z \to Alb(X'_z)$ of X'_z which is an algebraic fiber space as $\kappa(X'_z) = 0$. As K is an abelian variety, $X'_z \to K$ factors through α_z by a surjective map $Alb(X'_z) \to K$. In particular, the general fiber F_z of $\alpha_z : X'_z \to Alb(X'_z)$ is contained in F and hence has dimension no more than three. Then X'_z has a good minimal model by Theorem 4.2 as F_z does. Since X'_z is the general fiber of the Iitaka fibration of X, Theorem 4.4 implies X also has a good minimal model. □

Remark 4.6. In fact, we have

$$\dim(F_z) \leq \dim(F) - \kappa(X) + q(Z),$$

where $q(Z)$ is the irregularity of Z. Hence if $\dim(F) - \kappa(X) + q(Z) \leq 3$, then Theorem 4.5 still holds.

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