Available online at www.sciencedirect.com
SCIENCE DIRECT®

Advances in Applied Mathematics 34 (2005) 591–615

ADVANCES IN Applied Mathematics

www.elsevier.com/locate/yaama

# Reconstruction of cracks in an inhomogeneous anisotropic medium using point sources

Gen Nakamura [a,1], Gunther Uhlmann [b,2], Jenn-Nan Wang [c,*,3]

[a] *Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan*
[b] *Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA*
[c] *Department of Mathematics, National Taiwan University, Taipei 106, Taiwan*

Received 3 May 2004; accepted 3 November 2004

Available online 29 January 2005

**Abstract**

We consider the inverse problem, in two and three dimensions, of identifying elastic cracks embedded in an inhomogeneous anisotropic elastic medium using point sources. The observable data is given by the near-field measurements of the outgoing Green's function for the related stationary system. We give a reconstruction algorithm for this inverse problem.


## 1. Introduction

In this paper we consider the inverse problem of identifying cracks from the near-field measurements of the outgoing Green's function for the stationary elasticity system with inhomogeneous anisotropic medium in $\mathbb{R}^n$. Throughout the paper, we consider $n = 2$ or 3. This is a sequel to our earlier paper [15] in which we dealt with the same problem by using

* Corresponding author.
*E-mail address:* jnwang@math.ntu.edu.tw (J.-N. Wang).
[1] Partially supported by Grant-in-Aid for Scientific Research (B)(2)(No. 14340038) of Japan Society for Promotion of Science.
[2] Partially supported by NSF and a John Simon Guggenheim fellowship.
[3] Partially supported by the National Science Council of Taiwan.

0196-8858/$ – see front matter 
doi:10.1016/j.aam.2004.11.003

boundary measurements for the static system. Similar to the result in [15], our focus here is also on the reconstruction issue.

The inverse problem is described as follows. We put sources and receivers along a connected, closed curve ($n = 2$) or connected, closed surface ($n = 3$) which is large enough to include all inhomogeneities and cracks of the probed region. Each source on the curve or surface induces a scattered field due to the inhomogeneities and the cracks. The scattered field is then recorded by receivers on the same curve or surface. Suppose that we know all the information on the probed medium expect for the elastic cracks. The inverse problem is to reconstruct the cracks from these measurements. A related inverse problem for a scalar equation was considered in [7].

The solution of the inverse problem requires a full understanding of the related direct problem which is the existence and uniqueness of the outgoing Green's function for the inhomogeneous anisotropic elasticity system. As usual, we assume that the medium is homogeneous outside of a large ball. However, unlike most of the literature on elastic scattering, we assume that the homogeneous part is still *anisotropic*. Even the direct problem in this case is quite difficult given that we are considering the full anisotropic elastic system. Our method for obtaining the outgoing Green's function and its properties is to prove the existence and uniqueness in appropriate spaces of the scattering solution in the exterior of cracks for the inhomogeneous anisotropic elasticity system. To characterize the scattering solution, we need to impose certain radiation conditions at infinity. Since the medium is anisotropic outside of a large domain, the classical Sommerfeld–Kupradze radiation conditions are not applicable in this case. For the homogeneous anisotropic elasticity system in two and three dimensions, Natroshvili [19,20] established the generalized Sommerfeld–Kupradze radiation conditions by analyzing the radiation pattern of the oscillation equations under some restrictions on the slowness curves. Here we will impose the same restrictions on these curves and adopt the radiation conditions derived in [19] and [20].

In proving the existence and uniqueness of the scattering solution, we assume that the *unique continuation property* holds for the anisotropic elasticity system considered in the paper. It should be noted that the unique continuation property for general anisotropic elasticity systems is still an open problem. Nonetheless, in two dimensions, the unique continuation property has been proved for anisotropic elasticity with Lipschitz coefficients under some generic conditions [16,17]. We remark that if we assume the unique continuation property then the Runge approximation property is valid. The latter plays an important role in reconstructing the cracks. Combining the unique continuation property and the radiation conditions, we can prove the uniqueness of the scattering solution. We then show the existence of the scattering solution by a Fredholm-type theorem, which is inspired by Lax and Phillips' work [13].

In this paper we will only consider "insulating" cracks which means that the traction vanishes on the cracks. The scattering solution is generated by the source term. Thus the outgoing Green's function is the Schwartz kernel of the map from the source term to the scattering solution. We will first establish the scattering solution with the source term in $L^2_{\mathrm{comp}}$, i.e., compactly supported $L^2$ functions. However, for the purpose of studying the inverse problem, we need to allow the source term to be in a more general space, namely the dual space of $H^1_{\mathrm{comp}}$ (see the definition in Section 3).

Our strategy for identifying cracks from the near-field measurements of the outgoing Green's function at a fixed energy have two main steps. Firstly, we determine the Dirichlet-to-Neumann map for the stationary system from the near-field of the outgoing Green's function at a fixed energy. Secondly, using this Dirichlet-to-Neumann map, we can apply the method developed in [15] to identify cracks which are treated as buried in a finite anisotropic body.

There is an extensive literature investigating the crack determination problem by boundary measurements. We refer to a recent survey article [3] and references there for other developments. The inverse scattering problem from a crack for acoustic waves was studied in [1] and [11]. Extending techniques in those two papers to elastic waves were developed in [2] and [12], respectively. The elastic medium considered in [2] and [12] is homogeneous and isotropic.

The plan of this paper is as follows. In Section 2, we will describe the radiation conditions derived in Natroshvili's papers [19] and [20]. In Section 3, we prove the limiting absorption principle for the inhomogeneous anisotropic elasticity system in $\mathbb{R}^n$ based on the recent work [18]. Then the limiting absorption principle will be used to establish the existence of the scattering solution in the whole space. In Section 4, we prove the existence and uniqueness of the scattering problem in the presence of cracks. The inverse problem of determining cracks is discussed in Section 5.

## 2. Preliminaries

Let $C = (C_{pqrs})$ be a homogeneous elastic tensor satisfying the symmetry properties

$$C_{pqrs} = C_{qprs} = C_{rspq} \quad \forall p, q, r, s \tag{2.1}$$

and the strong convexity condition, i.e., there exists a $\delta > 0$ such that

$$CE \cdot E \geqslant \delta |E|^2 \tag{2.2}$$

for any symmetric matrix $E = (E_{rs})$, where

$$(CE)_{pq} = \sum_{rs} C_{pqrs} E_{rs} \quad \text{and} \quad A \cdot B = \sum_{pq} A_{pq} B_{pq} \quad \text{for matrices } A, B.$$

Here and below, unless otherwise indicated, all Roman indices except $i$ and $n$ are set to be from 1 to $n$, where $n = 2$ or 3. We reserve $i$ for the imaginary number $\sqrt{-1}$. Given $\mathbb{R} \ni \omega > 0$, define the matrix differential operator

$$L(D, \omega) = L(D) + \omega^2 I,$$

where $L(D) = (L_{pr}(D))$ with

$$L_{pr}(D) = \sum_{qs} C_{pqrs} \partial_q \partial_s. \tag{2.3}$$

The symbol of $L(D,\omega)$ is given by $(\omega^2 I - L(\xi)) =: L(\xi,\omega)$ with $L(\xi) = (L_{pr}(\xi))$ and

$$L_{pr}(\xi) = \sum_{qs} C_{pqrs}\xi_q\xi_s.$$

Let $\phi(\xi,\omega)$ be the common denominator of entries of $L^{-1}(\xi,\omega)$. For the isotropic case, it is not hard to compute that

$$\phi(\xi,\omega) = c_0\big(|\xi|^2 - \omega^2(\lambda+2\mu)^{-1}\big)\big(|\xi|^2 - \omega^2\mu^{-1}\big) \tag{2.4}$$

for some constant $c_0$. The formula (2.4) is valid for $n = 2$ or 3. Since we will focus on the "genuinely" anisotropic case in this paper, we assume that

$$\phi(\xi,\omega) = \det L(\xi,\omega).$$

Using the spherical coordinates $\xi = r\theta$ with $r \geqslant 0$ and $\theta \in \mathbb{S}^{n-1}$, we get that

$$\phi(\xi,\omega) = \phi\big(r\theta,\omega^2\big) = \det\big(\omega^2 - r^2 L(\theta)\big) = (-1)^n \det L(\theta)\det\big(r^2 - \omega^2 L^{-1}(\theta)\big).$$

Note that $L^{-1}(\theta)$ exists for all $\theta \in \mathbb{S}^{n-1}$ because of the strong convexity condition (2.2). Now we suppose that there exists $n$ functions $k_1(\theta),\dots,k_n(\theta)$ with

$$0 < \delta_1 < k_1(\theta) < \cdots < k_n(\theta) < \delta_2 \quad \forall\theta \in \mathbb{S}^{n-1} \tag{2.5}$$

such that

$$\phi(\xi,\omega) = (-1)^n \det L(\theta)\prod_{j=1}^{n}\big(r^2 - \omega^2 k_j^2(\theta)\big).$$

Let the surface (or curve) $S_j$ be defined by $\{(r,\theta)\colon\ r = \omega k_j(\theta)\}$. Obviously, $\phi(\xi,\omega)$ vanishes on the curve $S_j$. We now assume that

$$S_2,\dots,S_n \text{ are convex.} \tag{2.6}$$

Notice that the convexity of $S_1$ is also assumed in [19] and [20]. However, this assumption is redundant. The convexity of $S_1$ is a well-known property in the theory of anisotropic elastic waves (see [5] for detailed arguments). From (2.5) we can see that $\nabla\phi(\xi,\omega) \neq 0$ on $S_j$ for all $j$. Furthermore, it follows from (2.6) that for any $x \neq 0$ there exists a unique point $\xi^j$ on $S_j$ such that the unit outer normal vector $\vartheta(\xi^j)$ of $S_j$ at $\xi^j$ is parallel to $x$, i.e., $\vartheta(\xi^j) \parallel x$. Since $\phi(-\xi,\omega) = \phi(\xi,\omega)$, the normal vector $\vartheta(-\xi^j)$ is equal to $-\vartheta(\xi^j)$. In fact, in terms of $\phi(\xi,\omega)$, the unit normal vector $\vartheta(\xi^j)$ is given by

$$\vartheta\big(\xi^j\big) = (-1)^j\frac{\nabla\phi(\xi^j,\omega)}{|\nabla\phi(\xi^j,\omega)|}.$$

Based on conditions (2.5) and (2.6), Natroshvili constructed fundamental solutions for $L(D,\omega)$ by considering the limits of $\Gamma(x,\tau_\varepsilon)$ as $\pm\varepsilon \to 0$, where $\tau_\varepsilon = \omega + i\varepsilon$ and

$$L(D,\tau_\varepsilon)\Gamma(x,\tau_\varepsilon) = \bigl(L(D) + \tau_\varepsilon^2 I\bigr)\Gamma(x,\tau_\varepsilon) = \delta(x)I.$$

In fact, by the light of Fourier transform, $\Gamma(x,\tau_\varepsilon)$ is given by

$$\Gamma(x,\tau_\varepsilon) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} L^{-1}(\xi,\tau_\varepsilon)e^{ix\xi}\,\mathrm{d}\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \phi^{-1}(\xi,\tau_\varepsilon)L^*(\xi,\tau_\varepsilon)e^{ix\xi}\,\mathrm{d}\xi,$$

where $L(\xi,\tau_\varepsilon) = (\tau_\varepsilon^2 - L(\xi))$ and $L^*(\xi,\tau_\varepsilon)$ is the adjoint of the matrix $L(\xi,\tau_\varepsilon)$, i.e., the transpose of the cofactor of $L(\xi,\tau_\varepsilon)$. It is easy to see that the matrix $\Gamma(x,\tau_\varepsilon) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and, together with all its derivatives, decay exponentially in $|x|$ as $|x| \to \infty$. Here we only consider the outgoing fundamental solution, denoted by $\Gamma(x,\omega)$, corresponding to the limit of $\Gamma(x,\tau_\varepsilon)$ as $+\varepsilon \to 0$. It was proved in [19] and [20] that the limit

$$\lim_{+\varepsilon\to 0}\Gamma(x,\tau_\varepsilon) = \Gamma(x,\omega)$$

exist for all $x \neq 0$ and the limit exists uniformly in $|x| > a > 0$. On the other hand, for sufficiently large $|x|$, we have the following asymptotic formula

$$\Gamma(x,\omega) = \sum_j |x|^{-(n-1)/2} R_j e^{ix\xi^j} + O\bigl(|x|^{-(n+1)/2}\bigr), \tag{2.7}$$

where $\xi^j \in S_j$ with $\vartheta(\xi^j) \parallel x$ and

$$R_j(\eta) := R_j\bigl(\xi^j(\eta)\bigr) = (-1)^j \frac{c_n}{\sqrt{\aleph(\xi^j)}\,|\nabla\phi(\xi^j,\omega)|} L^*\bigl(\xi^j,\omega\bigr),$$

where $\eta = x/|x|$, $c_n$ is a constant depending on $n$, and $\aleph(\xi^j)$ is the Gaussian curvature of $S_j$ at $\xi^j$. Furthermore, for any $y$ in a compact set and any multi-indices $\alpha$, $\beta$, we have that

$$\begin{aligned}\partial_x^\alpha\partial_y^\beta\Gamma(x-y,\omega) = \sum_j |x|^{-(n-1)/2} R_j(\eta)\bigl(\xi^j\bigr)^\alpha\bigl(-\xi^j\bigr)^\beta e^{i(x-y)\xi^j}&\\ + O\bigl(|x|^{-(n+1)/2}\bigr)&\end{aligned} \tag{2.8}$$

as $|x| \to \infty$.

Now we are ready the radiation condition for the anisotropic elasticity system $L(D,\omega)$. Let the function $u(x)$ be $C^1$ for large $|x|$, then $u(x)$ is said to satisfy the *generalized Sommerfeld–Kupradze* (*outgoing*) *radiation conditions* if

$$\begin{cases} u(x) = \sum_j u^{(j)}(x), \quad u^{(j)} = O(|x|^{-(n-1)/2}),\\ \partial_l u^{(j)}(x) - i\xi_l^j u^{(j)}(x) = O(|x|^{-(n+1)/2}), \quad j,l = 1,\dots,n, \end{cases} \tag{2.9}$$

hold, where $\xi^j \in S_j$ satisfies $\vartheta(\xi^j) \parallel x$. When $u(x)$ is a vector or matrix function, we say that $u(x)$ satisfies (2.9) if each component of $u$ satisfies (2.9). It is easy to see that $\Gamma(x,\omega)$ satisfies the radiation conditions (2.9). Similar to the isotropic case, a function $u$ in $\mathbb{R}^n \setminus \overline{O}$ satisfying (2.9) has an integral representation formula, where $O$ is an open bounded smooth domain. More precisely, let $u \in H^2_{\mathrm{loc}}(\mathbb{R}^n \setminus \overline{O}) \cap C^1(\mathbb{R}^n \setminus \overline{O})$ satisfy the radiation conditions (2.9) and $L(D,\omega)u$ be compactly supported, then

$$u(x) = \int_{\mathbb{R}^n\setminus\overline{O}} \Gamma(x-y,\omega)L(D_y,\omega)u(y)\,\mathrm{d}y + \int_{\partial O} \Big\{\Gamma(x-y,\omega)\big[T\big(D_y,\eta(y)\big)u(y)\big] - \big[T\big(D_y,\eta(y)\big)\Gamma(x-y,\omega)\big]^t u(y)\Big\}\,\mathrm{d}S \quad \forall x \in \mathbb{R}^n \setminus \overline{O} \tag{2.10}$$

and

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y,\omega)L(D_y,\omega)u(y)\,\mathrm{d}y \quad \forall x \in \mathbb{R}^n$$

if $O = \emptyset$, where $T(D,\eta)$ is the boundary traction operator defined by

$$\big(T(D,\eta)\big)_{pr} = \sum_{qs} C_{pqrs}\eta_q\partial_s$$

with $\eta = [\eta_1,\ldots,\eta_n]^t$ being the unit outer normal of $\partial O$ (see [19] and [20]).

## 3. The scattering problem in the whole space

In this section we would like to discuss the scattering problem for the inhomogeneous anisotropic elasticity system in the whole space. We aim to solve the following scattering problem

$$\begin{cases} \mathcal{L}u + \omega^2 u = h \in L^2_{\mathrm{comp}}(\mathbb{R}^n) \quad \text{in } \mathbb{R}^n, \\ u \text{ satisfies the radiation conditions (2.9)}, \end{cases} \tag{3.1}$$

where

$$\mathcal{L}u = \mathrm{div}\big(C(x)\nabla u\big).$$

Throughout the paper, we assume that the elastic tensor $C(x) = (C_{pqrs}(x)) \in C^1(\mathbb{R}^n)$ satisfies the full symmetry properties

$$C_{pqrs}(x) = C_{qprs}(x) = C_{rspq}(x) \quad \forall p,q,r,s \text{ and } x \in \mathbb{R}^n \tag{3.2}$$

and the strong convexity condition, i.e., there exists a $\tilde{\delta} > 0$ such that for all $x \in \mathbb{R}^n$ and symmetric matrix $E$

$$C(x)E \cdot E \geqslant \tilde{\delta}|E|^2. \tag{3.3}$$

Moreover, there exists a $R > 0$ such that $C(x) = C$ for $|x| > R$, where $C$ is a homogeneous anisotropic elastic tensor. As mentioned in the Introduction, we suppose that the unique continuation property holds for any $H^2_{\mathrm{loc}}(\mathbb{R}^n)$ solution $u(x)$ of $\mathcal{L}u + \omega^2 u = 0$.

We now formulate (3.1) in a weak sense. To this end, let us introduce a weighted Sobolev space. Let $\varrho = (1 + |x|^2)^{-d/2}$ with $d > 1/2$ and the weighted inner product

$$\langle f, g\rangle_{H^{1,-d}(\mathbb{R}^n)} = \langle \varrho f, \varrho g\rangle + \langle \varrho \nabla f, \varrho \nabla g\rangle \quad \text{for } f, g \in H^1_{\mathrm{loc}}\big(\mathbb{R}^n\big).$$

Denote

$$H^{1,-d}\big(\mathbb{R}^n\big) = \text{completion of } H^1\big(\mathbb{R}^n\big) \text{ with respect to } \|\cdot\|_{H^{1,-d}(\mathbb{R}^n)} = \langle \cdot, \cdot\rangle^{1/2}_{H^{1,-d}(\mathbb{R}^n)}.$$

**Definition 3.1.** A function $u \in H^{1,-d}(\mathbb{R}^n)$ which is $C^1$ smooth for sufficiently large $|x|$ is called a scattering solution of (3.1) if $u$ satisfies the radiation conditions (2.9) and

$$\forall \psi(x) \in C_0^\infty\big(\mathbb{R}^n\big), \quad -F(u, \psi) + \omega^2\langle u, \psi\rangle = \langle h, \psi\rangle.$$

Here the sesquilinear form $F(\cdot, \cdot)$ is defined by

$$\begin{aligned} F(u, v) &= \int_{\mathbb{R}^n} \sum_{pqrs} C_{pqrs}(x)\partial_s u_r \overline{\partial_q v_p}\,\mathrm{d}x = \int_{\mathbb{R}^n} \sum_{pq} \sigma_{pq} \overline{\partial_q v_p}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} C(x)\varepsilon(u) \cdot \overline{\varepsilon(v)}\,\mathrm{d}x, \end{aligned}$$

where $\varepsilon(u)_{pq} = (1/2)(\partial_q u_p + \partial_p u_q)$ is known as the strain tensor.

We will prove the existence of a scattering solution to (3.1) by the limiting absorption principle. To establish the limiting absorption principle, we first prove the uniqueness whose proof relies heavily on the radiation conditions and the unique continuation property.

**Theorem 3.2.** *There exists at most one solution to* (3.1).

**Proof.** It suffices show that a homogeneous solution of (3.1) is trivial, namely, any solution $u$ to (3.1) with $h = 0$ must be zero. First of all, we show that a solution $u$ of (3.1) with $h = 0$ decays at a rate of $|x|^{-(n+1)/2}$ at infinity. The same phenomenon was proved for the homogeneous anisotropic elasticity system in [19] and [20]. Since $h = 0$, in light of Green's formula, we have

$$\int\limits_{S_{R'}} \left\{T\big(D, \eta(x)\big)u(x) \cdot \overline{u(x)} - u(x) \cdot \overline{T\big(D, \eta(x)\big)u(x)}\right\} \mathrm{d}S = 0 \quad \forall R' > R, \tag{3.4}$$

where $S_{R'} = \{|x| = R'\}$ and $\eta = x/|x|$. With the identity (3.4) at hand, the same arguments given in [19, Lemma 12] and [20, Lemma 4.1] provide the proof of $u = O(|x|^{-(n+1)/2})$ as $|x| \to \infty$.

The decaying property of $u$ clearly implies

$$u(x) = o\big(|x|^{-(n-1)/2}\big) \quad \text{as } |x| \to \infty.$$

Let $\chi(x) = \chi(|x|) \in C^\infty(\mathbb{R}^n)$ satisfy

$$\chi(x) = \begin{cases} 0 & \text{in } |x| \leqslant R, \\ 1 & \text{in } |x| > 3R/2, \end{cases}$$

and set $v(x) := \chi(x)u(x)$. Then it is readily seen that $v(x) \in C^\infty(\mathbb{R}^n)$ satisfies

$$v(x) = o\big(|x|^{-(n-1)/2}\big) \quad \text{as } |x| \to \infty \tag{3.5}$$

and

$$L(D, \omega) = \big(L(D) + \omega^2 I\big)v(x) = g(x),$$

where $\operatorname{supp}(g) \subseteq \{|x| \leqslant 3R/2\}$. Define the differential operator $L^*(D, \omega)$ with symbol $L^*(\xi, \omega)$. It is readily seen that

$$L^*(D, \omega)L(D, \omega)v = \phi(D, \omega)v = L^*(D, \omega)g =: \tilde{g},$$

where $\phi(D, \omega)$ is the differential operator (scalar) with symbol $\phi(\xi, \omega)$. Likewise, we have $\operatorname{supp}(\tilde{g}) \subseteq \{|x| \leqslant 3R/2\}$. Having conditions (2.5), (2.6) and the asymptotic behavior (3.5) in mind, we now apply Littman's result [14], which is a generalization of Rellich's result, to conclude that

$$u(x) = v(x) = 0 \quad \text{in } \big\{|x| > 3R/2\big\}.$$

Now using the unique continuation property, we get that $u(x) \equiv 0$ in $\mathbb{R}^n$. $\square$

Having obtained the uniqueness result, we can use the same method in [18] to prove the limiting absorption principle for the operator $\mathcal{L}$ in $\mathbb{R}^n$. In [18], the limiting absorption principle for $\mathcal{L}$ with Lipschitz elastic tensor in $\mathbb{R}^2$ were established. The method can be extended to $\mathbb{R}^3$ without essential modifications.

**Theorem 3.3** (Limiting absorption principle). *Let $\mathbb{C} \supset Q^+ := (\omega_0, \omega_1) \times i(0, \varepsilon)$, where $0 < \omega_0 < \omega_1$ and $\varepsilon > 0$. Assume $h \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$. Then the map*

$$\mathcal{R}(\cdot) : Q^+ \to H^{1,-d}(\Omega)$$

*defined by $\mathcal{R}(z)h = (\mathcal{L} + z)^{-1}h$ is uniformly continuous.*

By virtue of the limiting absorption principle, the solution of (3.1) is now given by

$$u = \left(\mathcal{L} + \omega^2 I\right)^{-1} h := \lim_{+\varepsilon \to 0} \left(\mathcal{L} + \left(\omega^2 + i\varepsilon\right) I\right)^{-1} h, \tag{3.6}$$

where the limit exists in $H^{1,-d}(\mathbb{R}^n)$. Since $C(x) \in C^1(\mathbb{R}^n)$, in view of the elliptic regularity theorem (see [6] for example), we have that $u(x) \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$.

## 4. The scattering problem in the presence of cracks

In this section we will face our target problem—the scattering problem in the exterior of the crack. Since the present paper is a continuation of [15], we will follow the notations used there. To describe the crack, we assume that $\widetilde{\Sigma} \subset \mathbb{R}^n$ is a $C^2$ closed Jordan curve ($n = 2$) or closed connected surface ($n = 3$) and $\Sigma \subset \widetilde{\Sigma}$ is an open curve or surface. When $n = 3$ we suppose that the boundary $\partial\Sigma$ of $\Sigma$ is $C^2$. Here $\Sigma$ will be considered as a crack. We assume that $\Sigma \subset B_R$, namely, the crack lies in the inhomogeneous part. We can have several number of cracks. For this case our theory also works without any essential change. Let $\Omega_-$ be the open subset of $\mathbb{R}^n$ with boundary $\widetilde{\Sigma}$ and $\Omega_+ := \mathbb{R}^n \setminus \overline{\Omega}_-$. The trace operator from $\Omega_\pm$ to $\widetilde{\Sigma}$ is denoted by $\gamma_\pm$, respectively. The direction of the unit normal $\nu$ to $\widetilde{\Sigma}$ is directed into $\Omega_+$.

We now introduce two Sobolev spaces $\overline{H}^k(\Sigma)$ and $\dot{H}^k(\overline{\Sigma})$, which are subspaces of $H^k(\widetilde{\Sigma})$, defined by

$$\overline{H}^k(\Sigma) = H^k\left(\widetilde{\Sigma}\right)\big|_{\Sigma}$$

and

$$u \in \dot{H}^k\left(\overline{\Sigma}\right) \quad \text{iff } u \in H^k\left(\widetilde{\Sigma}\right) \text{ and } \operatorname{supp}(u) \subseteq \overline{\Sigma},$$

respectively. To deal with the exterior problem, we also need a weighted Sobolev space in the cracked domain $\Omega := \mathbb{R}^n \setminus \overline{\Sigma}$

$$H^{1,-d}(\Omega) := \Big\{u \in \mathfrak{D}'\left(\mathbb{R}^n\right):\ u_\pm := u|_{\Omega_\pm},\ \text{where } u_- \in H^1\left(\mathbb{R}^n\right)\big|_{\Omega_-},\ u_+ \in H^{1,-d}(\Omega_+);$$
$$[u] := \gamma_+ u_+ - \gamma_- u_- = 0 \text{ on } \widetilde{\Sigma} \setminus \overline{\Sigma}\Big\},$$

where $\|u\|_{H^{1,-d}(\Omega)} := \|u_-\|_{H^1(\Omega_-)} + \|u_+\|_{H^{1,-d}(\Omega_+)}$ and $H^{1,-d}(\Omega_+) := H^{1,-d}(\mathbb{R}^n)|_{\Omega_+}$. Also, we define

$$H^1(\Omega) := \{u \in \mathfrak{D}'(\mathbb{R}^n):\ u_\pm := u|_{\Omega_\pm},\ \text{where } u_- \in H^1(\Omega_-),\ u_+ \in H^1(\Omega_+);\\ [u] := \gamma_+ u_+ - \gamma_- u_- = 0 \text{ on } \widetilde{\Sigma} \setminus \overline{\Sigma}\}$$

and

$$H^1_{\text{comp}}(\Omega) := \{u \in H^1(\Omega) \text{ and } \operatorname{supp}(u) \text{ is compact}\}.$$

Now we consider the scattering problem in the exterior of the crack

$$\begin{cases} \mathcal{L}v + \omega^2 v = f \quad \text{in } \Omega, \\ T(D, \nu)v = 0 \quad \text{on } \Sigma, \\ v \text{ satisfies the radiation conditions (2.9)}, \end{cases} \tag{4.1}$$

where $f \in L^2_{\text{comp}}(\Omega)$ and the traction operator $T$ is defined in terms of $C(x)$ and $T(D,\nu)v = 0$ is interpreted as $T(D,\nu)v = \gamma_- T(D,\nu)v = \gamma_+ T(D,\nu)v = 0$. In the weak formulation, solving (4.1) is equivalent to finding $v \in H^{1,-d}(\Omega)$ such that

$$\begin{cases} -F_\Omega(v,\psi) + \omega^2\langle v,\psi\rangle = \langle f,\psi\rangle \quad \forall \psi \in H^1_{\text{comp}}(\Omega), \\ v \text{ satisfies the radiation conditions (2.9)}, \end{cases} \tag{4.2}$$

where

$$F_\Omega(v,\psi) = \int_\Omega C(x)\varepsilon(v)\cdot\overline{\varepsilon(\psi)}\,\mathrm{d}x.$$

Note that by the standard elliptic regularity theorem, $v$ possesses enough smoothness to make sense of the radiation conditions. Also, the condition $T(D,\nu)v|_\Sigma = 0$ is implicitly enforced in (4.2). The task now is to prove the uniqueness and existence of $v$ to (4.2). We begin with the uniqueness.

**Theorem 4.1.** *There exists at most one solution to* (4.2).

**Proof.** The proof of this theorem is similar to that of Theorem 3.2. We will show that if $f = 0$ then $v = 0$ in $\Omega$. Choosing $\widetilde{R} > R$ and using Green's formula given in [15] (see the formula (A.1) there) over $B_{\widetilde{R}} \setminus \overline{\Sigma}$, we can derive that

$$0 = \int_{B_{\widetilde{R}}\setminus\Sigma} \{\bar{v}(\mathcal{L}+\omega^2)v - v\overline{(\mathcal{L}+\omega^2)v}\}\,\mathrm{d}x = \int_{B_{\widetilde{R}}\setminus\Sigma} \{\bar{v}\mathcal{L}v - v\overline{\mathcal{L}v}\}\,\mathrm{d}x$$
$$= \int_{S_{\widetilde{R}}} \{T(D,\eta)v(x)\cdot\overline{v(x)} - v(x)\cdot\overline{T(D,\eta)v(x)}\}\,\mathrm{d}S - \int_\Sigma [T(D,\nu)v]\gamma_+\bar{v}\,\mathrm{d}S$$

$$- \int\limits_{\Sigma} \gamma_- T(D,\nu)v[\bar{v}]\,\mathrm{d}S + \int\limits_{\Sigma} \bigl[\overline{T(D,\nu)v}\bigr]\gamma_+ v\,\mathrm{d}S$$

$$+ \int\limits_{\Sigma} \gamma_- \overline{T(D,\nu)v}[v]\,\mathrm{d}S. \tag{4.3}$$

Here we want to remark that the traction operator $T$ is defined in terms of the inhomogeneous elastic tensor $C(x)$, which is homogeneous on $\Gamma_{\widetilde{R}}$ for all $\widetilde{R} > R$. Taking advantage of $T(D,\nu)v = 0$ on $\Sigma$, we obtain from (4.3) that

$$\int\limits_{S_{\widetilde{R}}} \bigl\{T(D,\eta)v(x)\cdot\overline{v(x)} - v(x)\cdot\overline{T(D,\eta)v(x)}\bigr\}\,\mathrm{d}S = 0,$$

which is the same integral as (3.4). Using the arguments in [19, Lemma 12] and [20, Lemma 4.1] again, we have $v = O(|x|^{-(n+1)/2})$ as $|x| \to \infty$. Now combining Littman's theorem and the unique continuation property, we conclude that $v = 0$ in $\Omega$. □

We now turn our attention to the existence of (4.2). The line of argument is to reduce the exterior scattering problem to an interior problem. The idea is due to Lax and Phillips [13]. Let $U_h(x) \in H^{1,-d}(\mathbb{R}^n)$ be the solution of (3.1) with right-hand side $h \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $\operatorname{supp}(h) \subset \overline{B_{R_1}}$ for some $R_1 > 0$. Define

$$V(x) = U_h(x) - \varphi(x)W(x),$$

where $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ equals 1 in a neighborhood of $\Sigma$ and zero in $\mathbb{R}^n \setminus B_{R_1}$. Then we can deduce that

$$\mathcal{L}V + \omega^2 V = h - \bigl(\mathcal{L} + \omega^2\bigr)(\varphi W).$$

Therefore, $V$ is a scattering solution of (4.2) if $W$ solves

$$\begin{cases} f = h - (\mathcal{L} + \omega^2)(\varphi W) & \text{in } B_{R_1} \setminus \overline{\Sigma}, \\ T(D,\nu)U_h = T(D,\nu)W \quad \text{on } \Sigma, & W = 0 \quad \text{on } S_{R_1}. \end{cases} \tag{4.4}$$

Note that the problem (4.4) is interpreted in the weak sense.

The idea now is to relate $W$ to $h$. There are many ways to do this. Here we choose $W$ to be the solution of

$$\begin{cases} \mathcal{L}W = 0 \quad \text{in } B_{R_1} \setminus \overline{\Sigma}, \\ T(D,\nu)W = T(D,\nu)U_h \quad \text{on } \Sigma, \qquad W = 0 \quad \text{on } S_{R_1}. \end{cases} \tag{4.5}$$

As usual, (4.5) is understood in the weak sense. The well-posedness of (4.5) was already proved in [15] and we have that $W \in H^1(B_{R_1} \setminus \overline{\Sigma})$, where

$$H^1\big(B_{R_1}\setminus\overline{\Sigma}\big):=\big\{u\in\mathfrak{D}'(B_{R_1}):\ u_\pm:=u|_{\Omega_\pm},\ \text{where } u_-\in H^1(\Omega_-),\ u_+\in H^1\big(B_{R_1}\setminus\overline{\Omega_-}\big);\\ [u]:=\gamma_+u_+-\gamma_-u_-=0 \text{ on } \widetilde{\Sigma}\setminus\overline{\Sigma}\big\}.$$

It is easy to see that the map $Q:h\to(\mathcal{L}+\omega^2)(\varphi W)$ is linear and determined by the compositions

$$h\to U_h\to T(D,\nu)U_h|_\Sigma\to W\to\big(\mathcal{L}+\omega^2\big)(\varphi W). \tag{4.6}$$

We observe that

$$\big(\mathcal{L}+\omega^2\big)(\varphi W)=\varphi\mathcal{L}W+[\mathcal{L},\varphi]W+\omega^2\varphi=[\mathcal{L},\varphi]W+\omega^2\varphi W,$$

where $[\mathcal{L},\varphi]$ is the commutator of $\mathcal{L}$ and $\varphi$. It should be noted that $[\mathcal{L},\varphi]$ contains only first derivatives. Recall that $U_h\in H^2_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, the spaces associated with the compositions (4.6) are

$$L^2(B_{R_1})\to H^2(B_{R_1})\hookrightarrow H^1(B_{R_1})\to\overline{H}^{-1/2}(\Sigma)\to H^1\big(B_{R_1}\setminus\overline{\Sigma}\big)\to L^2(B_{R_1}).$$

From the compact embedding property of $H^2(B_{R_1})\hookrightarrow H^1(B_{R_1})$, we get: $Q:L^2(B_{R_1})\to L^2(B_{R_1})$ is compact. In turn, it remains to solve the Fredholm-type equation in $L^2(B_{R_1})$

$$(I-Q)h=f. \tag{4.7}$$

Therefore, to complete the proof of existence, we only need to show the injectivity of $(I-Q)$.

So if $f=0$, then $V$ solves (4.2) with homogeneous data. By virtue of Theorem 4.1, we have that $V\equiv0$ in $\Omega$ and therefore, $U_h=\varphi W$ in $B_{R_1}\setminus\overline{\Sigma}$. Since $\varphi(x)$ is 1 near $\Sigma$, $U_h=W$ near $\Sigma$, namely, $W$ is $H^2$ near $\Sigma$. It turns out $W$ solves

$$\mathcal{L}W=0\quad\text{in } B_{R_1},\qquad W=0\quad\text{on } S_{R_1}.$$

Consequently, $W$ is trivial and hence $h=f+(\mathcal{L}+\omega^2)(\varphi W)=0$ in $B_{R_1}$. Since $h\in L^2_{\mathrm{comp}}(\mathbb{R}^n)$ and $\operatorname{supp}(h)\subset\overline{B_{R_1}}$, we have $h\equiv0$. Thus, we have shown that

**Theorem 4.2.** *There exists a solution to* (4.2).

## 5. Inverse problem

This section is devoted to the study of the inverse problem. We first precisely formulate the inverse problem we have in mind. Let $v_f$ be the scattering solution of (4.1) for $f\in L^2_{\mathrm{comp}}(\Omega)$. We can write

$$v_f(x)=\mathcal{G}f(x):=\int_\Omega G_\Sigma(x,y,\omega)f(y)\,\mathrm{d}y,$$

where $G_\Sigma(x, y, \omega)$ is called the outgoing Green's function. It is clear that the integral operator $\mathcal{G}$ maps $L^2_{\text{comp}}(\Omega)$ to $H^{1,-d}(\Omega)$. In this paper we consider the following inverse problem

**Inverse Problem.** Identify $\Sigma$ from $G_\Sigma(x, y, \omega)$ for $x, y \in S_R$ at a fixed $\omega > 0$.

As stated in the introduction, we are concerned with giving reconstruction formulas for this inverse problem. A reconstruction algorithm of our method will be given at the end of this section.

### *5.1. Near-field measurements to the Dirichlet-to-Neumann map*

The first step in our method for this inverse problem is to convert the near-field measurement at a fixed energy to the Dirichlet-to-Neumann map, or the displacement-to-traction map, on $S_R$. In order to do so, we would like to extend the mapping property of $\mathcal{G}$. More precisely, we will show that $\mathcal{G}$ maps $H^{-1}_{\text{comp},\Sigma}(\Omega)$ to $H^{1,-d}(\Omega)$, where

$$H^{-1}_{\text{comp},\Sigma}(\Omega) := \left\{u \in \left(H^1_{\text{comp}}(\Omega)\right)^*\text{: supp}(u) \text{ is compact and } \operatorname{supp}(u) \cap \overline{\Sigma} = \emptyset\right\}.$$

The proof of this fact is based on the following a priori estimate.

**Lemma 5.1.** *Let $u \in H^{1,-d}(\Omega)$ satisfy $T(D, \nu)u = 0$ on $\Sigma$, the radiation conditions* (2.9), *and $f := (\mathcal{L} + \omega^2)u \in (H^1_{\text{comp}}(\Omega))^*$. Then there exists a constant $c > 0$ such that*

$$\|u\|_{H^{1,-d}(\Omega)} \leqslant c\left(\|f\|_{(H^1_{\text{comp}}(\Omega))^*} + \|u\|_{L^{2,-d}(\Omega)}\right), \tag{5.1}$$

*where $\|u\|^2_{L^{2,-d}(\Omega)} = \int_\Omega |\rho u|^2\,\mathrm{d}x$.*

**Proof.** This lemma can be proved by straightforward computations. Indeed, let $\chi(r) \in C_0^\infty(\mathbb{R})$ with $0 \leqslant \chi \leqslant 1$ satisfy

$$\chi(r) = \begin{cases} 1 & r \leqslant 1, \\ 0 & r \geqslant 2, \end{cases}$$

and define $\chi_\varepsilon(r) = \chi(\varepsilon r)$. In what follows we denote $|x| = r$. It is readily seen that $(1 + r^2)^{-d}\chi_\varepsilon(r)^2 u \in H^1_{\text{comp}}(\Omega)$. Therefore, using the weak formulation we have that

$$\begin{aligned}\left\langle f, \left(1 + r^2\right)^{-d}\chi_\varepsilon^2 u\right\rangle = &-\int_\Omega C(x)\varepsilon(u)\cdot\varepsilon\left(\left(1 + r^2\right)^{-d}\chi_\varepsilon^2\bar{u}\right)\mathrm{d}x \\ &+ \int_\Omega \omega^2 u\cdot\left(1 + r^2\right)^{-d}\chi_\varepsilon^2\bar{u}\,\mathrm{d}x.\end{aligned} \tag{5.2}$$

We now treat the first term on the right side of (5.2). Observe that

$$
\begin{aligned}
&C(x)\varepsilon(u)\cdot\varepsilon\big((1+r^2)^{-d}\chi_\varepsilon^2\bar{u}\big)\\
&\quad= C(x)\varepsilon(u)\cdot\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\varepsilon\big((1+r^2)^{-d/2}\chi_\varepsilon\bar{u}\big)\\
&\qquad+C(x)\varepsilon(u)\cdot\frac{1}{2}\Big[\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\otimes(1+r^2)^{-d/2}\chi_\varepsilon\bar{u}\\
&\qquad+(1+r^2)^{-d/2}\chi_\varepsilon\bar{u}\otimes\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\Big]\\
&\quad= C(x)\varepsilon\big((1+r^2)^{-d/2}\chi_\varepsilon u\big)\cdot\varepsilon\big((1+r^2)^{-d/2}\chi_\varepsilon\bar{u}\big)\\
&\qquad-C(x)\left\{\frac{1}{2}\big[\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\otimes u+u\otimes\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\big]\right\}\\
&\qquad\times\left\{(1+r^2)^{-d/2}\chi_\varepsilon\varepsilon(\bar{u})+\frac{1}{2}\big[\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\otimes u+u\otimes\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\big]\right\}\\
&\qquad+C(x)\varepsilon(u)\cdot\frac{1}{2}\Big[\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\otimes(1+r^2)^{-d/2}\chi_\varepsilon\bar{u}\\
&\qquad+(1+r^2)^{-d/2}\chi_\varepsilon\bar{u}\otimes\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)\Big].
\end{aligned}
\tag{5.3}
$$

In view of the strong convexity condition and Korn's inequality, we get that

$$
\begin{aligned}
&\int_\Omega C(x)\varepsilon\big((1+r^2)^{-d/2}\chi_\varepsilon u\big)\cdot\varepsilon\big((1+r^2)^{-d/2}\chi_\varepsilon\bar{u}\big)\,\mathrm{d}x\\
&\qquad\geqslant c_1\int_\Omega\big|\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon u\big)\big|^2\,\mathrm{d}x-c_2\int_\Omega(1+r^2)^{-d}\chi_\varepsilon^2|u|^2\,\mathrm{d}x
\end{aligned}
\tag{5.4}
$$

for some positive constants $c_1$ and $c_2$. It should be noted that Korn's inequality holds in the cracked domain $\Omega$ because of our assumptions on $\Sigma$. It is useful to compute

$$
\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon u\big)=u\otimes\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)+(1+r^2)^{-d/2}\chi_\varepsilon\nabla u \tag{5.5}
$$

and

$$
\nabla\big((1+r^2)^{-d/2}\chi_\varepsilon\big)=-d(1+r^2)^{-d/2-1}x\chi_\varepsilon+(1+r^2)^{-d/2}\nabla\chi_\varepsilon. \tag{5.6}
$$

On the other hand, we can see that

$$
\big|\big\langle f,(1+r^2)^{-d}\chi_\varepsilon^2u\big\rangle\big|\leqslant\|f\|_{(H^1_{\mathrm{comp}}(\Omega))^*}\big\|(1+r^2)^{-d}\chi_\varepsilon^2u\big\|_{H^1(\Omega)}. \tag{5.7}
$$

To obtain the estimate (5.1)–(5.7), use the inequality

$$
|ab|\leqslant\tilde{\varepsilon}|a|^2+\frac{1}{4\tilde{\varepsilon}}|b|^2 \tag{5.8}
$$

for small constant $\tilde{\varepsilon} > 0$, and let $\varepsilon \to 0$. Note that all terms containing $\nabla \chi_\varepsilon$ will tend to zero as $\varepsilon \to 0$. $\square$

Now we can prove that

**Theorem 5.2.** *There exists one and only one scattering solution of* (4.1) *for any* $f \in H^{-1}_{\mathrm{comp},\Sigma}(\Omega)$.

**Proof.** The uniqueness has been shown in Section 4. So we focus on the existence. It is easy to see that $H^{-1}_{\mathrm{comp},\Sigma}(\Omega) \subset H^{-1}_{\mathrm{comp}}(\mathbb{R}^n)$. Thus, if $f \in H^{-1}_{\mathrm{comp},\Sigma}(\Omega)$, then there exist a sequence of functions $f_j \in L^2_{\mathrm{comp}}(\Omega)$ such that

$$f_j \to f \quad \text{in } \left(H^1_{\mathrm{comp}}(\Omega)\right)^*.$$

Denote $v_j \in H^{1,-d}(\Omega)$ be the scattering solution of (4.1) associated with $f_j$. Note that $v_j$ exists from the results in Section 4. $\square$

**Claim.** $\sup_j \|v_j\|_{H^{1,-d}(\Omega)} < \infty$.

We assume the claim for this moment. Observe that the embedding $H^{1,-d}(\Omega) \to L^{2,-d'}(\Omega)$ is compact for any $1/2 < d < d'$. To see why this is true, we use the usual Rellich's theorem for $H^1(\Omega_-)$ and the compact embedding result for the weighted Sobolev space $H^{1,-d}(\Omega_+)$ in [4]. Therefore, there exist $v \in L^{2,-d'}(\Omega)$ and a subsequence of $\{v_j\}$, still denoted by $\{v_j\}$, such that

$$\|v_j - v\|_{L^{2,-d'}(\Omega)} \to 0.$$

By the a priori estimate (5.1), we see that $\{v_j\}$ is a Cauchy sequence in $H^{1,-d'}(\Omega)$. So, $v \in H^{1,-d'}(\Omega)$ and $v_j \to v$ in $H^{1,-d'}(\Omega)$. To verify that $v$ is a scattering solution, we recall the weak formulation for $v_j$

$$-F_\Omega(v_j, \psi) + \omega^2 \langle v_j, \psi\rangle = \langle f_j, \psi\rangle \quad \forall \psi \in H^1_{\mathrm{comp}}(\Omega).$$

Taking $j \to \infty$ yields that $v$ satisfies

$$-F_\Omega(v, \psi) + \omega^2 \langle v, \psi\rangle = \langle f, \psi\rangle \quad \forall \psi \in H^1_{\mathrm{comp}}(\Omega).$$

It remains to show that $v$ satisfies the radiation conditions (2.9). To this end, we choose a cut-off function $\varphi \in C^\infty(\mathbb{R}^n)$ such that

$$\varphi(x) = \begin{cases} 0 & |x| < R', \\ 1 & |x| > R' + 1, \end{cases}$$

where $R' > 0$ is large enough so that $\overline{B_R \cup \bigcup_j \operatorname{supp}(f_j)} \subset B_{R'}$. Note that $\bigcup_j \operatorname{supp}(f_j)$ is a bounded set. We now have that

$$\varphi v_j = \int_{\mathbb{R}^n} \Gamma(x - y, \omega) g_j(y)\,\mathrm{d}y,$$

where $g_j(y) := g_j(v_j, \nabla v_j, \nabla\varphi, \nabla^2\varphi)(y)$ is supported in $K := \{R' \leqslant |x| \leqslant R' + 1\}$ and $g_j \to g$ in $L^2(K)$. Therefore, letting $j \to \infty$, we get that

$$\varphi v = \int_{\mathbb{R}^n} \Gamma(x - y, \omega) g(y)\,\mathrm{d}y,$$

which implies that $v$ satisfies the radiation conditions.

To complete the proof, we need to prove the claim. Assume the claim is false, namely, there exist a subsequence $\{v_j\}$ such that $\lim_j \|v_j\|_{H^{1,-d}(\Omega)} = \infty$. Set $w_j = v_j/\|v_j\|_{H^{1,-d}(\Omega)}$ and thus $\|w_j\|_{H^{1,-d}(\Omega)} = 1$. Using the compactness theorem, we can find a subsequence of $\{w_j\}$, denoted by $\{w_j\}$ as usual, such that $w_j \to w$ in $L^{2,-d'}(\Omega)$ and $w \in L^{2,-d'}(\Omega)$ for $1/2 < d < d'$. By same arguments as above, we get that $w \in H^{1,-d'}(\Omega)$ and $w$ is a radiation solution with zero source term (since $f_j/\|v_j\|_{H^{1,-d}(\Omega)} \to 0$). By the uniqueness, $w \equiv 0$ and we have the contradiction. □

We are ready to show that the Dirichlet-to-Neumann map on $S_R$ can be constructed by the measurements $G_\Sigma(x, y, \omega)$ for all $x, y \in S_R$ and one fixed $\omega > 0$. Similar arguments are also used in [10] (or [8]). Define the Dirichlet-to-Neumann map $\Lambda_\Sigma : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ by

$$\Lambda_\Sigma(g) = T(D, \eta)v|_{S_R},$$

where $v$ is the solution of

$$\begin{cases} (\mathcal{L} + \omega^2)v = 0 \quad \text{in } B_R \setminus \overline{\Sigma}, \\ T(D, \nu)v = 0 \quad \text{on } \Sigma, \\ v = g \in H^{1/2}(S_R) \quad \text{on } S_R. \end{cases} \tag{5.9}$$

To make sure that $\Lambda_\Sigma$ is well defined, we assume that the boundary value problem (5.9) with $g = 0$ has only trivial solution. On the other hand, let $v^e$ be the solution of

$$\begin{cases} (L + \omega^2)v^e = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{B_R}, \\ v^e = g \quad \text{on } S_R, \\ v^e \text{ satisfies the radiation conditions,} \end{cases}$$

where $L$ denotes the operator $\mathcal{L}$ with homogeneous elastic tensor $C$ (see (2.3)). Define

$$\Lambda^e(g) = T(D, \eta)v^e|_{S_R}.$$

Now given $g \in H^{1/2}(S_R)$, we can define $M_g \in H^{-1}_{\mathrm{comp},\Sigma}(\Omega)$ by

$$\langle M_g, \phi\rangle = \langle g, \phi|_{S_R}\rangle \quad \forall \phi \in H^1_{\mathrm{comp}}(\Omega).$$

Let $v_g$ be the scattering solution of (4.1) with the source term $-M_g$. Define

$$\Psi g = v_g|_{S_R}.$$

Formally, $\Psi g$ is given by

$$\Psi g(x) = -\int_{\Omega} G_{\Sigma}(x, y, \omega) M_g(y)\,\mathrm{d}y = -\int_{S_R} G_{\Sigma}(x, y, \omega) g(y)\,\mathrm{d}S_R, \quad x \in S_R.$$

The following lemma plays a key role in constructing the Dirichlet-to-Neumann map from the near-field measurement.

**Lemma 5.3.**

(i) $\Lambda_{\Sigma} - \Lambda^e : H^{1/2}(S_R) \to H^{-1/2}(S_R)$ *is injective.*
(ii) $(\Lambda_{\Sigma} - \Lambda^e)\Psi = I$.

**Proof.** (i) This is an easy consequence of the uniqueness for the scattering solution. So we aim to prove (ii). From the definition of $v_g$, we obtain that

$$\begin{aligned}
\langle g, \phi|_{S_R}\rangle = \langle M_g, \phi\rangle &= \int_{\Omega} \big(C(x)\varepsilon(v_g)\cdot\varepsilon\big(\overline{\phi}\big) - \omega^2 v_g\overline{\phi}\big)\,\mathrm{d}x \\
&= \Bigg(\int_{B_R\setminus\overline{\Sigma}} + \int_{\mathbb{R}^n\setminus B_R}\Bigg)\big(C(x)\varepsilon(v_g)\cdot\varepsilon\big(\overline{\phi}\big) - \omega^2 v_g\overline{\phi}\big)\,\mathrm{d}x \\
&= \big\langle \Lambda_{\Sigma}(v_g|_{S_R}), \phi|_{S_R}\big\rangle - \big\langle \Lambda^e(v_g|_{S_R}), \phi|_{S_R}\big\rangle \\
&= \big\langle\big(\Lambda_{\Sigma} - \Lambda^e\big)\Psi g, \phi|_{S_R}\big\rangle.
\end{aligned} \tag{5.10}$$

In deriving (5.10), we have used the variational formulations of $\Lambda_{\Sigma}$ and $\Lambda^e$. □

It is clear that $\Psi$ is determined by $G_{\Sigma}(x, y, \omega)$ for all $x, y, \in S_R$. Therefore, in view of Lemma 5.3, we can construct $\Lambda_{\Sigma}$ by near-field measurements $G_{\Sigma}(x, y, \omega)$ for $x, y \in S_R$ using the formula

$$\Lambda_{\Sigma} = \Lambda^e - \Psi^{-1}. \tag{5.11}$$

### 5.2. Identifying the crack from the Dirichlet-to-Neumann map

Having the Dirichlet-to-Neumann map $\Lambda_\Sigma$, we now want to construct $\Sigma$ from it. For this part of inverse problem, we will use the same ideas as in [15].

To begin, let $r := \{r(t) \in \overline{B_R}\colon\ 0 \leqslant t \leqslant 1\}$ be a non-selfintersecting continuous curve joining $r(0), r(1) \in \Gamma$ with $r(t) \in B_R$ for $0 < t < 1$. This curve $r$ is called a needle. Define

$$T(r, \Sigma) := \sup\bigl\{t\colon\ 0 < t < 1,\ r(s) \notin \overline{\Sigma} \text{ for } 0 < s < t\bigr\}.$$

Physically, $T(r, \Sigma)$ can be interpreted as the first hitting time of the needle $r$ to $\Sigma$. It is clear that if $T(r, \Sigma) = 1$ then the needle $r$ does not touch the crack $\Sigma$. For any given needle $r$, we would like to find a characterization of $T(r, \Sigma)$. To do so, we define the indicator function $I(t, r)$ by

$$I(t, r) := \lim_{j\to\infty}\bigl\langle g_j, (\Lambda_\emptyset - \Lambda_\Sigma)g_j\bigr\rangle, \tag{5.12}$$

where $\Lambda_\emptyset$ is the Dirichlet-to-Neumann map in the absence of cracks. The Dirichlet data $g_j$ requires further explanations. Assume that $\omega^2$ is not a Dirichlet eigenvalue of $\mathcal{L}$ in $B_R$. Let $v_j \in H^1(B_R)$ $(j \in \mathbb{N})$ satisfy

$$\begin{cases} (\mathcal{L} + \omega^2)v_j = 0 & \text{in } B_R,\\ v_j \to G(\cdot, r(t))\ (j \to \infty) & \text{in } H^1_{\mathrm{loc}}(B_R \setminus r_t), \end{cases} \tag{5.13}$$

where $r_t\colon = \{r(s)\colon\ 0 < s \leqslant t\}$. Here the distribution $G(\cdot, x^0)$ in $x^0 \in B_R$ satisfies

$$\bigl(\mathcal{L} + \omega^2\bigr)G\bigl(\cdot, x^0\bigr) + \delta\bigl(x - x^0\bigr)b = 0$$

and

$$\bigl(G\bigl(\cdot, x^0\bigr) - E\bigl(\cdot, x^0\bigr)b\bigr)_{x^0\in B_R} \text{ is bounded} \quad \text{in } H^1(B_R),$$

where $0 \neq b \in \mathbb{C}$ and the distribution $E(x, x^0)$ in $x^0 \in \mathbb{R}^n$ satisfies

$$\bigl(\mathcal{L}_{C(x^0)} + \omega^2\bigr)E\bigl(x, x^0\bigr) + \delta\bigl(x - x^0\bigr)I = 0.$$

Note that $C(x^0)$ is a homogeneous elastic tensor with $C(x) = C(x^0)$ for all $x \in \mathbb{R}^n$. The existence of $v_j$ is guaranteed by the Runge approximation property which is an easy consequence of the unique continuation property. The existence of $G(\cdot, x^0)$ can be proved by the same method in [8] or [9]. Now the Dirichlet data $g_j$ is given by

$$g_j = v_j|_{S_R}.$$

To further study the indicator function $I(t,r)$, we would like to rewrite it in an integral form containing the so-called *reflected solution* defined as follows. Let $u_j \in H^1(B_R \setminus \overline{\Sigma})$ be the solution of

$$\begin{cases} (\mathcal{L}+\omega^2)u_j = 0 & \text{in } B_R \setminus \overline{\Sigma}, \\ T(D,\nu)u_j = 0 & \text{on } \Sigma, \\ u_j = g_j & \text{on } S_R \end{cases}$$

and $w_j = u_j - v_j \in H^1(B_R \setminus \overline{\Sigma})$, then we can show that

**Lemma 5.4** (Reflected solution). *If $r_t \cap \overline{\Sigma} = \emptyset$, then $w_j \to w$ in $H^1(B_R \setminus \overline{\Sigma})$ and $w \in H^1(B_R \setminus \overline{\Sigma})$ satisfies*

$$\begin{cases} (\mathcal{L}+\omega^2)w = 0 & \text{in } B_R \setminus \overline{\Sigma}, \\ T(D,\nu)w = -T(D,\nu)G(\cdot, r(t)) & \text{on } \Sigma, \\ w = 0 & \text{on } S_R. \end{cases} \tag{5.14}$$

Lemma 5.4 can be proved in the same way as in [15, Lemma 3.1]. With the reflected solution $w$, we can give another form of the indicator function $I(t,r)$.

**Lemma 5.5.** *Assume $r_t \cap \overline{\Sigma} = \emptyset$. Then we have*

$$I(t,r) = \int\limits_{B_R \setminus \overline{\Sigma}} \sigma(w) \cdot \varepsilon(\overline{w})\,\mathrm{d}x - \omega^2 \int\limits_{B_R \setminus \overline{\Sigma}} |w|^2\,\mathrm{d}x. \tag{5.15}$$

**Proof.** In view of Lemma 5.4 and the definition of $I(t,r)$, it suffices to show that

$$\langle g_j, (\Lambda_\emptyset - \Lambda_\Sigma)g_j \rangle = \int\limits_{B \setminus \overline{\Sigma}} \sigma(w_j) \cdot \varepsilon(\overline{w_j})\,\mathrm{d}x - \omega^2 \int\limits_{B_R \setminus \overline{\Sigma}} |w_j|^2\,\mathrm{d}x. \tag{5.16}$$

The derivation of (5.16) is based on Green's formula (A.1) in [15]. By means of Green's formula and boundary conditions, we have that

$$\begin{aligned} &\int\limits_{B \setminus \overline{\Sigma}} \sigma(w_j) \cdot \varepsilon(\overline{w_j})\,\mathrm{d}x \\ &\quad = \int\limits_{B_R \setminus \overline{\Sigma}} \sigma(u_j - v_j) \cdot \varepsilon(\overline{u_j} - \overline{v_j})\,\mathrm{d}x \\ &\quad = \int\limits_{B_R \setminus \overline{\Sigma}} \sigma(u_j) \cdot \varepsilon(\overline{u_j} - \overline{v_j})\,\mathrm{d}x - \int\limits_{B_R \setminus \overline{\Sigma}} \sigma(v_j) \cdot \varepsilon(\overline{u_j} - \overline{v_j})\,\mathrm{d}x \end{aligned}$$

$$= -\int_{B_R\setminus\overline{\Sigma}} \mathcal{L}u_j\cdot\big(\overline{u_j}-\overline{v_j}\big)+\big\langle T(D,\eta)u_j,(u_j-v_j)\big\rangle-\int_{B_R\setminus\overline{\Sigma}}\sigma(v_j)\cdot\varepsilon\big(\overline{u_j}-\overline{v_j}\big)\,\mathrm{d}x$$

$$=\omega^2\int_{B_R\setminus\overline{\Sigma}} u_j\cdot\big(\overline{u_j}-\overline{v_j}\big)\,\mathrm{d}x-\int_{B_R\setminus\overline{\Sigma}}\sigma(v_j)\cdot\varepsilon\big(\overline{u_j}\big)\,\mathrm{d}x+\int_{B_R\setminus\overline{\Sigma}}\sigma(v_j)\cdot\varepsilon\big(\overline{v_j}\big)\,\mathrm{d}x$$

$$=\omega^2\int_{B_R\setminus\overline{\Sigma}} u_j\cdot\big(\overline{u_j}-\overline{v_j}\big)\,\mathrm{d}x-\int_{B_R\setminus\overline{\Sigma}}\varepsilon(v_j)\cdot\sigma\big(\overline{u_j}\big)\,\mathrm{d}x+\int_{B_R\setminus\overline{\Sigma}}\varepsilon(v_j)\cdot\sigma\big(\overline{v_j}\big)\,\mathrm{d}x$$

$$=\omega^2\int_{B_R\setminus\overline{\Sigma}} u_j\cdot\big(\overline{u_j}-\overline{v_j}\big)\,\mathrm{d}x+\int_{B_R\setminus\overline{\Sigma}} v_j\cdot\overline{\mathcal{L}u_j}\,\mathrm{d}x-\big\langle v_j|_{S_R},T(D,\eta)u_j|_{S_R}\big\rangle$$

$$-\int_{B_R\setminus\overline{\Sigma}} v_j\cdot\overline{\mathcal{L}v_j}\,\mathrm{d}x+\big\langle v_j|_{S_R},T(D,\nu)v_j|_{S_R}\big\rangle$$

$$=\omega^2\int_{B_R\setminus\overline{\Sigma}} u_j\cdot\big(\overline{u_j}-\overline{v_j}\big)\,\mathrm{d}x-\omega^2\int_{B_R\setminus\overline{\Sigma}} v_j\cdot\big(\overline{u_j}-\overline{v_j}\big)\,\mathrm{d}x+\big\langle g_j,(\Lambda_\emptyset-\Lambda_\Sigma)g_j\big\rangle$$

$$=\omega^2\int_{B_R\setminus\overline{\Sigma}}|w_j|^2\,\mathrm{d}x+\big\langle g_j,(\Lambda_\emptyset-\Lambda_\Sigma)g_j\big\rangle$$

and (5.16) follows. □

With the help of the expression (5.15) of $I(t,r)$, we want to show that

**Theorem 5.6.** *If $r(T(r,\Sigma))\in\Sigma$, then $|I(t,r)|\to\infty$ as $t\to T(r,\Sigma)$.*

In view of Theorem 5.6, we can construct the crack $\Sigma$ by examining the behavior of $I(t,r)$. On the other hand, using (5.12) we can determine the indicator function $I(t,r)$ by the Dirichlet-to-Neumann map.

Theorem 5.6 can be proved in the same way as in [15] where the authors treated the static case ($\omega=0$). The heart of the method in [15] lies in analyzing the singularity of the reflected solution at the tip of the needle. Now let $w_0$ be the reflected solution corresponding to $\omega=0$, i.e.,

$$\begin{cases}\mathcal{L}w_0=0 \quad \text{in } B_R\setminus\overline{\Sigma},\\ T(D,\nu)w_0=-T(D,\nu)G_0(\cdot,r(t)) \quad \text{on } \Sigma,\\ w_0=0 \quad \text{on } S_R,\end{cases}\tag{5.17}$$

where the distribution $G_0(\cdot,x^0)$ satisfies

$$\mathcal{L}G_0\big(\cdot,x^0\big)+\delta\big(x-x^0\big)b=0$$

and

$$\left(G_0(\cdot, x^0) - E_0(\cdot, x^0)b\right)_{x^0 \in \Omega} \text{ is bounded} \quad \text{in } H^1(\Omega), \tag{5.18}$$

where the distribution $E_0(x, x^0)$ in $x^0 \in \mathbb{R}^n$ satisfies

$$\mathcal{L}_{C(x^0)} E_0(x, x^0) + \delta(x - x^0) I = 0. \tag{5.19}$$

Since $w$ and $w_0$ have the same singularity at the tip of the needle, we argue that in analyzing the blow-up behavior of $I(t, r)$ we can replace $w$ by $w_0$ in (5.15). Our aim now is to justify this claim rigorously.

To begin, we want to show that

**Lemma 5.7.** *There exists a constant $c > 0$ such that for $r_t \cap \overline{\Sigma} = \emptyset$ we have*

$$\int_{B_R \setminus \overline{\Sigma}} |w_0|^2 \, \mathrm{d}x < c. \tag{5.20}$$

This lemma indicates that the $L^2$ norm of $w_0$ over $B_R \setminus \overline{\Sigma}$ stays bounded as the tip of the needle approaching the crack.

It suffices to prove Lemma 5.7 as the tip of the needle is sufficiently close to the crack. To this end, let $x^0 = r(t) \in B_R \setminus \overline{\Sigma}$ and $a = x(T(r, \Sigma))$. Assume that $x^0$ is sufficiently close to $a$. In other words, let $B_\varepsilon(a)$ be an open ball of radius $0 < \varepsilon \ll 1$ centered at $a$, then $x^0 \in B_\varepsilon(a)$. In order to prove (5.20), we would like to know the behavior of $w_0$ in $B_\varepsilon(a)$. Now we assume that $\varepsilon$ is so small that $B_{2\varepsilon}(a) \cap \Sigma =: \Sigma_\varepsilon \subset \Sigma$. Without loss of generality, we suppose $x^0 \in \Omega_-$. Since $\Sigma \in C^2$, we can find a domain $\Omega_-^\varepsilon$ with $\partial\Omega_-^\varepsilon \in C^2$ such that $(B_{2\varepsilon}(a) \cap \Omega_-) \subset \Omega_-^\varepsilon \subset \Omega_-$ and $\Sigma_\varepsilon \subset \partial\Omega_-^\varepsilon$ (choosing $\varepsilon$ smaller if necessary). Let $\widetilde{w}_- \in H^1(\Omega_-^\varepsilon)$ be the solution of

$$\begin{cases} \mathcal{L}\widetilde{w}_- = 0 \quad \text{in } \Omega_-^\varepsilon, \\ T(D, \nu)\widetilde{w}_- = -T(D, \nu) G_0(\cdot, x^0) \quad \text{on } \Sigma_\varepsilon, \\ \widetilde{w}_- = 0 \quad \text{on } \partial\Omega_-^\varepsilon \setminus \overline{\Sigma}_\varepsilon. \end{cases} \tag{5.21}$$

We now claim that

**Lemma 5.8.** $\int_{\Omega_-^\varepsilon} |\widetilde{w}_-|^2 \, \mathrm{d}x < c < \infty$ *as $x^0 \to a$.*

**Proof.** We adopt arguments in [7] to our case here. Let $v$ be the solution of

$$\begin{cases} \mathcal{L}v = \widetilde{w}_- \quad \text{in } \Omega_-^\varepsilon, \\ T(D, \nu)v = 0 \quad \text{on } \Sigma_\varepsilon, \\ v = 0 \quad \text{on } \partial\Omega_-^\varepsilon \setminus \overline{\Sigma}_\varepsilon. \end{cases}$$

The standard elliptic regularity theorem implies that

$$\|v\|_{H^2(\Omega_-^\varepsilon)} \leqslant c\|\widetilde{w}_-\|_{L^2(\Omega_-^\varepsilon)}$$

for some constant $c > 0$. By the light of the Sobolev embedding theorem, we have that

$$\begin{aligned}
&\big|v(x) - v(y)\big| \leqslant c|x-y|^{1/2}\|\widetilde{w}_-\|_{L^2(\Omega_-^\varepsilon)}, \quad \forall x, y \in \Omega_-^\varepsilon \quad \text{and} \\
&\|v\|_{L^\infty(\Omega_-^\varepsilon)} \leqslant c\|\widetilde{w}_-\|_{L^2(\Omega_-^\varepsilon)}.
\end{aligned} \tag{5.22}$$

Using Green's formula, we obtain

$$\begin{aligned}
\int_{\Omega_-^\varepsilon} |\widetilde{w}_-|^2\,\mathrm{d}x &= \int_{\Omega_-^\varepsilon} \mathcal{L}v \cdot \overline{\widetilde{w}}_-\,\mathrm{d}x = \int_{\Omega_-^\varepsilon} \big(\mathcal{L}v \cdot \overline{\widetilde{w}}_- - v \cdot \mathcal{L}\overline{\widetilde{w}}_-\big)\,\mathrm{d}x \\
&= \int_{\Sigma_\varepsilon} v \cdot T(D,\nu)\overline{G}_0\big(x, x^0\big)\,\mathrm{d}S_x \\
&= \int_{\Sigma_\varepsilon} \big(v(x) - v\big(x^0\big)\big) \cdot T(D,\nu)\overline{G}_0\big(x, x^0\big)\,\mathrm{d}S_x \\
&\quad + v\big(x^0\big) \cdot \int_{\Sigma_\varepsilon} T(D,\nu)\overline{G}_0\big(x, x^0\big)\,\mathrm{d}S_x.
\end{aligned} \tag{5.23}$$

On the other hand, we note that

$$\mathcal{L}G_0\big(x, x^0\big) = 0$$

for $x \in B_{2\varepsilon}(a) \cap \Omega_+$, therefore,

$$v\big(x^0\big) \cdot \int_{\Sigma_\varepsilon} T(D,\nu)\overline{G}_0\big(x, x^0\big)\,\mathrm{d}S_x = -v\big(x^0\big) \cdot \int_{\partial B_{2\varepsilon}(a)\cap\Omega_+} T(D,\nu)\overline{G}_0\big(x, x^0\big)\,\mathrm{d}S_x. \tag{5.24}$$

Substituting (5.24) into (5.23) and taking advantage of (5.22), we have that

$$\begin{aligned}
\|\widetilde{w}_-\|^2_{L^2(\Omega_-^\varepsilon)} &= \int_{\Omega_-^\varepsilon} |\widetilde{w}_-|^2\,\mathrm{d}x \\
&\leqslant c\Bigg\{\int_{\Sigma_\varepsilon} \big|x - x^0\big|^{1/2}\big|T(D,\nu)G_0\big(x, x^0\big)\big|\,\mathrm{d}S_x \\
&\quad + \int_{\partial B_{2\varepsilon}(a)\cap\Omega_+} \big|T(D,\nu)\overline{G}_0\big(x, x^0\big)\big|\,\mathrm{d}S_x\Bigg\}\|\widetilde{w}_-\|_{L^2(\Omega_-^\varepsilon)}.
\end{aligned} \tag{5.25}$$

In view of (5.18) and the singularity of $E_0(x, x^0)$, we deduce that

$$\int\limits_{\Sigma_\varepsilon} \left|x - x^0\right|^{1/2} \left|T(D,\nu)G_0\left(x, x^0\right)\right| \mathrm{d}S_x + \int\limits_{\partial B_{2\varepsilon}(a)\cap\Omega_+} \left|T(D,\nu)\overline{G}_0\left(x, x^0\right)\right| \mathrm{d}S_x \leqslant c, \tag{5.26}$$

where $c$ is a uniform constant as $x^0 \to a$. This lemma now follows from (5.25) and (5.26). □

We are now ready to prove Lemma 5.7.

**Proof of Lemma 5.7.** Define

$$\widetilde{w}_0 := \begin{cases} G_0(x, x^0) & \text{for } x \in B_{2\varepsilon}(a) \cap \Omega_+, \\ \widetilde{w}_- & \text{for } x \in B_{2\varepsilon}(a) \cap \Omega_-. \end{cases}$$

Recall that $x^0 \in B_\varepsilon(a) \cap \Omega_-$. So we have

$$\mathcal{L}\widetilde{w}_0 = 0 \quad \text{in } B_{2\varepsilon}(a) \setminus \overline{\Sigma}_\varepsilon$$

and

$$\int\limits_{B_{2\varepsilon}(a)\setminus\overline{\Sigma}_\varepsilon} \left|\widetilde{w}_0\right|^2 \mathrm{d}x < c \quad \text{as } x^0 \to a. \tag{5.27}$$

Let $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function with $\operatorname{supp}(\psi) \subset B_{2\varepsilon}(a)$ and $\psi = 1$ on $B_\varepsilon(a)$. Then $w_0' := w_0 - \psi\widetilde{w}_0$ satisfies

$$\begin{cases} \mathcal{L}w_0' = -\operatorname{div}(C(x)(\widetilde{w}_0 \otimes \nabla\psi)) - (C(x)\nabla\widetilde{w}_0)\nabla\psi =: F & \text{in } B_R \setminus \overline{\Sigma}, \\ T(D,\nu)w_0' = -T(D,\nu)(G_0(x,x^0) - \psi G_0(x,x^0)) & \text{on } \Sigma, \\ w_0' = 0 & \text{on } B_R. \end{cases} \tag{5.28}$$

Note that $F = 0$ in $B_\varepsilon(a) \setminus \overline{\Sigma}$ and $T(D,\nu)(G(x,x^0) - \psi G_0(x,x^0)) = 0$ on $B_\varepsilon(a) \cap \Sigma$. It is readily seen that $\|F\|_{L^2(B_R\setminus\overline{\Sigma})}$ and $\|T(D,\nu)(G_0(x,x^0) - \psi G_0(x,x^0))\|_{\overline{H}^{-1/2}(\Sigma)}$ are uniformly bounded as $x^0 \to a$. Therefore, by the regularity theorem for (5.28), we have that

$$\left\|w_0'\right\|_{H^1(B_R\setminus\overline{\Sigma})} < c < \infty$$

uniformly as $x^0 \to a$, which immediately gives

$$\left\|w_0'\right\|_{L^2(B_R\setminus\overline{\Sigma})} < c \quad \text{as } x^0 \to a. \tag{5.29}$$

Now the estimate (5.20) is an easy consequence of (5.27) and (5.29). □

Now we are able to complete the proof of Theorem 5.6.

**Proof of Theorem 5.6.** Let $w'' = w - w_0$, then $w''$ satisfies

$$\begin{cases} (\mathcal{L} + \omega^2)w'' = -\omega^2 w_0 \quad \text{in } B_R \setminus \overline{\Sigma}, \\ T(D,\nu)w'' = -T(D,\nu)(G(\cdot, r(t)) - G_0(\cdot, r(t))) \quad \text{on } \Sigma, \\ w'' = 0 \quad \text{on } S_R. \end{cases} \tag{5.30}$$

It is not hard to check that $G(\cdot, r(t)) - G_0(\cdot, r(t)) \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, Lemma 5.7 implies that $\omega^2 w_0 \in L^2(B_R \setminus \overline{\Sigma})$ uniformly as $x^0 \to a$. Therefore, the elliptic estimate for (5.30) leads to

$$\|w - w_0\|_{H^1(B_R\setminus\overline{\Sigma})} = \|w''\|_{H^1(B_R\setminus\overline{\Sigma})} < c \quad \text{uniformly as } x^0 \to a \tag{5.31}$$

for some $c > 0$. Note that

$$\begin{aligned} \sigma(w)\cdot\varepsilon(\overline{w}) &= \sigma(w_0)\cdot\varepsilon(\overline{w}_0) + \sigma(w_0)\cdot\varepsilon(\overline{w - w_0}) + \sigma(w - w_0)\cdot\varepsilon(\overline{w}_0) \\ &\quad + \sigma(w - w_0)\cdot\varepsilon(\overline{w - w_0}) \\ &= C(x)\varepsilon(w_0)\cdot\varepsilon(\overline{w}_0) + \varepsilon(w_0)\cdot C(x)\varepsilon(\overline{w - w_0}) + C(x)\varepsilon(w - w_0)\cdot\varepsilon(\overline{w}_0) \\ &\quad + C(x)\varepsilon(w - w_0)\cdot\varepsilon(\overline{w - w_0}). \end{aligned} \tag{5.32}$$

By the strong convexity condition and the inequality (5.8) with $0 < \tilde{\varepsilon} < \tilde{\delta}/4$, we get from (5.32) that

$$\sigma(w)\cdot\varepsilon(\overline{w}) > \frac{\tilde{\delta}}{2}\left|\varepsilon(w_0)\right|^2 - \left(\frac{1}{2\tilde{\varepsilon}}\|C\|_{L^\infty(B_R)} + \tilde{\delta}\right)\left|\varepsilon(w - w_0)\right|^2. \tag{5.33}$$

In view of (5.20), (5.31) and (5.33), analyzing the blow-up behavior of $I(t, r)$ is equivalent to estimating

$$\int_{B_R\setminus\overline{\Sigma}} \left|\varepsilon(w_0)\right|^2 \mathrm{d}x$$

as $x^0 \to a$. It was shown in [15] that

$$\int_{B_R\setminus\overline{\Sigma}} \left|\varepsilon(w_0)\right|^2 \mathrm{d}x \to \infty \quad \text{when } x^0 \to a.$$

So Theorem 5.6 is proved. □

We now summarize the reconstruction algorithm of our method.

**Reconstruction Algorithm.**

**Step 1.** Compute the Dirichlet-to-Neumann $\Lambda_\Sigma$ map on $S_R$ from $G_\Sigma(x, y, \omega)$ for all $x, y \in S_R$ using the formula (5.11).

**Step 2.** Given a needle $r = \{r(t)\colon\ 0 \leqslant t \leqslant 1\}$ and construct $v_j$ satisfying (5.13).

**Step 3.** Compute $g_j = v_j|_{S_R}$ and evaluate the indicator function $I(t, r) := \lim_{j\to\infty}\langle g_j, (\Lambda_\emptyset - \Lambda_\Sigma)g_j\rangle$.

**Step 4.** Increase $t$ and search for $t$ where $|I(r, t)|$ becomes very large. Denote this $t$ by $t_a(r, \Sigma)$.

**Step 5.** Choose many needles $r$ and repeat all previous steps. Draw some surface $\Sigma_a$ which is close enough to the points $t_a(r, \Sigma)$ for these $r$. $\Sigma_a$ gives an approximation of $\Sigma$.

## References


[1] C. Alves, T. Ha-Duong, On inverse scattering by screens, Inverse Problems 13 (1997) 1161–1176.
[2] C. Alves, T. Ha-Duong, Inverse scattering for elastic plane cracks, Inverse Problems 15 (1999) 91–97.
[3] K. Bryan, M. Vogelius, A review of selected works on crack identification, in: Proceedings of the IMA Workshop on Geometric Methods in Inverse Problems, in press.
[4] Y. Choquet-Bruhat, D. Christodoulou, Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity, Acta Math. 146 (1981) 129–150.
[5] G.F.D. Duff, The Cauchy problem for elastic waves in an anisotropic medium, Philos. Trans. Roy. Soc. London Ser. A 252 (1960) 249–273.
[6] M. Giaquinta, Introduction to Regularity Theory for Nonlinear Elliptic Systems, Birkhäuser, Boston, 1993.
[7] M. Ikehata, Reconstruction of obstacle from boundary measurements, Wave Motion 30 (1999) 205–223.
[8] M. Ikehata, Reconstruction of inclusion from boundary measurements, J. Inverse Ill-posed Probl. Ser. 10 (2002) 37–65.
[9] M. Ikehata, G. Nakamura, Reconstruction of cavity from boundary measurements, Preprint.
[10] M. Ikehata, G. Makarakis, G. Nakamura, Inverse boundary value problem for ocean acoustics using point sources, Math. Methods Appl. Sci., in press.
[11] R. Kress, Inverse scattering from an open arc, Math. Methods Appl. Sci. 18 (1995) 267–293.
[12] R. Kress, Inverse elastic scattering from a crack, Inverse Problems 12 (1996) 667–684.
[13] P. Lax, R. Phillips, Scattering theory for the acoustic equation in an even number of space dimensions, Indiana Univ. Math. J. 22 (1972/73) 101–134.
[14] W. Littman, Decay at infinity of solutions to partial differential equations with constant coefficients, Trans. Amer. Math. Soc. 123 (1966) 449–459.
[15] G. Nakamura, G. Uhlmann, J.-N. Wang, Reconstruction of cracks in an inhomogeneous anisotropic elastic medium, J. Math. Pures Appl. 82 (2003) 1251–1276.
[16] G. Nakamura, G. Uhlmann, J.-N. Wang, Unique continuation property for elliptic systems and crack determination in anisotropic elasticity, Contemp. Math., in press.
[17] G. Nakamura, J.-N. Wang, Unique continuation for the two-dimensional anisotropic elasticity system and its applications to inverse problems, Preprint.
[18] G. Nakamura, J.-N. Wang, The limiting absorption principle for the two-dimensional inhomogeneous anisotropic elasticity system, Trans. Amer. Math. Soc., in press.
[19] D. Natroshvili, Two-dimensional steady-state oscillation problems of anisotropic elasticity, Georgian Math. J. 3 (1996) 239–262.
[20] D. Natroshvili, Boundary integral equation method in the steady state oscillation problems for anisotropic bodies, Math. Methods Appl. Sci. 20 (1997) 95–199.