Routing Functions – an Effective Approach to Deriving One-to-Many Disjoint Paths

by

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Abstract

This paper introduces a new concept called routing functions, which have a close relation to one-to-many disjoint paths in networks. By using a minimal routing function, a maximal number of disjoint paths whose maximal length is minimized in the worst case can be constructed in the hypercube and the folded hypercube. The end nodes of the paths constructed for the hypercube are not necessarily distinct. As a byproduct, the strong Rabin number of the hypercube and the Rabin number of the folded hypercube are computed. The latter provides a solution to an open problem raised by Liaw and Chang. A maximal number of disjoint paths whose total length is minimized can also be constructed in the hypercube by the use of a routing function.

Index Terms: Disjoint paths, folded hypercube, hypercube, optimization problem, Rabin number, strong Rabin number

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1 Introduction

A k-dimensional hypercube (abbreviated to a k-cube) consists of 2^k nodes that are labeled with 2^k binary numbers from 0 to 2^k-1. Two nodes are connected with a link if and only if their labels differ by exactly one bit. The diameter of a k-cube is k. On the other hand, a k-dimensional folded hypercube (abbreviated to a k-fcube) [12] is basically a k-cube augmented with 2^{k-1} complement links. Each *complement link* connects two nodes whose labels are complement to each other. Figure 1 shows the structure of a 3-fcube. There are four complement links, i.e., (000, 111), (001, 110), (010, 101), and (011, 100), in a 3-fcube. The both end nodes of each complement link are the *complement nodes* of each other. Due to the complement links, the diameter of a k-fcube is reduced to $\lceil k/2 \rceil$.

Routing is a process of transmitting messages among processors, and efficient routing is crucial to the performance of a multiprocessor system. In the past decade, routing with internally node-disjoint paths (disjoint paths for short) has received much attention because disjoint paths have the advantages of efficiency and fault tolerance. A set of disjoint paths between two nodes was named a *container* [16]. Containers in a variety of interconnection networks (networks for short) can be found in the literature [2, 3, 5-9, 11, 19, 21].

The *(node) connectivity* of a network is the minimum number of nodes whose removal can cause the network disconnected or trivial [1]. Suppose that W is a network with connectivity k and s, d_1, d_2, …, d_k are any k+1 distinct nodes of W. It was shown (see Theorem 2.6 in [1]) that there exist k disjoint paths from s to d_1, d_2, …, d_k, respectively, in W. The connectivities of a k-cube and a k-fcube are k and k+1, respectively. In [20], Rabin constructed k disjoint paths from s to d_1, d_2, …, d_k in a k-cube. The path from s to d_i has length $d_H(s, d_i)$ if $d_H(s, d_i)=k$, and $\leq d_H(s, d_i)+2$ if $d_H(s, d_i)<k$, where $1\leq i\leq k$ and $d_H(s, d_i)$ is the *Hamming distance* between s and d_i, i.e., the number of different bits between s and d_i. Disjoint paths from s to d_1, d_2, …, d_k in a (k+1)-dimensional star network can be found in [4], where the path from s to d_i has length $\leq d_H(s, d_i)+6$. The interested readers may consult [10, 17, 18] for more examples.

With the concern of minimal transmission delay, we wish to construct the disjoint paths from s to d_1, d_2, …, d_k so that the maximal length is minimized. This led to an optimization problem, named the Rabin number problem (see [16]). The *Rabin number* of W was defined to be the minimal l so that for any s, d_1, d_2, …, d_k, there exist k disjoint paths from s to d_1, d_2, …, d_k whose maximal length is at most l. The *Rabin number problem* on W is to compute the Rabin number of W. Computing Rabin number of a non-trivial network is really a hard problem. As shown in [17], it is NP-complete to decide that given a graph G, two integer m and l, and m+1 distinct nodes of G, whether there exist m disjoint paths from one node to the others whose maximal length is at most l. Recently, Liaw and Chang [18] defined the *strong Rabin number* all the same as the Rabin number, except that $\{d_1, d_2, \ldots, d_k\}$ is allowed to be a multiset. A *multiset* is a collection of elements in which multiple occurrences of the same element are allowed [15].

In this paper, we introduce a new concept of routing functions, which can be used to construct one-to-many disjoint paths in networks. In Section 2, we show that any minimal routing function can be used to construct k disjoint paths from s to $d_1, d_2, \ldots, d_k$ in a k-cube whose maximal length is minimized in the worst case, where $\{d_1, d_2, \ldots, d_k\}$ is a multiset. Then, in Section 3, we extend the result of Section 2 to construct k+1 disjoint paths in a k-fcube, where the end nodes of these paths are all distinct. The paths have lengths $\lceil k/2 \rceil+1$ at most, where $\lceil k/2 \rceil$ is the diameter of a k-fcube. Since the maximal length of the paths is minimized in the worst case, the construction is optimal. In Section 4, this paper concludes with some remarks. It was indicated that there exist routing functions which can be used to construct k disjoint paths with minimum total length in a k-cube. The strong Rabin number of a k-cube and the Rabin number of a k-fcube are obtained with the results of Sections 2 and 3, respectively. Throughout this paper, all disjoint paths mean internally node-disjoint paths. That is, all nodes of these paths are distinct, except their end nodes.

2 Disjoint Paths in the Hypercube

Suppose that $s, d_1, d_2, \ldots, d_m$ are arbitrary m+1 distinct nodes of a k-cube, where $m \leq k$. Since the hypercube is node symmetric, we assume $s=\overbrace{00...0}^{k}=0^k$ without loss of generality. Hence, $d_H(s, d_i)=|d_i|$, which is the number of 1's contained in d_i, where $1 \leq i \leq m$. There are two purposes in this section. One is to construct disjoint paths from s to $d_1, d_2, \ldots, d_m$ so that their maximal length is minimized. For this purpose, we use F: $\{d_1, d_2, \ldots, d_m\} \rightarrow \{n_1, n_2, \ldots, n_m\}$ to represent a routing function, where $n_1, n_2, \ldots, n_m$ denote m distinct dimensions of a k-cube ($1 \leq n_j \leq k$ for all $1 \leq j \leq m$). Let $e_b=0^{b-1}10^{k-b}$, where $1 \leq b \leq k$. The intuitive meaning of $F(d_i)=n_j$ is that the immediate successor of s in the path to d_i is the node e_{n_j}. Obviously F is one-to-one.

The other is to determine F so that $d_{i,F(d_i)}=1$ assures that the path to d_i is shortest, where $d_i=d_{i,1}d_{i,2}\ldots d_{i,k}$ is assumed. For this purpose, we would like to have $d_{i,F(d_i)} \geq d_{j,F(d_j)}$ when $|d_i|<|d_j|$, because otherwise ($|d_i|<|d_j|$ and $d_{i,F(d_i)}=0<1=d_{j,F(d_j)}$) it may happen that $d_{i,F(d_i)}=1$ and every disjoint path to d_i is not shortest. For example, assume $m=k=5$. When $(d_1, d_2, d_3, d_4, d_5)$=(00011, 00101, 00111, 00110, 11100) and $(F(d_1), F(d_2), F(d_3), F(d_4), F(d_5))$=(5, 2, 3, 4, 1), we have $|d_2|<|d_3|$ but

$d_{2,F(d_2)} = d_{2,2} = 0 < 1 = d_{3,3} = d_{3,F(d_3)}$. Every disjoint path from s to d_3 has length at least 5, although $d_{3,F(d_3)} = 1$.

In order to find such a F, we define $V_F=(v_1, v_2, \ldots, v_k)$, where each v_g ($1 \le g \le k$) is the number of d_i's with $|d_i|=g$ and $d_{i,F(d_i)} = 0$. For the example above, we have $V_F=(0, 1, 0, 0, 0)$. We say $(v_1, v_2, \ldots, v_k)<(v'_1, v'_2, \ldots, v'_k)$ if $v_1<v'_1$ or $v_1=v'_1$, $v_2=v'_2$, …, $v_{l-1}=v'_{l-1}$, and $v_l<v'_l$ for some $2 \le l \le k$, and $(v_1, v_2, \ldots, v_k) \le (v'_1, v'_2, \ldots, v'_k)$ if either $(v_1, v_2, \ldots, v_k)<(v'_1, v'_2, \ldots, v'_k)$ or $(v_1, v_2, \ldots, v_k)=(v'_1, v'_2, \ldots, v'_k)$. F is said to be *minimal* if $V_F \le V_{F'}$ for every routing function F': $\{d_1, d_2, \ldots, d_m\} \to \{n_1, n_2, \ldots, n_m\}$. We would like to have a minimal F, because intuitively F' has higher probability than F'' to serve the second purpose if $V_{F'}<V_{F''}$. Actually, a minimal F will serve the second purpose, which is shown in this section.

2.1 Two procedures to construct disjoint paths using a minimal F

Conveniently, a k-cube can be represented with $\overbrace{**\ldots*}^{k} = *^k$, where $* \in \{0, 1\}$. Thus, $*^{k-1}1$ and $*^{k-1}0$, which contain the nodes of a k-cube whose rightmost bits are 1 and 0, respectively, represent two disjoint $(k-1)$-cubes that are contained in the k-cube. For each node $x=x_1x_2...x_k$ of a k-cube, we define $x^{(k)}=x_1x_2...x_{k-1}(1-x_k)$. That is, x differs from $x^{(k)}$ only in bit x_k.

In order to obtain disjoint paths from s to $d_1, d_2, \ldots, d_m$, we first construct disjoint paths that connect $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$, and then augment these paths with links (s, e_{n_1}), (s, e_{n_2}), …, (s, e_{n_m}). The following is a recursive procedure, named *Paths0*, with inputs F, m, k, D, and I, where F is a minimal routing function from $D=\{d_1, d_2, \ldots, d_m\}$ to $I=\{n_1, n_2, \ldots, n_m\}$. It will be shown in Section 2.2 that *Paths0*(F, m, k, D, I) can produce m disjoint paths, denoted by $P_1, P_2, \ldots, P_m$, in a k-cube that connect $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$, where P_i is the path to d_i for all $1 \le i \le m$. When $d_{i,F(d_i)} = 1$, P_i has minimal length $|d_i|-1$. When $d_{i,F(d_i)} = 0$, P_i has length $|d_i|+1$. Besides, if t_i is the immediate predecessor of d_i in P_i, then $|t_i|=|d_i|+1$ (i.e., the subpath to t_i is shortest) when $d_{i,F(d_i)} = 0$.

Procedure *Paths0*(F, m, k, D, I).

/* We use $*^k$ to represent the k-cube where $P_1, P_2, \ldots, P_m$ are located. */

Step 1. If $k=2$, then return $P_1, P_2, \ldots, P_m$ which are m disjoint paths in a 2-cube that connect $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$.

Step 2. Determine d_c so that $|d_c|\le|d_i|$ for all $1\le i\le m$, where $1\le c\le m$. If there are multiple candidates for d_c, then select arbitrary one with $d_{c,F(d_c)}=1$, or any if all have $d_{c,F(d_c)}=0$. Without loss of generality, we assume $c=1$, $n_c=n_1=k$, and $F(d_i)=n_i$ for all $1\le i\le m$.

Step 3. Partition D into D' and D'', where $D'=\{d_j \mid d_{j,F(d_c)}=d_{j,F(d_1)}=d_{j,k}=0$ and $1\le j\le m\}$ and $D''=\{d_j \mid d_{j,F(d_c)}=d_{j,F(d_1)}=d_{j,k}=1$ and $1\le j\le m\}$.

Step 4. Construct $P_1, P_2, \ldots, P_m$ according to the following four cases.

Case 1. $d_{1,k}=0$.

/* Without loss of generality, suppose $D'=\{d_1, d_2, \ldots, d_r\}$ and $D''=\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y': $\{d_2, d_3, \ldots, d_r\}\rightarrow\{n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$. Let $1\le u<k$ with $d_{1,u}=1$, and define Y'': $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}\rightarrow\{u, n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_1^{(k)})=u$ and $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$. */

Construct $P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths0*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$). /* $P_2, P_3, \ldots, P_r$ connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. */

Construct $P'_1, P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths0*(Y'', $m-r+1$, $k-1$, $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{u, n_{r+1}, n_{r+2}, \ldots, n_m\}$), where P'_1 is the path to $d_1^{(k)}$. /* $P'_1, P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e_u^{(k)}, e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}$. */

Construct P_1 by augmenting P'_1 with the link $(d_1^{(k)}, d_1)$.

Augment $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ with links $(e_k, e_u^{(k)})$, $(e_{n_{r+1}}, e_{n_{r+1}}^{(k)})$, $(e_{n_{r+2}}, e_{n_{r+2}}^{(k)})$, …, $(e_{n_m}, e_{n_m}^{(k)})$.

Case 2. $d_{1,k}=1$ and $|d_1|=1$.

/* Suppose $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y': $\{d_2, d_3, \ldots, d_r\}\rightarrow\{n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$, and Y'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\}\rightarrow\{n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$. */

Construct $P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths0*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$). /* $P_2, P_3, \ldots, P_r$ connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. */

Construct $P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths0*(Y'', $m-r$, $k-1$, $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{n_{r+1}, n_{r+2}, \ldots, n_m\}$). /* $P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$. */

Construct P_1 as (e_k, d_1). /* P_1 has length 0. */

Augment $P_{r+1}, P_{r+2}, \ldots, P_m$ with links $(e_{n_{r+1}}, e^{(k)}_{n_{r+1}})$, $(e_{n_{r+2}}, e^{(k)}_{n_{r+2}})$, …, $(e_{n_m}, e^{(k)}_{n_m})$.

Case 3. $d_{1,k}=1$, $|d_1|>1$, and $d_{1,a}=1$ for some $a \in \{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$.

/* Suppose $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y': $\{d_2, d_3, \ldots, d_r\} \to \{n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$. Define Y'': $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{a, n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_1)=a$ and $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$. */

Construct $P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths0*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$). /* $P_2, P_3, \ldots, P_r$ connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. */

Construct $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths0*(Y'', $m-r+1$, $k-1$, $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{a, n_{r+1}, n_{r+2}, \ldots, n_m\}$). /* $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e^{(k)}_a, e^{(k)}_{n_{r+1}}, e^{(k)}_{n_{r+2}}, \ldots, e^{(k)}_{n_m}\}$ and $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$. */

Augment $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ with links $(e_k, e^{(k)}_a)$, $(e_{n_{r+1}}, e^{(k)}_{n_{r+1}})$, $(e_{n_{r+2}}, e^{(k)}_{n_{r+2}})$, …, $(e_{n_m}, e^{(k)}_{n_m})$.

Case 4. $d_{1,k}=1$, $|d_1|>1$, and $d_{1,a}=0$ for all $a \in \{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$.

/* Suppose $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, where $1\le r\le m$. Define Y'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $Y''(d_i)=F(d_i)=n_i$ for all $r+1\le i\le m$. */

Construct $P_{r+1}, P_{r+2}, \ldots, P_m$ in $*^{k-1}1$ by executing *Paths0*(Y'', $m-r$, $k-1$, $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{n_{r+1}, n_{r+2}, \ldots, n_m\}$). /* $P_{r+1}, P_{r+2}, \ldots, P_m$ connect $\{e^{(k)}_{n_{r+1}}, e^{(k)}_{n_{r+2}}, \ldots, e^{(k)}_{n_m}\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$. */

/* Define W'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $W''(d_i)=n_j$ if P_i begins at $e^{(k)}_{n_j}$ for all $r+1\le i\le m$, where $r+1\le j\le m$. */

If $d_1 \notin P_i$ for all $r+1\le i\le m$, then {

> /* Suppose $1\le u<k$ with $d_{1,u}=1$. Define Y': $\{d^{(k)}_1, d_2, d_3, \ldots, d_r\} \to \{u, n_2, n_3, \ldots, n_r\}$ as follows: $Y'(d^{(k)}_1)=u$ and $Y'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$. */
>
> Construct $P'_1, P_2, P_3, \ldots, P_r$ in $*^{k-1}0$ by executing *Paths0*(Y', r, $k-1$, $\{d^{(k)}_1, d_2, d_3, \ldots, d_r\}$, $\{u, n_2, n_3, \ldots, n_r\}$), where P'_1 is the path to $d^{(k)}_1$. /* $P'_1, P_2, P_3, \ldots, P_r$ connect $\{e_u, e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d^{(k)}_1, d_2, d_3, \ldots, d_r\}$. */
>
> Construct P_1 by augmenting P'_1 with the link $(d^{(k)}_1, d_1)$.

/* Suppose $u=n_h$ for some $r+1\le h\le m$. */

Augment P_{r+1}, P_{r+2}, …, P_m with links $(e_{n_{r+1}}, e_{n_{r+1}}^{(k)})$, $(e_{n_{r+2}}, e_{n_{r+2}}^{(k)})$, …, $(e_{n_{h-1}}, e_{n_{h-1}}^{(k)})$, $(e_k, e_u^{(k)})$, $(e_{n_{h+1}}, e_{n_{h+1}}^{(k)})$, …, $(e_{n_m}, e_{n_m}^{(k)})$. }

If $d_1 \in P_l$ for some $r+1\le l\le m$ and $d_l^{(k)} \notin \{d_2, d_3, \dots, d_r\}$, then {

/* Define Q': $\{d_l^{(k)}, d_2, d_3, \dots, d_r\} \to \{W''(d_l), n_2, n_3, \dots, n_r\}$ as follows: $Q'(d_l^{(k)})=W''(d_l)$ and $Q'(d_i)=F(d_i)=n_i$ for all $2\le i\le r$. */

Construct P'_l, P_2, P_3, …, P_r in $*^{k-1}0$ by executing *Paths0*$(Q', r, k-1, \{d_l^{(k)}, d_2, d_3, \dots, d_r\}, \{W''(d_l), n_2, n_3, \dots, n_r\})$, where P'_l is the path to $d_l^{(k)}$. /* P'_l, P_2, P_3, …, P_r connect $\{e_{W''(d_l)}, e_{n_2}, e_{n_3}, \dots, e_{n_r}\}$ and $\{d_l^{(k)}, d_2, d_3, \dots, d_r\}$. */

Construct P_1 as the subpath of P_l that goes from $e_{W''(d_l)}^{(k)}$ to d_1.

Reconstruct P_l as the path obtained by augmenting P'_l with the link $(d_l^{(k)}, d_l)$.

Augment P_1, P_{r+1}, P_{r+2}, …, P_{l-1}, P_{l+1}, …, P_m with links $(e_{n_{r+1}}, e_{n_{r+1}}^{(k)})$, $(e_{n_{r+2}}, e_{n_{r+2}}^{(k)})$, …, $(e_{W''(d_l)-1}, e_{W''(d_l)-1}^{(k)})$, $(e_k, e_{W''(d_l)}^{(k)})$, $(e_{W''(d_l)+1}, e_{W''(d_l)+1}^{(k)})$, …, $(e_{n_m}, e_{n_m}^{(k)})$. }

If $d_1 \in P_l$ for some $r+1\le l\le m$ and $d_l^{(k)} \in \{d_2, d_3, \dots, d_r\}$, then {

/* Suppose $d_l^{(k)}=d_h$ for some $2\le h\le r$. Define Y: $\{d_1, d_2, \dots, d_m\} \to \{n_1, n_2, \dots, n_m\}$ as follows: $Y(d_l)=F(d_h)$, $Y(d_h)=W''(d_l)$, $Y(d_1)=F(d_1)=k$, $Y(d_i)=F(d_i)$ for all $2\le i\le r$ and $i\ne h$, and $Y(d_j)=W''(d_j)$ for all $r+1\le j\le m$ and $j\ne l$. */

Construct P_1, P_2, …, P_m all the same as Case 3. /* Substitute Y for the F in Case 3. */ }

Step 5. Return P_1, P_2, …, P_m.

The following procedure, named *Paths1*, can produce m disjoint paths from s to d_1, d_2, …, d_m, respectively, in a k-cube.

Procedure *Paths1*(F, m, k, D, I).

/* F is a minimal routing function from $D=\{d_1, d_2, \dots, d_m\}$ to $I=\{n_1, n_2, \dots, n_m\}$. */

Step 1. Obtain P_1, P_2, …, P_m in a k-cube by executing *Paths0*(F, m, k, D, I).

Step 2. Augment P_1, P_2, …, P_m with links (s, e_{n_1}), (s, e_{n_2}), …, (s, e_{n_m}).

Step 3. Return $P_1, P_2, \ldots, P_m$.

2.2 Correctness of *Paths0*(F, m, k, D, I)

We show that *Paths0*(F, m, k, D, I) can produce m disjoint paths $P_1, P_2, \ldots, P_m$ that connect $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$. Besides, when $d_{i,F(d_i)}=1$, P_i has minimal length $|d_i|-1$, and when $d_{i,F(d_i)}=0$, P_i has length $|d_i|+1$ and $|t_i|=|d_i|+1$, where t_i is the immediate predecessor of d_i in P_i.

2.2.1 Properties

First some fundamental properties of F are introduced. Suppose that F': $D'\rightarrow I'$ and F'': $D''\rightarrow I''$ are two routing functions, where $\{D', D''\}$ is a partition of $D=\{d_1, d_2, \ldots, d_m\}$ and $\{I', I''\}$ is a partition of $I=\{n_1, n_2, \ldots, n_m\}$. Then, F: $D\rightarrow I$ is said to be the *union* of F' and F'', denoted by $F=F'\cup F''$, if $F(d_i)=F'(d_i)$ for all $d_i\in D'$ and $F(d_j)=F''(d_j)$ for all $d_j\in D''$. Clearly we have $V_F=V_{F'}+V_{F''}$, i.e., $V_F=(v'_1+v''_1, v'_2+v''_2, \ldots, v'_k+v''_k)$ if $V_{F'}=(v'_1, v'_2, \ldots, v'_k)$ and $V_{F''}=(v''_1, v''_2, \ldots, v''_k)$. The following two lemmas are immediate.

Lemma 1. Suppose $F=F'\cup F''$. If $d_{i,F'(d_i)}=1$ for every $d_i\in D'$, then $V_F=V_{F''}$.

Lemma 2. Suppose that $F=F'\cup F''$ is minimal. Then F' and F'' are minimal.

Lemma 3. Suppose $d_{i,F(d_i)}=0$, $d_{j,F(d_j)}=0$, and $d_{i,F(d_j)}=1$, where $1\le i\le m$, $1\le j\le m$, and $i\ne j$. Then F is not minimal.

Proof. We define Y: $D\rightarrow I$ as follows: $Y(d_i)=F(d_j)$, $Y(d_j)=F(d_i)$, and $Y(d_r)=F(d_r)$ for all $r\in\{1, 2, \ldots, m\}-\{i, j\}$. Suppose $V_F=(v_1, v_2, \ldots, v_k)$ and $V_Y=(v'_1, v'_2, \ldots, v'_k)$. If $|d_i|\ne|d_j|$, then $v'_{|d_i|}=v_{|d_i|}-1$, $v'_{|d_j|}\le v_{|d_j|}$, and $v'_l=v_l$ for all $l\in\{1, 2, \ldots, k\}-\{|d_i|, |d_j|\}$. If $|d_i|=|d_j|$, then $v'_{|d_i|}\le v_{|d_i|}-1$ and $v'_l=v_l$ for all $1\le l\le k$ and $l\ne|d_i|$. Hence $V_Y<V_F$. □

Lemma 4. Suppose $d_{j,F(d_j)}=0$, $d_{j,F(d_i)}=1$, and $d_{i,F(d_j)}=1$, where $1\le i\le m$, $1\le j\le m$, and $i\ne j$. Then F is not minimal.

Proof. We define Y, V_Y, and V_F all the same as the above. If $d_{i,F(d_i)}=0$, then F is not minimal by Lemma 3. If $d_{i,F(d_i)}=1$, then $v'_{|d_j|}=v_{|d_j|}-1$ and $v'_l=v_l$ for all $1\le l\le k$ and $l\ne|d_j|$, which implies $V_Y<V_F$. □

Lemma 5. Suppose $d_{i,F(d_i)}=0$, $d_{i,F(d_j)}=1$, and $|d_i|<|d_j|$, where $1\le i\le m$, $1\le j\le m$, and $i\ne j$. Then F is not minimal.

Proof. We define Y, V_Y, and V_F all the same as the above. Then $v'_{|d_i|} = v_{|d_i|} - 1$ and $v'_l = v_l$ for all $1 \le l < |d_i|$. Hence $V_Y < V_F$. □

Lemma 6. If F is minimal and $d_{i,F(d_i)} = 0$, then $d_{i,1}d_{i,2}...d_{i,F(d_i)-1}1d_{i,F(d_i)+1}...d_{i,k} \notin \{d_1, d_2, \ldots, d_m\}$ ($d_{i,1}d_{i,2}...d_{i,F(d_i)-1}1d_{i,F(d_i)+1}...d_{i,k}$ is the node obtained by changing the bit $d_{i,F(d_i)}$ of d_i to 1).

Proof. Suppose conversely $d_{i,1}d_{i,2}...d_{i,F(d_i)-1}1d_{i,F(d_i)+1}...d_{i,k} = d_r$ for some $1 \le r \le m$ and $r \ne i$. That is, $d_{r,F(d_i)} = 1$ and $d_{r,j} = d_{i,j}$ for all $1 \le j \le k$ and $j \ne F(d_i)$. Then $F(d_r) \in \{n_1, n_2, \ldots, n_m\}$. If $d_{r,F(d_r)} = 0$, then F is not minimal by Lemma 3, which is a contradiction. If $d_{r,F(d_r)} = d_{i,F(d_r)} = 1$, then F is not minimal by Lemma 4, which is again a contradiction. □

The following lemma will be proved in Section 2.2.2.

Lemma 7. Suppose F: $D = \{d_1, d_2, \ldots, d_m\} \to I = \{n_1, n_2, \ldots, n_m\}$ is a minimal routing function. *Paths0*(F, m, k, D, I) can produce m disjoint paths $P_1, P_2, \ldots, P_m$ in a k-cube that connect $\{e_{n_1}, e_{n_2}, \ldots, e_{n_m}\}$ and $\{d_1, d_2, \ldots, d_m\}$, where $k \ge 2$, $m \le k$, and P_i is the path to d_i for all $1 \le i \le m$. When $d_{i,F(d_i)} = 1$, P_i has length $|d_i|-1$. When $d_{i,F(d_i)} = 0$, P_i has length $|d_i|+1$ and $|t_i| = |d_i|+1$, where t_i is the immediate predecessor of d_i in P_i.

2.2.2 The proof of Lemma 7

Lemma 7 can be proved by induction on k. It is not difficult to check that Lemma 7 holds for a 2-cube (i.e., k=2). Suppose that Lemma 7 holds for a (k–1)-cube, where k>2 is assumed. The situation for a k-cube is discussed below. We need to show that $P_1, P_2, \ldots, P_m$ meet our requirements.

Define two routing functions F': $D' \to I'$ and F'': $D'' \to I''$ so that $F = F' \cup F''$. That is, $F'(d_i) = F(d_i)$ for all $d_i \in D'$ and $F''(d_j) = F(d_j)$ for all $d_j \in D''$. Clearly $I' \cup I'' = I$. By Lemma 2 F' and F'' are minimal. $P_1, P_2, \ldots, P_m$ were obtained according to the following four cases.

2.2.2.1 Case 1: $d_{1,k}$=0

We have assumed $D' = \{d_1, d_2, \ldots, d_r\}$ and $D'' = \{d_{r+1}, d_{r+2}, \ldots, d_m\}$ (hence, $I' = \{n_1, n_2, \ldots, n_r\}$ and $I'' = \{n_{r+1}, n_{r+2}, \ldots, n_m\}$). Since $d_1 \in D'$, we have $|D''| \le k-1$. Actually we have $|D''| \le k-2$ for the following reason. If $|D''| = k-1$, then $m = k$, $\{n_1, n_2, \ldots, n_m\} = \{1, 2, \ldots, k\}$, and $D'' = \{d_2, d_3, \ldots, d_k\}$. Since $|d_1| \ge 1$ and $d_{1,k}$=0, we have $d_{1,v}$=1 for some $1 \le v \le k-1$. Suppose $v = F(d_l)$, where $l \ne 1$. If $|d_1| = |d_l|$, then $d_{l,F(d_l)} = 0$ as a consequence of Step 2. By Lemma 3 ($d_{1,F(d_1)} = d_{1,k} = 0$, $d_{l,F(d_l)} = 0$, and $d_{1,F(d_l)} = d_{1,v} = 1$), F is not minimal, which is a contradiction. If $|d_1|<|d_l|$, then F is not minimal by Lemma 5 ($d_{1,F(d_1)} = d_{1,k} = 0$, $d_{1,F(d_l)} = d_{1,v} = 1$, and $|d_1|<|d_l|$), which is again a contradiction.

Since Y' is a minimal routing function by Lemma 2, by induction hypothesis $P_2, P_3, \ldots, P_r$ that were produced by executing $Paths0(Y', r-1, k-1, \{d_2, d_3, \ldots, d_r\}, \{n_2, n_3, \ldots, n_r\})$ are $r-1=|D'|-1$ disjoint paths in $*^{k-1}0$ that connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. For all $2\le i\le r$, P_i has length $|d_i|-1$ if $d_{i,Y'(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $|d_i|+1$ and $|t_i|=|d_i|+1$ if $d_{i,Y'(d_i)} = d_{i,F(d_i)} = 0$.

Similarly, $P'_1, P_{r+1}, P_{r+2}, \ldots, P_m$ that were produced by executing $Paths0(Y'', m-r+1, k-1, \{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}, \{u, n_{r+1}, n_{r+2}, \ldots, n_m\})$ are $m-r+1=|D''|+1\le k-1$ disjoint paths in $*^{k-1}1$ that connect $\{e_u^{(k)}, e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}$, provided $e_k \notin \{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}$ and Y'' is a minimal routing function. Recall that we have assumed $s \notin \{d_1, d_2, \ldots, d_m\}$. Here e_k in $*^{k-1}1$ corresponds to s in $*^k$.

We have $e_k \notin \{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\}$, as explained below. Obviously $e_k\ne d_1^{(k)}$. If $e_k=d_l$ for some $r+1\le l\le m$, then $d_{l,F(d_l)} = 0$ as a consequence of Step 2. By Lemma 3 ($d_{l,F(d_l)} = 0$, $d_{1,F(d_1)} = d_{1,k} = 0$, and $d_{l,F(d_1)} = d_{l,k} = 1$), F is not minimal, which is a contradiction.

Y'' is a minimal routing function, as explained below. First, Y'' is a routing function if $d_1^{(k)} \notin \{d_{r+1}, d_{r+2}, \ldots, d_m\}$ and $u \notin \{n_{r+1}, n_{r+2}, \ldots, n_m\}$. The former holds as a consequence of Lemma 6 (F is minimal and $d_{1,F(d_1)} = d_{1,k} = 0$). The latter holds for the following reason. If $u \in \{n_{r+1}, n_{r+2}, \ldots, n_m\}$, then $u=F(d_l)$ for some $r+1\le l\le m$ and hence F is not minimal by Lemma 4 ($d_{1,F(d_1)} = d_{1,k} = 0$, $d_{1,F(d_l)} = d_{1,u} = 1$, and $d_{l,F(d_1)} = d_{l,k} = 1$), which is a contradiction.

Further, Y'' is minimal for the following reason. Suppose conversely that Y'' is not minimal. Then there exists a routing function Y: $\{d_1^{(k)}, d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{u, n_{r+1}, n_{r+2}, \ldots, n_m\}$ so that $V_Y<V_{Y''}$. Since $d^{(k)}_{1,Y''(d_1^{(k)})} = d_{1,u}^{(k)}=d_{1,u}=1$, we have $V_{Y''}=V_{F''}$ by Lemma 1 ($D'=\{d_1^{(k)}\}$). If $u=Y(d_1^{(k)})$, then define a routing function P'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $P''(d_j)=Y(d_j)$ for all $r+1\le j\le m$. Since $d^{(k)}_{1,Y(d_1^{(k)})} = d_{1,u}^{(k)}=d_{1,u}=1$, we have $V_{P''}=V_Y$ by Lemma 1 ($D'=\{d_1^{(k)}\}$). Hence $V_{P''}<V_{F''}$, which is a contradiction. On the other hand, if $u=Y(d_l)$ for some $r+1\le l\le m$, then define a routing function Q': $\{d_1, d_2, \ldots, d_r\} \to \{Y(d_1^{(k)}), n_2, n_3, \ldots, n_r\}$ as follows: $Q'(d_1)=Y(d_1^{(k)})$ and $Q'(d_j)=F'(d_j)$ for all $2\le j\le r$. Since $d_{1,F'(d_1)} = d_{1,k} = 0$, we have $V_{Q'}\le V_{F'}$. We define another routing function Q'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{k,$

n_{r+1}, n_{r+2}, …, n_m}−{ $Y(d_1^{(k)})$} as follows: $Q''(d_l)=k$ and $Q''(d_j)=Y(d_j)$ for all $r+1 \le j \le m$ and $j \ne l$. Since $d_{l,Q''(d_l)} = d_{l,k} = 1$, we have $V_{Q''} \le V_Y$. Hence, $V_{Q'}+V_{Q''} \le V_{F'}+V_Y < V_{F'}+V_{Y''} = V_{F'}+V_{F''} = V_F$, which is a contradiction because F is minimal and $Q' \cup Q''$ is a routing function from D to I.

Consequently, for all $r+1 \le i \le m$, P_i has length $(|d_i|-1)-1=|d_i|-2$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $(|d_i|-1)+1=|d_i|$ and $|t_i|=|d_i|+1$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 0$. P'_1 has length $(|d_1^{(k)}|-1)-1=|d_1|-1$ because $d^{(k)}_{1,Y''(d_1^{(k)})} = d^{(k)}_{1,u} = 1$. P_1 has length $(|d_1|-1)+1=|d_1|$, and $|t_1|=|d_1^{(k)}|=|d_1|+1$. Besides, if d_1 is not included in P_2, P_3, …, P_r, then P_1, P_2, …, P_m are disjoint. In the following we show that d_1 is not included in P_2, P_3, …, P_r.

Suppose conversely that d_1 is contained in P_h, where $2 \le h \le r$. We define a routing function W': {d_2, d_3, …, d_r} $\rightarrow$ {n_2, n_3, …, n_r} as follows: $W'(d_i)=n_j$ if P_i begins at e_{n_j}, where $2 \le i \le r$ and $2 \le j \le r$. That is, P_i goes from $e_{W'(d_i)}$ to d_i. We have $d_{1,W'(d_h)} = 1$ for the following reason. If $d_{h,Y'(d_h)} = 1$, then P_h has length $|d_h|-1$, which implies $d_{1,W'(d_h)} = 1$. If $d_{h,Y'(d_h)} = 0$, then P_h has length $|d_h|+1$ and $|t_h|=|d_h|+1$. It is meant that the subpath from $e_{W'(d_h)}$ to t_h has length $|d_h|=|t_h|-1$. Since $d_1 \ne d_h$, we have $d_{1,W'(d_h)} = 1$.

For all $2 \le i \le r$, we have $d_{i,W'(d_i)} = d_{i,Y'(d_i)}$ for the following reason. If $d_{i,Y'(d_i)} = 1$, then $d_{i,W'(d_i)} = 1$ because P_i has length $|d_i|-1$. Consequently, if $d_{l,Y'(d_l)} = 0$ and $d_{l,W'(d_l)} = 1$ for some $2 \le l \le r$, then $V_{W'} < V_{Y'}$, which is a contradiction because Y' is minimal. Hence, $d_{i,Y'(d_i)} = 0$ implies $d_{i,W'(d_i)} = 0$.

We define another routing function P': {d_1, d_2, …, d_r} $\rightarrow$ {n_1, n_2, …, n_r} as follows: $P'(d_1)=F'(d_1)=n_1=k$ and $P'(d_i)=W'(d_i)$ for all $2 \le i \le r$. Since $d_{i,W'(d_i)} = d_{i,Y'(d_i)}$ and $F'(d_i)=Y'(d_i)$ for all $2 \le i \le r$, we have $d_{i,P'(d_i)} = d_{i,F'(d_i)}$ for all $1 \le i \le r$, which implies $V_{P'}=V_{F'}$. Hence P' is also minimal. If $|d_1|<|d_h|$, then P' is not minimal by Lemma 5 ($d_{1,P'(d_1)} = d_{1,k} = 0$, $d_{1,P'(d_h)} = d_{1,W'(d_h)} = 1$, and $|d_1|<|d_h|$), which is a contradiction. If $|d_1|=|d_h|$, then P_h has length greater than $|d_h|-1$. Hence P_h has length $|d_h|+1$, which implies $d_{h,Y'(d_h)} = 0$. By Lemma 3 ($d_{1,P'(d_1)} = d_{1,k} = 0$, $d_{h,P'(d_h)} = d_{h,W'(d_h)} = d_{h,Y'(d_h)} = 0$, and $d_{1,P'(d_h)} = d_{1,W'(d_h)} = 1$), P' is not minimal, which is again a contradiction.

Finally, the lengths of P_1, P_{r+1}, P_{r+2}, …, P_m increase by one after adding links (e_k, $e_u^{(k)}$), ($e_{n_{r+1}}$, $e^{(k)}_{n_{r+1}}$), ($e_{n_{r+2}}$, $e^{(k)}_{n_{r+2}}$), …, (e_{n_m}, $e^{(k)}_{n_m}$). It is not difficult to see that P_1, P_2, …, P_m meet our requirements.

2.2.2.2 Case 2: $d_{1,k}$=1 and $|d_1|$=1

We have assumed $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$ (hence, $I'=\{n_2, n_3, \ldots, n_r\}$ and $I''=\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$), where $d_1=e_k$ and $|D'|\le k-1$. Since $Y'=F'$ is a minimal routing function, by induction hypothesis $P_2, P_3, \ldots, P_r$ that were produced by executing *Paths0*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$) are $r-1=|D'|$ disjoint paths in $*^{k-1}0$ that connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. For all $2\le i\le r$, P_i has length $|d_i|-1$ if $d_{i,Y'(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $|d_i|+1$ and $|t_i|=|d_i|+1$ if $d_{i,Y'(d_i)} = d_{i,F(d_i)} = 0$.

Similarly, since Y'' is a minimal routing function by Lemma 2, $P_{r+1}, P_{r+2}, \ldots, P_m$ that were produced by executing *Paths0*(Y'', $m-r$, $k-1$, $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{n_{r+1}, n_{r+2}, \ldots, n_m\}$) are $m-r=|D''|-1$ disjoint paths in $*^{k-1}1$ that connect $\{e^{(k)}_{n_{r+1}}, e^{(k)}_{n_{r+2}}, \ldots, e^{(k)}_{n_m}\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$. Further, since $e_k=d_1$, $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ are $|D''|$ disjoint paths in $*^{k-1}1$ that connect $\{e_k, e^{(k)}_{n_{r+1}}, e^{(k)}_{n_{r+2}}, \ldots, e^{(k)}_{n_m}\}$ and $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$. For all $r+1\le i\le m$, P_i has length $(|d_i|-1)-1=|d_i|-2$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $(|d_i|-1)+1=|d_i|$ and $|t_i|= |d_i|+1$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 0$.

After adding links $(e_{n_{r+1}}, e^{(k)}_{n_{r+1}})$, $(e_{n_{r+2}}, e^{(k)}_{n_{r+2}})$, $\ldots$, $(e_{n_m}, e^{(k)}_{n_m})$, the lengths of $P_{r+1}, P_{r+2}, \ldots, P_m$ increase by one. It is not difficult to see that $P_1, P_2, \ldots, P_m$ meet our requirements.

2.2.2.3 Case 3: $d_{1,k}=1$, $|d_1|>1$, and $d_{1,a}=1$ for some $a \in \{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$

We have assumed $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$ (hence, $I'=\{n_2, n_3, \ldots, n_r\}$ and $I''=\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$), where $|D'|\le k-1$. We have $|D''|\le k-1$ for the following reason. If $|D''|=k$, then $I''=\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}=\{1, 2, \ldots, k\}$. Hence $\{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$ is empty, which is a contradiction. Since $Y'=F'$ is a minimal routing function, by induction hypothesis $P_2, P_3, \ldots, P_r$ that were produced by executing *Paths0*(Y', $r-1$, $k-1$, $\{d_2, d_3, \ldots, d_r\}$, $\{n_2, n_3, \ldots, n_r\}$) are $r-1=|D'|$ disjoint paths in $*^{k-1}0$ that connect $\{e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_2, d_3, \ldots, d_r\}$. For all $2\le i\le r$, P_i has length $|d_i|-1$ if $d_{i,Y'(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $|d_i|+1$ and $|t_i|=|d_i|+1$ if $d_{i,Y'(d_i)} = d_{i,F(d_i)} = 0$.

Similarly, $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ that were produced by executing *Paths0*(Y'', $m-r+1$, $k-1$, $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{a, n_{r+1}, n_{r+2}, \ldots, n_m\}$) are $m-r+1=|D''|\le k-1$ disjoint paths in $*^{k-1}1$ that connect $\{e^{(k)}_a, e^{(k)}_{n_{r+1}}, e^{(k)}_{n_{r+2}}, \ldots, e^{(k)}_{n_m}\}$ and $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$, provided $e_k \notin \{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$ and Y'' is a minimal routing function.

Since $|d_1|>1$, we have $e_k \notin \{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$ as a consequence of Step 2. Y'' is a minimal routing function for the following reason. First, Y'' is a routing function because $a \notin \{n_{r+1}, n_{r+2}, \ldots, n_m\}$. Suppose conversely that W'': $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\} \rightarrow \{a, n_{r+1}, n_{r+2}, \ldots, n_m\}$ is a minimal routing function, where $V_{W''}<V_{Y''}$. We further assume $W''(d_l)=a$ for some $l \in \{1, r+1, r+2, \ldots, m\}$. Define a routing function P'': $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\} \rightarrow \{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$ as follows: $P''(d_l)=k$ and $P''(d_j)=W''(d_j)$ for all $j \in \{1, r+1, r+2, \ldots, m\}-\{l\}$. Since $d_{l,P''(d_l)} = d_{l,k} = 1$, we have $V_{P''} \leq V_{W''}$. Since $d_{1,F''(d_1)} = d_{1,k} = 1$ and $d_{1,Y''(d_1)} = d_{1,a} = 1$, we have $V_{Y''}=V_{F''}$. Hence $V_{P''} \leq V_{W''}<V_{Y''}=V_{F''}$, which is a contradiction because F'' is minimal.

Consequently, for all $i \in \{1, r+1, r+2, \ldots, m\}$, P_i has length $(|d_i|-1)-1=|d_i|-2$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $(|d_i|-1)+1=|d_i|$ and $|t_i|=|d_i|+1$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 0$. After adding links $(e_k, e_a^{(k)})$, $(e_{n_{r+1}}, e_{n_{r+1}}^{(k)})$, $(e_{n_{r+2}}, e_{n_{r+2}}^{(k)})$, …, $(e_{n_m}, e_{n_m}^{(k)})$, the lengths of $P_1, P_{r+1}, P_{r+2}, \ldots, P_m$ increase by one. It is not difficult to see that $P_1, P_2, \ldots, P_m$ meet our requirements.

2.2.2.4 Case 4: $d_{1,k}=1$, $|d_1|>1$, and $d_{1,a}=0$ for all $a \in \{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$

We have assumed $D'=\{d_2, d_3, \ldots, d_r\}$ and $D''=\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$ (hence, $I'=\{n_2, n_3, \ldots, n_r\}$ and $I''=\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}$), where $|D'| \leq k-1$. Actually we have $|D'| \leq k-2$ for the following reason. If $|D'|=k-1$, then $D''=\{d_1\}$ and $I''=\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}=\{k\}$. Hence $\{1, 2, \ldots, k-1\}-\{k, n_{r+1}, n_{r+2}, \ldots, n_m\}=\{1, 2, \ldots, k-1\}$. This implies $|d_1|=1$, which is a contradiction.

Since $|d_1|>1$, we have $e_k \notin \{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}$ as a consequence of Step 2. Y'' is a minimal routing function by Lemma 2. Hence, by induction hypothesis $P_{r+1}, P_{r+2}, \ldots, P_m$ that were produced by executing *Paths0*(Y'', $m-r$, $k-1$, $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$, $\{n_{r+1}, n_{r+2}, \ldots, n_m\}$) are $m-r=|D''|-1 \leq k-1$ disjoint paths in $*^{k-1}1$ that connect $\{e_{n_{r+1}}^{(k)}, e_{n_{r+2}}^{(k)}, \ldots, e_{n_m}^{(k)}\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\}$. For all $r+1 \leq i \leq m$, P_i has length $(|d_i|-1)-1=|d_i|-2$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $(|d_i|-1)+1=|d_i|$ and $|t_i|=|d_i|+1$ if $d_{i,Y''(d_i)} = d_{i,F(d_i)} = 0$.

$P_1, P_2, \ldots, P_m$ were obtained according to three situations. The first situation is $d_1 \notin P_i$ for all $r+1 \leq i \leq m$. If Y' is a minimal routing function, then by induction hypothesis $P'_1, P_2, P_3, \ldots, P_r$ that were produced by executing *Paths0*(Y', r, $k-1$, $\{d_1^{(k)}, d_2, d_3, \ldots, d_r\}$, $\{u, n_2, n_3, \ldots, n_r\}$) are $r=|D'|+1 \leq k-1$ disjoint paths in $*^{k-1}0$ that connect $\{e_u, e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_1^{(k)}, d_2, d_3, \ldots, d_r\}$. Y' is a routing function for the following reason. We have $d_1^{(k)} \notin \{d_2, d_3, \ldots, d_r\}$ as a consequence of Step 2, because $|d_1^{(k)}|<|d_1|$. We have $u \notin \{n_2, n_3, \ldots, n_r\}$ because $d_{1,u}=1$ and $1 \leq u<k$ imply $u \in I''-\{k\}=\{n_{r+1}, n_{r+2}, \ldots, n_m\}$. Further, Y' is minimal as explained below.

Suppose conversely that W': $\{d_1^{(k)}, d_2, d_3, \ldots, d_r\} \to \{u, n_2, n_3, \ldots, n_r\}$ is a minimal routing function, where $V_{W'}<V_{Y'}$. Since $d^{(k)}_{1,Y'(d_1^{(k)})}=d^{(k)}_{1,u}=1$, we have $V_{Y'}=V_{F'}$ by Lemma 1 ($D'=\{d_1^{(k)}\}$). Define a routing function P': $\{d_1, d_2, d_3, \ldots, d_r\} \to \{u, n_2, n_3, \ldots, n_r\}$ as follows: $P'(d_1)=W'(d_1^{(k)})$ and $P'(d_i)=W'(d_i)$ for all $2 \le i \le r$. Since $d_{1,v}=d^{(k)}_{1,v}$ for all $1 \le v \le k-1$ and $u \ne k$, we have $d_{1,P'(d_1)}=d_{1,W'(d_1^{(k)})}=d^{(k)}_{1,W'(d_1^{(k)})}$. Thus $V_{P'}=V_{W'}$. Since $u \in \{n_{r+1}, n_{r+2}, \ldots, n_m\}$, we have $u=W''(d_l)$ for some $r+1 \le l \le m$. Define another routing function P'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} \to \{k, n_{r+1}, n_{r+2}, \ldots, n_m\}-\{u\}$ as follows: $P''(d_l)=k$ and $P''(d_i)=W''(d_i)$ for all $r+1 \le i \le m$ and $i \ne l$. Since $d_{l,P''(d_l)}=d_{l,k}=1$, we have $V_{P''} \le V_{W''}$.

We have $V_{W''}=V_{Y''}$ if $d_{i,W''(d_i)}=d_{i,Y''(d_i)}$ for all $r+1 \le i \le m$. The latter holds for the following reason. If $d_{i,Y''(d_i)}=1$, then P_i has length $|d_i|-2$. Since P_i goes from $e^{(k)}_{W''(d_i)}$ to d_i and $|e^{(k)}_{W''(d_i)}|=2$, we have $d_{i,W''(d_i)}=1$. On the other hand, if $d_{i,Y''(d_i)}=0$, then $d_{i,W''(d_i)}=0$, for otherwise ($d_{l,Y''(d_l)}=0$ and $d_{l,W''(d_l)}=1$ for some $r+1 \le l \le m$) $V_{W''}<V_{Y''}$, which is a contradiction because Y'' is minimal.

Since $d_{1,F''(d_1)}=d_{1,k}=1$, we have $V_{Y''}=V_{F''}$ by Lemma 1 ($D'=\{d_1\}$). Hence, $V_{P'}+V_{P''} \le V_{W'}+V_{W''}<V_{Y'}+V_{Y''}= V_{F'}+V_{F''}=V_F$, which is a contradiction because $P' \cup P''$ is a routing function from D to I and F is minimal. Hence Y' is minimal.

Consequently, for all $2 \le i \le r$, P_i has length $|d_i|-1$ if $d_{i,Y'(d_i)}=d_{i,F(d_i)}=1$, and P_i has length $|d_i|+1$ and $|t_i|=|d_i|+1$ if $d_{i,Y'(d_i)}=d_{i,F(d_i)}=0$. Since $d^{(k)}_{1,Y'(d_1^{(k)})}=d^{(k)}_{1,u}=1$, P'_1 has length $|d_1^{(k)}|-1=|d_1|-2$. Hence P_1 has length $|d_1|-1$. After adding links $(e_{n_{r+1}}, e^{(k)}_{n_{r+1}})$, $(e_{n_{r+2}}, e^{(k)}_{n_{r+2}})$, …, $(e_{n_{h-1}}, e^{(k)}_{n_{h-1}})$, $(e_k, e^{(k)}_u)$, $(e_{n_{h+1}}, e^{(k)}_{n_{h+1}})$, …, $(e_{n_m}, e^{(k)}_{n_m})$, the lengths of $P_{r+1}, P_{r+2}, \ldots, P_m$ increase by one (since $u \in \{n_{r+1}, n_{r+2}, \ldots, n_m\}$, there exists $r+1 \le h \le m$ so that $u=n_h$). It is not difficult to see that $P_1, P_2, \ldots, P_m$ meet our requirements.

The second situation is $d_1 \in P_l$ for some $r+1 \le l \le m$ and $d_l^{(k)} \notin \{d_2, d_3, \ldots, d_r\}=D'$. If Q' is a minimal routing function, then by induction hypothesis $P'_l, P_2, P_3, \ldots, P_r$ that were produced by executing *Paths0*$(Q', r, k-1, \{d_l^{(k)}, d_2, d_3, \ldots, d_r\}, \{W''(d_l), n_2, n_3, \ldots, n_r\})$ are $r=|D'|+1 \le k-1$ disjoint paths in $*^{k-1}0$ that connect $\{e_{W''(d_l)}, e_{n_2}, e_{n_3}, \ldots, e_{n_r}\}$ and $\{d_l^{(k)}, d_2, d_3, \ldots, d_r\}$. Since $d_l^{(k)} \notin \{d_2, d_3, \ldots, d_r\}$ and $W''(d_l) \notin \{n_2, n_3, \ldots, n_r\}$, Q' is a routing function. Further, Q' is minimal, as explained below.

Suppose conversely that $\hat{A}'$: $\{d_l^{(k)}, d_2, d_3, \ldots, d_r\} \to \{W''(d_l), n_2, n_3, \ldots, n_r\}$ is a minimal routing function, where $V_{\hat{A}'}<V_{Q'}$. We have $d_{l,W''(d_l)}=1$ for the following reason. Define a routing function $\acute{A}''$: $\{d_1,$

d_{r+1}, d_{r+2}, …, $d_m\} \rightarrow \{k$, n_{r+1}, n_{r+2}, …, $n_m\}$ as follows: $Á''(d_1)=k$ and $Á''(d_i)=W''(d_i)$ for all $r+1 \leq i \leq m$. Since $d_{1,\Im''(d_1)} = d_{1,k} = 1$, we have $V_{Á''}=V_{W''}$ by Lemma 1 ($D'=\{d_1\}$). Hence $Á''$ is minimal because $V_{W''}=V_{Y''}=V_{F''}$ and F'' is minimal. If $d_{l,W''(d_l)} = 0$, then $d_{l,Y''(d_l)} = 0$ which implies that P_l has length $(|d_l|-1)+1=|d_l|$ and $|t_l|=|d_l|+1$. Since $d_1 \in P_l$, $|e^{(k)}_{W''(d_l)}|=2$, and the subpath of P_l from $e^{(k)}_{W''(d_l)}$ to t_l has length $|d_l|-1=|t_l|-2$, we have $d_{1,W''(d_l)} = 1$, which implies $d_{1,\Im''(d_l)} = 1$. By Lemma 4 ($d_{l,\Im''(d_l)} = d_{l,W''(d_l)} = 0$, $d_{l,\Im''(d_1)} = d_{l,k} = 1$, and $d_{1,\Im''(d_l)} = d_{1,W''(d_l)} = 1$), $Á''$ is not minimal, which is a contradiction.

Since $W''(d_l) \neq k$ and $d_{l,W''(d_l)} = 1$, we have $d^{(k)}_{l,Q'(d_l^{(k)})} = d^{(k)}_{l,W''(d_l)} = d_{l,W''(d_l)} = 1$. By Lemma 1 ($D'=\{d_l^{(k)}\}$) we have $V_{Q'}=V_{F'}$. Define a routing function $\tilde{A}'$: $\{d_l, d_2, d_3, \ldots, d_r\} \rightarrow \{W''(d_l), n_2, n_3, \ldots, n_r\}$ as follows: $\tilde{A}'(d_l)=\hat{A}'(d_l^{(k)})$ and $\tilde{A}'(d_j)=\hat{A}'(d_j)$ for all $2 \leq j \leq r$. Since $\tilde{A}'(d_l) \neq k$, we have $d_{l,\wp'(d_l)} = d^{(k)}_{l,\wp'(d_l)} = d^{(k)}_{l,\Re'(d_l^{(k)})}$. If $d_{l,\wp'(d_l)} = d^{(k)}_{l,\Re'(d_l^{(k)})} = 1$, then $V_{\tilde{A}'}=V_{\hat{A}'}$. If $d_{l,\wp'(d_l)} = d^{(k)}_{l,\Re'(d_l^{(k)})} = 0$, then $V_{\tilde{A}'}<V_{\hat{A}'}$ because $|d_l^{(k)}|<|d_l|$. Hence we have $V_{\tilde{A}'} \leq V_{\hat{A}'}$. Define another routing function $\tilde{A}''$: $\{d_1, d_{r+1}, d_{r+2}, \ldots, d_m\}-\{d_l\} \rightarrow \{k, n_{r+1}, n_{r+2}, \ldots, n_m\}-\{W''(d_l)\}$ as follows: $\tilde{A}''(d_1)=k$ and $\tilde{A}''(d_j)=W''(d_j)$ for all $r+1 \leq j \leq m$ and $j \neq l$. Since $d_{1,\wp''(d_1)} = d_{1,k} = 1$ and $d_{l,W''(d_l)} = 1$, we have $V_{\tilde{A}''}=V_{W''}$. Hence, $V_{\tilde{A}'}+V_{\tilde{A}''} \leq V_{\hat{A}'}+V_{W''}<V_{Q'}+V_{Y''}=V_{F'}+V_{F''}=V_F$, which is a contradiction because $\tilde{A}' \cup \tilde{A}''$ is a routing function from D to I and F is minimal. Hence Q' is minimal.

Consequently, for all $2 \leq i \leq r$, P_i has length $|d_i|-1$ if $d_{i,Q'(d_i)} = d_{i,F(d_i)} = 1$, and P_i has length $|d_i|+1$ and $|t_i|=|d_i|+1$ if $d_{i,Q'(d_i)} = d_{i,F(d_i)} = 0$. P'_l has length $|d_l^{(k)}|-1=|d_l|-2$ because $d^{(k)}_{l,Q'(d_l^{(k)})} = d^{(k)}_{l,W''(d_l)} = d_{l,W''(d_l)} = 1$. On the other hand, (old) P_l has length $(|d_l|-1)-1=|d_l|-2$ because $d_{l,Y''(d_l)} = d_{l,W''(d_l)} = 1$. P_1 has length $|d_1|-2$ because $d_1 \in P_l$, P_l goes from $e^{(k)}_{W''(d_l)}$ to d_l, and $|e^{(k)}_{W''(d_l)}|=2$. The newly constructed P_l has length $|d_l|-1$. After adding links $(e_{n_{r+1}}, e^{(k)}_{n_{r+1}})$, $(e_{n_{r+2}}, e^{(k)}_{n_{r+2}})$, …, $(e_{W''(d_l)-1}, e^{(k)}_{W''(d_l)-1})$, $(e_k, e^{(k)}_{W''(d_l)})$, $(e_{W''(d_l)+1}, e^{(k)}_{W''(d_l)+1})$, …, $(e_{n_m}, e^{(k)}_{n_m})$, the lengths of P_1, P_{r+1}, P_{r+2}, …, P_{l-1}, P_{l+1}, …, P_m increase by one. It is not difficult to see that P_1, P_2, …, P_m meet our requirements.

The third situation is $d_1 \in P_l$ for some $r+1 \leq l \leq m$ and $d_l^{(k)} \in \{d_2, d_3, \ldots, d_r\}=D'$. We observe that P_1, P_2, …, P_m can be obtained all the same as Case 3, provided $d_{i,Y(d_i)} = d_{i,F(d_i)}$ for all $1 \leq i \leq m$ and $d_{1,a}=1$ for some $a \in \{1, 2, \ldots, k-1\}-Y(D'')$. The two conditions actually hold, as explained below.

Suppose $d_l^{(k)}=d_h$ for some $2 \leq h \leq r$ and $h \neq l$. We have $d_{h,v}=d_{l,v}$ for all $1 \leq v<k$. In order to show $d_{i,Y(d_i)} =$

$d_{i,F(d_i)}$ for all $1 \le i \le m$, it suffices to show $d_{j,Y(d_j)} = d_{j,F(d_j)}$ for all $j \in \{1, 2, \ldots, m\} - (\{2, 3, \ldots, r\} - \{h\}) = \{1, h, r+1, r+2, \ldots, m\}$. It is not difficult to check that $d_{1,Y(d_1)} = d_{1,k} = d_{1,F(d_1)}$ and $d_{v,Y(d_v)} = d_{v,W''(d_v)} = d_{v,Y''(d_v)} = d_{v,F(d_v)}$ for all $r+1 \le v \le m$ and $v \ne l$. Hence we only need to show $d_{j,Y(d_j)} = d_{j,F(d_j)}$ for $j \in \{h, l\}$. Since $W''(d_l) \ne k$, we have $d_{h,Y(d_h)} = d_{h,W''(d_l)} = d_{l,W''(d_l)} = 1$. Also we have $d_{h,F(d_h)} = 1$ for the following reason. Since $F(d_l) \ne k$, we have $d_{h,F(d_l)} = d_{l,F(d_l)} = d_{l,Y''(d_l)} = d_{l,W''(d_l)} = 1$. If $d_{h,F(d_h)} = 0$, then by Lemma 5 ($d_{h,F(d_h)} = 0$, $d_{h,F(d_l)} = 1$, and $|d_h| < |d_l|$) F is not minimal, which is a contradiction. Hence we have $d_{h,Y(d_h)} = d_{h,F(d_h)} = 1$. Further, since $F(d_h) \ne k$, we have $d_{l,Y(d_l)} = d_{l,F(d_h)} = d_{h,F(d_h)} = 1 = d_{l,W''(d_l)} = d_{l,Y''(d_l)} = d_{l,F(d_l)}$.

On the other hand, let $a = W''(d_l)$. We have $a \in \{1, 2, \ldots, k-1\} - Y(D'')$ because $W''(d_l) = Y(d_h) \ne k$ and $d_h \in D'$. Since $d_{l,Y''(d_l)} = d_{l,W''(d_l)} = 1$, P_l has length $(|d_l|-1)-1 = |d_l|-2$. We have $d_{1,a} = d_{1,W''(d_l)} = 1$ because $d_1 \in P_l$, $|e^{(k)}_{W''(d_l)}| = 2$, and P_l goes from $e^{(k)}_{W''(d_l)}$ to d_l.

2.3 When $\{d_1, d_2, \ldots, d_m\}$ is a multiset

When $\{d_1, d_2, \ldots, d_m\}$ is a multiset, Procedure *Paths0*(F, m, k, D, I) still works, provided Step 4 is modified as follows.

Let $T = \{d_j \mid d_j = e_k \text{ and } r+1 \le j \le m\}$. In Case 2, Y'' is changed to Y'': $\{d_{r+1}, d_{r+2}, \ldots, d_m\} - T \to \{n_{r+1}, n_{r+2}, \ldots, n_m\} - \{F(d_j) \mid d_j \in T \text{ and } r+1 \le j \le m\}$ so that $Y''(d_i) = F(d_i) = n_i$ for all $r+1 \le i \le m$ and $d_i \ne e_k$. Instead of executing *Paths0*$(Y'', m-r, k-1, \{d_{r+1}, d_{r+2}, \ldots, d_m\}, \{n_{r+1}, n_{r+2}, \ldots, n_m\})$, we execute *Paths0*$(Y'', m-r-|T|, k-1, \{d_{r+1}, d_{r+2}, \ldots, d_m\}-T, \{n_{r+1}, n_{r+2}, \ldots, n_m\} - \{F(d_j) \mid d_j \in T \text{ and } r+1 \le j \le m\})$, in order to construct $m-r-|T|$ disjoint paths in $*^{k-1}1$ that connect $\{e^{(k)}_{n_{r+1}}, e^{(k)}_{n_{r+2}}, \ldots, e^{(k)}_{n_m}\} - \{e^{(k)}_{F(d_j)} \mid d_j \in T \text{ and } r+1 \le j \le m\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_m\} - T$. Additionally, we construct another $|T|$ disjoint paths of length one that connect $e^{(k)}_{F(d_j)}$ and d_j for all $d_j \in T$. In Case 4, the conditions for the second and third situations are changed to "If $d_1 \in P_l$ for some $r+1 \le l \le m$, $d_1 \notin \{d_{r+1}, d_{r+2}, \ldots, d_m\}$, and $d_l^{(k)} \notin \{d_2, d_3, \ldots, d_r\}$" and "If $d_1 \in P_l$ for some $r+1 \le l \le m$, $d_1 \notin \{d_{r+1}, d_{r+2}, \ldots, d_m\}$, and $d_l^{(k)} \in \{d_2, d_3, \ldots, d_r\}$", respectively. One more situation whose condition is "If $d_1 \in P_l$ for some $r+1 \le l \le m$ and $d_1 \in \{d_{r+1}, d_{r+2}, \ldots, d_m\}$" needs to be added. The new situation constructs $P_1, P_2, \ldots, P_r$ in $*^{k-1}0$ all the same as the first situation (i.e., $d_1 \notin P_i$ for all $r+1 \le i \le m$).

Lemma 7 is also valid. However, the proof needs to be adapted to the changes above. First we note that the definitions of F, V_F, and minimal F remain unchanged, and Lemmas 1 to 6 remain valid. In Case 1, the proof for d_1 not included in $P_2, P_3, \ldots, P_r$ is needed only when $d_1 \notin \{d_2, d_3, \ldots, d_r\}$. In Case 2, that the new Y'' is a minimal routing function can be assured by Lemma 2. In Case 4, the proof for the newly added situation (i.e., $d_1 \in P_l$ for some $r+1 \le l \le m$ and $d_1 \in \{d_{r+1}, d_{r+2}, \ldots, d_m\}$) is the same as that for the first situation. The proofs for the other three situations remain the same.

According to the discussion above, we have the following lemma as a consequence of Lemma 7.

Lemma 8. Suppose F: $\{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is a minimal routing function, where $k \ge 2$ and $\{d_1, d_2, \ldots, d_k\}$ is a multiset. *Paths1*(F, k, k, $\{d_1, d_2, \ldots, d_k\}$, $\{1, 2, \ldots, k\}$) can produce k disjoint paths from s to $d_1, d_2, \ldots, d_k$ so that the path to d_i has length $|d_i|$ if $d_{i,F(d_i)} = 1$, and length $|d_i|+2$ if $d_{i,F(d_i)} = 0$, where $s \notin \{d_1, d_2, \ldots, d_k\}$. Moreover, if the path to d_i has length $|d_i|+2$, then $|t_i|=|d_i|+1$, where t_i is the immediate predecessor of d_i in the path.

Lemma 8 is further illustrated with Figure 2, where a path labeled with $*$ represents a shortest path. According to Lemma 8, the path from s to d_i has length k if $|d_i|=k$, and at most $k+1$ if $|d_i|<k$, which imposes an upper bound of $k+1$ on the strong Rabin number of a k-cube. On the other hand, a lower bound of $k+1$ was suggested in [10, 18]. Hence the strong Rabin number of a k-cube is equal to $k+1$.

2.4 Finding a minimal F

Thus far we have assumed the existence of a minimal routing function F. We now show how it can be determined in polynomial time. Actually the problem of finding a minimal F can be reduced to the problem of finding a maximum matching in a weighted bipartite graph. Suppose $G=(U, V, E)$ is a weighted bipartite graph, where U and V are two partite sets of nodes and E is the set of weighted links (each link of G is assigned a weight). A subset of E forms a *matching* in G if no two links share a common node. A matching in G is *maximum* if its total weight is maximum. A maximum matching in G can be found in $O(|U\cup V|(|E|+|U\cup V|\log|U\cup V|))$ time (see [13]).

Suppose F: $\{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is a routing function, where $\{d_1, d_2, \ldots, d_k\}$ is a multiset. We let $U=\{d_1, d_2, \ldots, d_k\}$, $V=\{1, 2, \ldots, k\}$, $E=\{(d_i, j) \mid 1\le i\le k \text{ and } 1\le j\le k.\}$, and assign link (d_i, j) a weight $(k+1)^{k-|d_i|+1}$ if $d_{i,j}=1$, and 1 if $d_{i,j}=0$. We note that a maximum matching M in G contains k links and its total weight can be expressed as $a_1*(k+1)^k+a_2*(k+1)^{k-1}+ \ldots +a_k*(k+1)+a_{k+1}$, where $0\le a_r\le k$ for all $1\le r\le k+1$. That is, there are a_r links in M whose weights are $(k+1)^{k-r+1}$ (or equivalently, M contains a_v links (d_i, j) with $|d_i|=v$ and $d_{i,j}=1$ for all $1\le v\le k$, and a_{k+1} links (d_i, j) with $d_{i,j}=0$).

A minimal F with $V_F=(r_1-a_1, r_2-a_2, \ldots, r_k-a_k)$ can be obtained from M as follows: $F(d_i)=j$ if $(d_i, j) \in M$, where r_v is the number of d_i's with $|d_i|=v$ for $1\le v\le k$ and $1\le i\le k$. If F is not minimal, then there exists $V_{F'}<V_F$ for some routing function F': $\{d_1, d_2, \ldots, d_k\} \rightarrow \{1, 2, \ldots, k\}$, which implies another matching in G whose total weight is greater than the total weight of M. This is a contradiction.

3 Disjoint Paths in the Folded Hypercube

Suppose that s, d_1, d_2, …, d_{k+1} are arbitrary k+2 distinct nodes of a k-fcube. In this section, we construct k+1 disjoint paths from s to d_1, d_2, …, d_{k+1} in the k-fcube whose lengths do not exceed $\lceil k/2\rceil+1$, where $\lceil k/2\rceil$ is the diameter of the k-fcube. Since the construction is easy for $k\le3$, we assume $k\ge4$ throughout this section.

3.1 A procedure to construct disjoint paths

Since the folded hypercube is node symmetric, we assume $s=0^k$ without loss of generality. We suppose $d_i=d_{i,1}d_{i,1}\ldots d_{i,k}$ for all $1\le i£k+1$, and let $\overline{x}=(1-x_1)(1-x_2)\ldots(1-x_k)$ denote the complement of a node $x=x_1x_2\ldots x_k$. The following is a procedure, named *Paths2*. It will be shown in Section 3.2 that *Paths2*(s, d_1, d_2, …, d_{k+1}) can produce k+1 disjoint paths from s to d_1, d_2, …, d_{k+1} in a k-fcube whose lengths are not greater than $\lceil k/2\rceil+1$. We use Q_1, Q_2, …, Q_{k+1} to denote the k+1 paths, where Q_i $(1\le i\le k+1)$ is the path to d_i. The intuition behind *Paths2*(s, d_1, d_2, …, d_{k+1}) is to construct k disjoint paths in a k-cube first, as we have done in Section 2. The k disjoint paths together with some complement links produce k of Q_1, Q_2, …, Q_{k+1}. The other path is determined in a particular way.

Procedure *Paths2*(s, d_1, d_2, …, d_{k+1}).

Step 1. Determine d_t so that $|d_t|\ge|d_i|$ for all $1\le i\le k+1$, where $1\le t\le k+1$. Without loss of generality, we assume $t=k+1$.

Step 2. Construct Q_1, Q_2, …, Q_{k+1} according to the following five cases.

Case 1. $|d_{k+1}|\le\lceil k/2\rceil-2$.

/* Suppose that F: $\{d_1, d_2, \ldots, d_k\} \rightarrow \{1, 2, \ldots, k\}$ is minimal. */

Construct Q_1, Q_2, …, Q_k in a k-cube by invoking *Paths1*(F, k, k, $\{d_1, d_2, \ldots, d_k\}$, $\{1, 2, \ldots, k\}$).

If $d_{k+1} \in Q_l$ for some $1\le l\le k$, then {

Construct Q_{k+1} as the subpath of Q_l from s to d_{k+1}.

Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$. }

If $d_{k+1} \notin Q_i$ for all $1 \le i \le k$, then

Construct Q_{k+1} as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_{k+1}}$ in a k-cube, and a link $(\overline{d_{k+1}}, d_{k+1})$.

Case 2. $|d_{k+1}| = \lceil k/2 \rceil - 1$.

If k is even, then

Construct $Q_1, Q_2, \ldots, Q_{k+1}$ all the same as Case 1.

If k is odd, then {

/* Suppose that F: $\{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. */

Construct $Q_1, Q_2, \ldots, Q_k$ in a k-cube by invoking *Paths1*$(F, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$.

If $d_{k+1} \in Q_l$ for some $1 \le l \le k$, then {

Construct Q_{k+1} as the subpath of Q_l from s to d_{k+1}.

If d_{k+1} is the immediate predecessor of d_l in Q_l, then

Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$.

If d_{k+1} is not the immediate predecessor of d_l in Q_l, then

/* Suppose that x is the immediate predecessor of d_l in Q_l. */

Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to x in a k-cube, and a link (x, d_l). }

If $d_{k+1} \notin Q_i$ for all $1 \le i \le k$, then {

Construct Q_{k+1} as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube.

If Q_{k+1} intersects with Q_h for some $1 \le h \le k$ at a node y $(\neq s)$, then {

Reconstruct Q_{k+1} as the combination of the subpath of Q_h from s to y and a shortest path from y to d_{k+1}.

/* Suppose that z is an adjacent node of d_h so that $|z| = |d_h| + 1$ and $z \notin Q_i$ for all $1 \le i \le k$. */

Reconstruct Q_h as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to z in a k-cube, and a link (z, d_h). }}}

Case 3. $|d_{k+1}|=\lceil k/2 \rceil$.

/* Suppose that F: $\{d_1, d_2, \ldots, d_k\} \rightarrow \{1, 2, \ldots, k\}$ is minimal. Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_r\}$ and $\{d_{r+1}, d_{r+2}, \ldots, d_k\}$ so that for all $1 \le i \le k$, $d_i \in \{d_1, d_2, \ldots, d_r\}$ if $d_{i,F(d_i)} = 0$ and $|d_i|=\lceil k/2 \rceil$, and $d_i \in \{d_{r+1}, d_{r+2}, \ldots, d_k\}$ else, where $0 \le r \le k$. Define a minimal routing function W: $\{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\} \rightarrow \{1, 2, \ldots, k\}$ as follows: $W(\overline{d_i})=F(d_i)$ for all $1 \le i \le r$ and $W(d_j)=F(d_j)$ for all $r+1 \le j \le k$. */

Construct $Q'_1, Q'_2, \ldots, Q'_r, Q_{r+1}, \ldots, Q_k$ in a k-cube by invoking *Paths1*$(W, k, k, \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\}, \{1, 2, \ldots, k\})$. /* Assume that Q'_i ends at $\overline{d_i}$ for all $1 \le i \le r$. */

For i=1, 2, …, r, {

/* Suppose that f_i is the immediate predecessor of $\overline{d_i}$ in Q'_i. */

If $\overline{d_i}=d_h$ for some $r+1 \le h \le k+1$, then

Construct Q_i as the combination of the subpath of Q'_i from s to f_i and two links $(f_i, \overline{f_i})$ and $(\overline{f_i}, d_i)$.

If $\overline{d_i} \neq d_j$ for all $r+1 \le j \le k+1$, then

Construct Q_i as the combination of Q'_i and a link $(\overline{d_i}, d_i)$. }

Construct Q_{k+1} as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube.

If Q_{k+1} intersects with Q_t for some $1 \le t \le r$ at a node x ($\neq s$), then {

/* Suppose that $\overline{d_t}=d_m$ for some $r+1 \le m \le k$ and g is the immediate predecessor of d_m in Q_m. */

Reconstruct Q_t as the combination of the subpath of Q_m from s to g and two links $(g, \bar{g})$ and $(\bar{g}, d_t)$.

Reconstruct Q_m as the combination of the subpath of (old) Q_t from s to $\bar{x}$ and a shortest path from $\bar{x}$ to d_m. }

If $d_{k+1} \in Q_l$ for some $r+1 \le l \le k$, then {

Reconstruct Q_{k+1} as the subpath of Q_l from s to d_{k+1}.

If k is even, then

Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$.

If k is odd, then {

/* Suppose that y is an adjacent node of d_l so that $|y|=|d_l|+1$ and $y \notin Q_i$ for all $1 \le i \le k$. */

Reconstruct Q_l as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to y in a k-cube, and a link (y, d_l).

If Q_l intersects with Q_v for some $1 \le v \le r$ at a node z $(\ne s)$, then {

/* Suppose that $\overline{d_v}=d_w$ for some $r+1 \le w \le k$ and $w \ne l$, and p is the immediate predecessor of d_w in Q_w. */

Reconstruct Q_v as the combination of the subpath of Q_w from s to p and two links $(p, \bar{p})$ and $(\bar{p}, d_v)$.

Reconstruct Q_w as the combination of the subpath of (old) Q_v from s to $\bar{z}$ and a shortest path from $\bar{z}$ to d_w. }}}

Case 4. $|d_{k+1}|=\lceil k/2 \rceil+1$.

/* Suppose that F: $\{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is minimal. Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_r\}$, $\{d_{r+1}, d_{r+2}, \ldots, d_u\}$, and $\{d_{u+1}, d_{u+2}, \ldots, d_k\}$ so that for all $1 \le i \le k$, $d_i \in \{d_1, d_2, \ldots, d_r\}$ if $d_{i,F(d_i)}=0$ and $|d_i|=\lceil k/2 \rceil+1$, $d_i \in \{d_{r+1}, d_{r+2}, \ldots, d_u\}$ if $d_{i,F(d_i)}=0$, $|d_i|=\lceil k/2 \rceil$, and $\overline{d_i} \notin \{d_1, d_2, \ldots, d_k\}$, and $d_i \in \{d_{u+1}, d_{u+2}, \ldots, d_k\}$ else, where $0 \le r \le u \le k$. Initially, let $D=\{\overline{d_1}, \ldots, \overline{d_u}, d_{u+1}, \ldots, d_k\}$ and define a minimal routing function Y: $D \to \{1, 2, \ldots, k\}$ as follows: $Y(\overline{d_j})=F(d_j)$ for all $1 \le j \le u$ and $Y(d_t)=F(d_t)$ for all $u+1 \le t \le k$. Whenever there is a $d_l \in D$ so that $u+1 \le l \le k$, $d_{l,Y(d_l)}=0$, $|d_l|=\lceil k/2 \rceil$, and $\overline{d_l} \in D$, D and Y need to be modified as follows. Assume that $p_l=p_{l,1}p_{l,2}\ldots p_{l,k}$ is an adjacent node of $\overline{d_l}$ so that $|p_l|=|\overline{d_l}|-1$, $p_l \notin D$, and $\overline{p_l} \notin D \cup \{d_{k+1}\}$. Also assume $\overline{d_l}=d_h$ for some $u+1 \le h \le k$ and $h \ne l$. First, swap $Y(d_l)$ and $Y(d_h)$ if $p_{l,Y(d_l)}=0$. Then, replace d_l with p_l in D, and assign $Y(p_l)$ with $Y(d_l)$. Without loss of generality, assume $D=\{\overline{d_1}, \ldots, \overline{d_u}, p_{u+1}, \ldots, p_v, d_{v+1}, \ldots, d_k\}$ finally, where $u \le v \le k$. */

Construct $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$ in a k-cube by invoking *Paths1*$(Y, k, k, D, \{1, 2, \ldots, k\})$.

/* Assume that Q'_j ends at $\overline{d_j}$ for all $1 \le j \le u$ and Q'_t ends at p_t for all $u+1 \le t \le v$. */

Construct Q_{k+1} as the combination of a link $(s, \overline{s})$ and a shortest path from $\overline{s}$ to d_{k+1} in a k-cube.

For i=1, 2, …, r, {

/* We use f_i to denote the immediate predecessor of $\overline{d_i}$ in Q'_i. */

If $\overline{d_i} = d_w$ for some $v+1 \le w \le k$, then {

If $\overline{f_i} \in Q_{k+1}$, then

Swap Q'_i and Q_w.

/* Now f_i denotes the immediate predecessor of $\overline{d_i}$ in (new) Q'_i. */

Construct Q_i as the combination of the subpath of Q'_i from s to f_i and two links $(f_i, \overline{f_i})$ and $(\overline{f_i}, d_i)$. }

If $\overline{d_i} \ne d_j$ for all $v+1 \le j \le k$, then

Construct Q_i as the combination of Q'_i and a link $(\overline{d_i}, d_i)$. }

For j=r+1, r+2, …, u,

Construct Q_j as the combination of Q'_j and a link $(\overline{d_j}, d_j)$.

For t=u+1, u+2, …, v,

Construct Q_t as the combination of Q'_t and two links $(p_t, \overline{p_t})$ and $(\overline{p_t}, d_t)$.

Case 5. $|d_{k+1}| \ge \lceil k/2 \rceil + 2$.

/* Without loss of generality, assume that $\{d_1, d_2, \ldots, d_k\}$ is partitioned into $\{d_1, d_2, \ldots, d_{u'}\}$ and $\{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ so that for all $1 \le i \le k$, $d_i \in \{d_1, d_2, \ldots, d_{u'}\}$ if $|d_i| \ge \lceil k/2 \rceil + 2$, and $d_i \in \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ else, where $0 \le u' \le k$. Initially, let $D = \{\overline{d_1}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ and determine a minimal routing function W: $D \to \{1, 2, \ldots, k\}$. Whenever there is a $\overline{d_h} \in D$ so that $1 \le h \le u'$, $|\overline{d_h}| = \lfloor k/2 \rfloor - 2$, $\overline{d_h} = d_l$ for some $u'+1 \le l \le k$, and $(\overline{d}_{h,\, W(\overline{d_h})} = 0$ or $\overline{d}_{h,\, W(d_l)} = 0)$, D and W need to be modified as follows. First, swap $W(\overline{d_h})$ and $W(d_l)$ if $\overline{d}_{h,\, W(\overline{d_h})} = 1$. Let $q_h = q_{h,1}q_{h,2}\ldots q_{h,k}$ be an adjacent node of $\overline{d_h}$ so that $q_h \notin D$, $\overline{q_h} \notin D$, and for some $w \in \{1, 2, \ldots, k\} - \{W(z) \mid z = z_1 z_2 \ldots z_k \in D$, $z_{W(z)} = 1$, and $|z| \le \lfloor k/2 \rfloor - 1\}$, $q_{h,w} = 1$, $\overline{d}_{h,w} = 0$, and $q_{h,i} = \overline{d}_{h,i}$ for all $1 \le i \le k$ and $i \ne w$. Then, replace $\overline{d_h}$ with q_h in D, and determine a new minimal routing function W arbitrarily. Without loss of generality, assume $D=\{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ finally, where $0 \le r' \le u'$. */

Construct $k+1$ disjoint paths, denoted by $Q'_1, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$, from s to $q_1, \ldots, q_{r'}$, $\overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_{k+1}$, respectively, by the construction method of Case 4 with an initial minimal W and destination nodes $q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_{k+1}$ (i.e., substituting W, $q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}$ for $F, d_1, d_2, \ldots, d_{u'}$ in Case 4, respectively).

/* Assume that Q'_i ends at q_i for all $1 \le i \le r'$ and Q'_j ends at $\overline{d_j}$ for all $r'+1 \le j \le u'$. */

For $i=1, 2, \ldots, r'$,

Construct Q_i as the combination of Q'_i and two links $(q_i, \overline{q_i})$ and $(\overline{q_i}, d_i)$.

For $j=r'+1, r'+2, \ldots, u'$, {

/* We use g_j to denote the immediate predecessor of $\overline{d_j}$ in Q'_j. */

If $\overline{d_j}=d_t$ for some $u'+1 \le t \le k$, then {

If $\overline{g_j} \in Q_{k+1}$, then

Swap Q'_j and Q_t.

/* Now g_j denotes the immediate predecessor of $\overline{d_j}$ in (new) Q'_j. */

Construct Q_j as the combination of the subpath of Q'_j from s to g_j and two links $(g_j, \overline{g_j})$ and $(\overline{g_j}, d_j)$. }

If $\overline{d_j} \ne d_v$ for all $u'+1 \le v \le k$, then

Construct Q_j as the combination of Q'_j and a link $(\overline{d_j}, d_j)$. }

3.2 Correctness of *Paths2*($s, d_1, d_2, \ldots, d_{k+1}$)

In this section, we show that *Paths2*($s, d_1, d_2, \ldots, d_{k+1}$) can produce $k+1$ disjoint paths from s to $d_1, d_2, \ldots, d_{k+1}$ in a k-fcube whose lengths do not exceed $\lceil k/2 \rceil+1$. We first introduce some fundamental properties.

3.2.1 Properties

Throughout this section, we suppose that F: $\{d_1, d_2, \ldots, d_k\} \rightarrow \{1, 2, \ldots, k\}$ is a minimal routing function, where $\{d_1, d_2, \ldots, d_k\}$ is a multiset.

Lemma 9. Suppose $d_{h,F(d_h)} = 0$, where $1 \le h \le k$. If $\overline{d_h} = d_l$ for some $1 \le l \le k$ and $l \ne h$, then $d_{l,F(d_l)} = 1$.

Proof. Suppose conversely $d_{l,F(d_l)} = 0$. We have $\overline{d}_{l,F(d_l)} = 1 = d_{h,F(d_l)}$. By Lemma 3 ($d_{h,F(d_h)} = 0$, $d_{l,F(d_l)} = 0$, and $d_{h,F(d_l)} = 1$), F is not minimal, which is a contradiction. □

Lemma 10. Suppose $d_{h,F(d_h)} = 0$, where $1 \le h \le k$. If $\overline{d_h} = d_l$ for some $1 \le l \le k$ and $l \ne h$, then the routing function Y: $\{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ that is defined as follows: $Y(d_h)=F(d_l)$, $Y(d_l)=F(d_h)$, and $Y(d_t)=F(d_t)$ for $t \in \{1, 2, \ldots, k\}-\{h, l\}$ (i.e., swap $F(d_l)$ and $F(d_h)$) is minimal. Besides, $d_{i,Y(d_i)} = d_{i,F(d_i)}$ for all $1 \le i \le k$.

Proof. It suffices to show $d_{h,Y(d_h)} = d_{h,F(d_h)}$ and $d_{l,Y(d_l)} = d_{l,F(d_l)}$. Since $d_{l,F(d_l)} = 1$ by Lemma 9, we have $d_{h,Y(d_h)} = d_{h,F(d_l)} = \overline{d}_{l,F(d_l)} = 0 = d_{h,F(d_h)}$ and $d_{l,Y(d_l)} = d_{l,F(d_h)} = \overline{d}_{h,F(d_h)} = 1 = d_{l,F(d_l)}$. □

Lemma 11. Suppose $d_{h,F(d_h)} = 0$, where $1 \le h \le k$. If $d_{h,F(d_l)} = 1$ for some $1 \le l \le k$ and $l \ne h$, then $|d_l| \le |d_h|$ and $d_{l,F(d_l)} = 1$.

Proof. If $|d_l| > |d_h|$, then by Lemma 5 ($d_{h,F(d_h)} = 0$, $d_{h,F(d_l)} = 1$, and $|d_h| < |d_l|$), F is not minimal, which is a contradiction. If $d_{l,F(d_l)} = 0$, then by Lemma 3 ($d_{h,F(d_h)} = 0$, $d_{l,F(d_l)} = 0$, and $d_{h,F(d_l)} = 1$), F is not minimal, which is a contradiction. □

Lemma 12. Suppose that $d_{i,F(d_i)} = 0$ and d'_i is a node of a k-fcube with $d'_{i,F(d_i)} = 1$ for all $1 \le i \le h$, where $1 \le h \le k$. Then the routing function Y:$\{d'_1, d'_2, \ldots, d'_h, d_{h+1}, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ that is defined as follows: $Y(d'_i)=F(d_i)$ for all $1 \le i \le h$ and $Y(d_j)=F(d_j)$ for all $h+1 \le j \le k$ is minimal.

Proof. Define routing functions Y_t: $\{d'_1, d'_2, \ldots, d'_t, d_{t+1}, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ as follows: $Y_t(d'_i)=F(d_i)$ for all $1 \le i \le t$ and $Y_t(d_j)=F(d_j)$ for all $t+1 \le j \le k$, where $0 \le t \le h$. Clearly, $Y_0=F$ and $Y_h=Y$. It suffices to show that for all $1 \le t \le h$, if Y_{t-1} is minimal, then Y_t is minimal.

Suppose conversely that Y_{t-1} is minimal and W: $\{d'_1, d'_2, \ldots, d'_t, d_{t+1}, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is a routing function with $V_W < V_{Y_t}$. Define another routing function P:$\{d'_1, d'_2, \ldots, d'_{t-1}, d_t, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ as follows: $P(d'_i)=W(d'_i)$ for all $1 \le i \le t-1$, $P(d_t)=W(d'_t)$, and $P(d_j)=W(d_j)$ for all $t+1 \le j \le k$. Suppose $V_{Y_{t-1}} = (v_1, v_2, \ldots, v_k)$ and $V_W=(v'_1, v'_2, \ldots, v'_k)$. Then, we have $V_{Y_t} = (v_1, v_2, \ldots, v_{|d_t|-1}, v_{|d_t|}-1, v_{|d_t|+1}, \ldots, v_k)$, because $d_{t,Y_{t-1}(d_t)} = d_{t,F(d_t)} = 0$ and $d'_{t,Y_t(d'_t)} = d'_{t,F(d_t)} = 1$, and $V_P \le (v'_1, v'_2, \ldots, v'_{|d_t|-1}, v'_{|d_t|}+1, v'_{|d_t|+1}, \ldots, v'_k)$

for the following reason. If $d_{t,P(d_t)}=1$, then $V_P \le V_W < (v'_1, v'_2, \ldots, v'_{|d_t|-1}, v'_{|d_t|}+1, v'_{|d_t|+1}, \ldots, v'_k)$. If $d_{t,P(d_t)}=0$, then $V_P \le (v'_1, v'_2, \ldots, v'_{|d_t|-1}, v'_{|d_t|}+1, v'_{|d_t|+1}, \ldots, v'_k)$.

Since $V_W=(v'_1, v'_2, \ldots, v'_k)<(v_1, v_2, \ldots, v_{|d_t|-1}, v_{|d_t|}-1, v_{|d_t|+1}, \ldots, v_k)=V_{Y_t}$, we have $V_P \le (v'_1, v'_2, \ldots, v'_{|d_t|-1}, v'_{|d_t|}+1, v'_{|d_t|+1}, \ldots, v'_k)<(v_1, v_2, \ldots, v_{|d_t|-1}, (v_{|d_t|}-1)+1, v_{|d_t|+1}, \ldots, v_k)=(v_1, v_2, \ldots, v_k)=V_{Y_{t-1}}$. This is a contradiction because Y_{t-1} is minimal. □

Lemma 13. Suppose that $d_{i,F(d_i)}=0$ and d'_i is a node of a k-fcube with $d'_{i,F(d_i)}=1$ for all $1 \le i \le h$, where $1 \le h \le k$. If $|d_i| \ge \lceil k/2 \rceil$ and either $d'_i=\overline{d_i}$ or d'_i is an adjacent node of $\overline{d_i}$ with $|d'_i|=|\overline{d_i}|-1$ for all $1 \le i \le h$, then each of $d_1, d_2, \ldots, d_h$ is not included internally in the k disjoint paths that are produced by executing *Paths1*$(Y, k, k, \{d'_1, d'_2, \ldots, d'_h, d_{h+1}, \ldots, d_k\}, \{1, 2, \ldots, k\})$, where $Y: \{d'_1, d'_2, \ldots, d'_h, d_{h+1}, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ is a minimal routing function defined as follows: $Y(d'_i)=F(d_i)$ for all $1 \le i \le h$ and $Y(d_j)=F(d_j)$ for all $h+1 \le j \le k$.

Proof. We first note that Y is minimal by Lemma 12. According to Lemma 8, *Paths1*$(Y, k, k, \{d'_1, d'_2, \ldots, d'_h, d_{h+1}, \ldots, d_k\}, \{1, 2, \ldots, k\})$ can produce k disjoint paths from s to $d'_1, d'_2, \ldots, d'_h, d_{h+1}, \ldots, d_k$ in a k-cube. For convenience, we use P'_i to denote the path to d'_i and P_j to denote the path to d_j, where $1 \le i \le h$ and $h+1 \le j \le k$.

Define a routing function $W: \{d'_1, d'_2, \ldots, d'_h, d_{h+1}, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ as follows: $W(d'_i)=u$ if e_u is contained in P'_i for all $1 \le i \le h$ and $W(d_j)=v$ if e_v is contained in P_j for all $h+1 \le j \le k$. We show $d'_{i,W(d'_i)}=d'_{i,Y(d'_i)}=1$ for all $1 \le i \le h$ and $d_{j,W(d_j)}=d_{j,Y(d_j)}$ for all $h+1 \le j \le k$ as follows. First, $d'_{i,Y(d'_i)}=d'_{i,F(d_i)}=1$. According to Lemma 8, P'_i has length $|d'_i|$, which means that P'_i is a shortest path from s to d'_i. Hence $d'_{i,W(d'_i)}=1$. Similarly, if $d_{j,Y(d_j)}=1$, then $d_{j,W(d_j)}=1$. On the other hand, if $d_{j,Y(d_j)}=0$, then $d_{j,W(d_j)}=0$, for otherwise ($d_{j,Y(d_j)}=0$ and $d_{j,W(d_j)}=1$) $V_W<V_Y$, which is a contradiction because Y is minimal.

Define another routing function $P: \{d_1, d_2, \ldots, d_k\} \to \{1, 2, \ldots, k\}$ as follows: $P(d_i)=W(d'_i)$ for all $1 \le i \le h$ and $P(d_j)=W(d_j)$ for all $h+1 \le j \le k$. P is minimal because $d_{w,P(d_w)}=d_{w,F(d_w)}$ for all $1 \le w \le k$, as explained below. Since $d'_{i,W(d'_i)}=1$ and either $d'_i=\overline{d_i}$ or d'_i is an adjacent node of $\overline{d_i}$ with $|d'_i|=|\overline{d_i}|-1$, we have $d_{i,P(d_i)}=d_{i,W(d'_i)}=0=d_{i,F(d_i)}$ for all $1 \le i \le h$. On the other hand, we have $d_{j,P(d_j)}=d_{j,W(d_j)}=d_{j,Y(d_j)}=d_{j,F(d_j)}$ for all $h+1 \le j \le k$.

It is time to show that $d_1, d_2, \ldots, d_h$ are not included internally in $P'_1, \ldots, P'_h, P_{h+1}, \ldots, P_k$. Since P'_i has length $|d'_i|\le\lfloor k/2\rfloor$ and $|d_i|\ge\lceil k/2\rceil$ for all $1\le i\le h$, $d_1, d_2, \ldots, d_h$ are not included internally in $P'_1, \ldots, P'_h$. On the other hand, suppose that d_r is included internally in P_l, where $1\le r\le h$ and $h+1\le l\le k$. That is, d_r is contained in the subpath of P_l that goes from s to t_l, where t_l is the immediate predecessor of d_l in P_l. By Lemma 8, the subpath is shortest, and hence $d_{r,W(d_l)}=1$. If $d_{l,F(d_l)}=0$, then by Lemma 3 ($d_{r,P(d_r)}=d_{r,F(d_r)}=0$, $d_{l,P(d_l)}=d_{l,W(d_l)}=d_{l,Y(d_l)}=d_{l,F(d_l)}=0$, and $d_{r,P(d_l)}=d_{r,W(d_l)}=1$), P is not minimal, which is a contradiction. If $d_{l,F(d_l)}=1$, then Lemma 8 implies $|d_r|<|d_l|$. By Lemma 5 ($d_{r,P(d_r)}=d_{r,F(d_r)}=0$, $d_{r,P(d_l)}=d_{r,W(d_l)}=1$, and $|d_r|<|d_l|$), P is not minimal, which is again a contradiction. Hence $d_1, d_2, \ldots, d_h$ are not included internally in $P_{h+1}, P_{h+2}, \ldots, P_k$. □

Lemma 14. Suppose that $d_{h,F(d_h)}=0$ and d_{k+1} is an arbitrary node of a k-fcube, where $1\le h\le k$. If $\overline{d_h}\in\{d_1, d_2, \ldots, d_k\}$ and $|d_h|=\lceil k/2\rceil$, then there is an adjacent node x of $\overline{d_h}$ so that $|x|=|\overline{d_h}|-1$, $x_{F(d_h)}=1$ or $x_{F(\overline{d_h})}=1$, $x\notin\{d_1, d_2, \ldots, d_k\}$, and $\overline{x}\notin\{d_1, d_2, \ldots, d_k\}\cup\{d_{k+1}\}$.

Proof. Let $q_1, q_2, \ldots, q_{\lfloor k/2\rfloor}$ be the $\lfloor k/2\rfloor$ adjacent nodes of $\overline{d_h}$ with $|q_j|=|\overline{d_h}|-1$ for all $1\le j\le\lfloor k/2\rfloor$, where $q_j= q_{j,1}q_{j,2}\ldots q_{j,k}$. We have $q_{j,F(d_h)}=1$ or $q_{j,F(\overline{d_h})}=1$ for all $1\le j\le\lfloor k/2\rfloor$, as explained below. Suppose $\overline{d_h}=d_l$ for some $1\le l\le k$ and $l\ne h$. We have $d_{l,F(d_l)}=1$ by Lemma 9. Since $\overline{d_h}_{,F(d_h)}=1$ and $\overline{d_h}_{,F(d_l)}=d_{l,F(d_l)}=1$, we have $q_{j,F(d_h)}=1$ or $q_{j,F(d_l)}=1$, because q_j and $\overline{d_h}$ are adjacent with $|q_j|=|\overline{d_h}|-1$.

Without loss of generality, we assume $d_{h,F(d_1)}=d_{h,F(d_2)}=\cdots=d_{h,F(d_{\lceil k/2\rceil})}=1$ and $d_{h,F(d_{\lceil k/2\rceil+1})}=d_{h,F(d_{\lceil k/2\rceil+2})}=\cdots=d_{h,F(d_k)}=0$. We show $\{d_1, d_2, \ldots, d_{\lceil k/2\rceil}, d_h, \overline{d_h}\}\cap\{q_1, q_2, \ldots, q_{\lfloor k/2\rfloor}, \overline{q_1}, \overline{q_2}, \ldots, \overline{q_{\lfloor k/2\rfloor}}\}$ empty by showing $d_{i,F(d_i)}=1\ne 0=q_{j,F(d_i)}$, $|d_i|<|\overline{q_j}|$, and $|q_j|<|\overline{d_h}|\le|d_h|<|\overline{q_j}|$ for all $1\le i\le\lceil k/2\rceil$ and $1\le j\le\lfloor k/2\rfloor$ as follows. We have $|d_i|\le|d_h|$ and $d_{i,F(d_i)}=1$ by Lemma 11. Since $\overline{d_h}_{,F(d_i)}=0$, we have $q_{j,F(d_i)}=0$, because q_j and $\overline{d_h}$ are adjacent with $|q_j|=|\overline{d_h}|-1$. Since $|d_h|=\lceil k/2\rceil$, $|\overline{d_h}|=\lfloor k/2\rfloor$, $|q_j|=\lfloor k/2\rfloor-1$, and $|\overline{q_j}|=\lceil k/2\rceil+1$, we have $|d_i|\le|d_h|<|\overline{q_j}|$ and $|q_j|<|\overline{d_h}|\le|d_h|<|\overline{q_j}|$.

Moreover, we show $\{d_h, \overline{d_h}\}\subseteq\{d_{\lceil k/2\rceil+1}, d_{\lceil k/2\rceil+2}, \ldots, d_k\}$ as follows. Since $d_{h,F(d_h)}=0$ and $d_{i,F(d_i)}=1$ for all $1\le i\le\lceil k/2\rceil$, we have $h\notin\{1, 2, \ldots, \lceil k/2\rceil\}$. Hence, $d_h\in\{d_{\lceil k/2\rceil+1}, d_{\lceil k/2\rceil+2}, \ldots, d_k\}$. On the other hand, since $\overline{d_h}_{,F(d_i)}=0\ne 1=d_{i,F(d_i)}$ for all $1\le i\le\lceil k/2\rceil$, we have $\overline{d_h}\notin\{d_1, d_2, \ldots, d_{\lceil k/2\rceil}\}$, which means $\overline{d_h} \in \{ d_{\lceil k/2 \rceil+1},\ d_{\lceil k/2 \rceil+2}, \ldots, d_k\}$.

According to the discussion above, x can be determined as $q_w \in \{q_1, q_2, \ldots, q_{\lfloor k/2 \rfloor}\}$, provided $q_w \notin \{d_1, d_2, \ldots, d_k\}$ and $\overline{q_w} \notin \{d_1, d_2, \ldots, d_k, d_{k+1}\}$. Since $\{d_1, d_2, \ldots, d_{\lceil k/2 \rceil}, d_h, \overline{d_h}\} \cap \{q_1, q_2, \ldots, q_{\lfloor k/2 \rfloor}, \overline{q_1}, \overline{q_2}, \ldots, \overline{q_{\lfloor k/2 \rfloor}}\}$ is empty and $\{d_h, \overline{d_h}\} \subseteq \{ d_{\lceil k/2 \rceil+1}, d_{\lceil k/2 \rceil+2}, \ldots, d_k\}$, it suffices to show $q_w \notin \{ d_{\lceil k/2 \rceil+1}, d_{\lceil k/2 \rceil+2}, \ldots, d_k\}-\{d_h, \overline{d_h}\}$ and $\overline{q_w} \notin \{ d_{\lceil k/2 \rceil+1}, d_{\lceil k/2 \rceil+2}, \ldots, d_k, d_{k+1}\}-\{d_h, \overline{d_h}\}$ for some $1 \le w \le \lfloor k/2 \rfloor$. We observe that $q_1, q_2, \ldots, q_{\lfloor k/2 \rfloor}, \overline{q_1}, \overline{q_2}, \ldots, \overline{q_{\lfloor k/2 \rfloor}}$ are all distinct and $|\{ d_{\lceil k/2 \rceil+1}, d_{\lceil k/2 \rceil+2}, \ldots, d_k, d_{k+1}\}-\{d_h, \overline{d_h}\}| = \lfloor k/2 \rfloor - 1$. Hence, there exists $\{q_w, \overline{q_w}\} \in \{\{q_1, \overline{q_1}\}, \{q_2, \overline{q_2}\}, \ldots, \{q_{\lfloor k/2 \rfloor}, \overline{q_{\lfloor k/2 \rfloor}}\}\}$ so that $q_w \notin \{ d_{\lceil k/2 \rceil+1}, d_{\lceil k/2 \rceil+2}, \ldots, d_k\}-\{d_h, \overline{d_h}\}$ and $\overline{q_w} \notin \{ d_{\lceil k/2 \rceil+1}, d_{\lceil k/2 \rceil+2}, \ldots, d_k, d_{k+1}\}-\{d_h, \overline{d_h}\}$. □

Lemma 15. Suppose $d_{h,F(d_h)} = 0$, where $1 \le h \le k$. If $|d_h| = \lfloor k/2 \rfloor - 2$, then there is an adjacent node x of d_h so that $|x| = |d_h|+1$, $x \notin \{d_1, d_2, \ldots, d_k\}$, $\overline{x} \notin \{d_1, d_2, \ldots, d_k\}$, and $x_w=1$ for some $w \in \{1, 2, \ldots, k\} - \{F(z) \mid z = z_1z_2\ldots z_k \in \{d_1, d_2, \ldots, d_k\},\ z_{F(z)} = 1, \text{ and } |z| \le \lfloor k/2 \rfloor - 1\}$.

Proof. We use H to denote the set $\{F(z) \mid z \in \{d_1, d_2, \ldots, d_k\},\ z_{F(z)} = 1, \text{ and } |z| \le \lfloor k/2 \rfloor - 1\}$. Without loss of generality, assume $H = \{F(d_1), F(d_2), \ldots, F(d_{|H|})\}$ and $\{1, 2, \ldots, k\} - H = \{t_1, t_2, \ldots, t_{k-|H|}\}$. Then we have $d_{h,t_j} = 0$ for all $1 \le j \le k-|H|$, as explained below. Suppose conversely $d_{h,t_l} = 1$ for some $1 \le l \le k-|H|$. We further assume $t_l = F(d_r)$, where $|H|+1 \le r \le k$. Since $F(d_r) \notin H$, either $|d_r| > \lfloor k/2 \rfloor - 1$ or $|d_r| \le \lfloor k/2 \rfloor - 1$ and $d_{r,F(d_r)} = 0$ holds. If $|d_r| > \lfloor k/2 \rfloor - 1$, then by Lemma 5 ($d_{h,F(d_h)} = 0$, $d_{h,F(d_r)} = d_{h,t_l} = 1$, and $|d_h| = \lfloor k/2 \rfloor - 2 < \lfloor k/2 \rfloor - 1 < |d_r|$) F is not minimal, which is a contradiction. Otherwise ($|d_r| \le \lfloor k/2 \rfloor - 1$ and $d_{r,F(d_r)} = 0$), if $r=h$, then $d_{h,F(d_h)} = d_{h,F(d_r)} = d_{h,t_l} = 1$, which contradicts to $d_{h,F(d_h)} = 0$. If $r \ne h$, then by Lemma 3 ($d_{h,F(d_h)} = 0$, $d_{r,F(d_r)} = 0$, and $d_{h,F(d_r)} = d_{h,t_l} = 1$) F is not minimal, which is again a contradiction.

Define $d_h^{(t_j)} = d_{h,1}d_{h,2}\ldots d_{h,t_j-1}\, 1\, d_{h,t_j+1} \ldots d_{h,k}$ for all $1 \le j \le k-|H|$ (we have shown $d_{h,t_j} = 0$ for all $1 \le j \le k-|H|$ above). We show $d_h^{(t_j)} \notin \{d_1, d_2, \ldots, d_k\}$ as follows. Suppose conversely $d_h^{(t_v)} = d_u$ for some $1 \le v \le k-|H|$ and $1 \le u \le k$. We have $d_{u,F(d_u)} = 0$ for the following reason. If $d_{u,F(d_u)} = 1$, then $F(d_u) \in H$ because $|d_u| = |d_h|+1 = \lfloor k/2 \rfloor - 1$. Since $t_v \notin H$, we have $F(d_u) \ne t_v$. Hence $d_{h,F(d_u)} = d_{u,F(d_u)} = 1$. By Lemma 5 ($d_{h,F(d_h)} = 0$, $d_{h,F(d_u)} = 1$, and $|d_h| < |d_u|$), F is not minimal, which is a contradiction.

We further assume $t_v=F(d_q)$, where $|H|+1\leq q\leq k$. Since $F(d_q)\notin H$, either $|d_q|>\lfloor k/2\rfloor-1$ or $|d_q|\leq\lfloor k/2\rfloor-1$ and $d_{q,F(d_q)}=0$ holds. If $|d_q|>\lfloor k/2\rfloor-1$, then by Lemma 5 ($d_{u,F(d_u)}=0$, $d_{u,F(d_q)}=d_{u,t_v}=d_{h,t_v}^{(t_v)}=1$, and $|d_u|=|d_h|+1=\lfloor k/2\rfloor-1<|d_q|$) F is not minimal, which is a contradiction. Otherwise ($|d_q|\leq\lfloor k/2\rfloor-1$ and $d_{q,F(d_q)}=0$), if $q=u$, then $d_{u,F(d_u)}=d_{u,F(d_q)}=d_{u,t_v}=d_{h,t_v}^{(t_v)}=1$, which contradicts to $d_{u,F(d_u)}=0$. If $q\neq u$, then by Lemma 3 ($d_{u,F(d_u)}=0$, $d_{q,F(d_q)}=0$, and $d_{u,F(d_q)}=d_{u,t_v}=d_{h,t_v}^{(t_v)}=1$) F is not minimal, which is again a contradiction. Hence we have $d_h^{(t_j)}\notin\{d_1, d_2, \ldots, d_k\}$ for all $1\leq j\leq k-|H|$.

Since $|d_h^{(t_j)}|=|d_h|+1$, $d_{h,t_j}^{(t_j)}=1$, and $d_h^{(t_j)}\notin\{d_1, d_2, \ldots, d_k\}$ for all $1\leq j\leq k-|H|$, x can be determined as $d_h^{(t_a)}\in\{d_h^{(t_1)}, d_h^{(t_2)}, \ldots, d_h^{(t_{k-|H|})}\}$, provided $\overline{d_h^{(t_a)}}\notin\{d_1, d_2, \ldots, d_k\}$. We have $\overline{d_h^{(t_j)}}\notin\{d_h, d_1, d_2, \ldots, d_{|H|}\}$ for all $1\leq j\leq k-|H|$, because $|\overline{d_h^{(t_j)}}|=\lceil k/2\rceil+1>\lfloor k/2\rfloor-2=|d_h|$ and $|\overline{d_h^{(t_j)}}|=\lceil k/2\rceil+1>\lfloor k/2\rfloor-1\geq|d_i|$ for all $1\leq i\leq|H|$. Hence we only need to show that there exists $\overline{d_h^{(t_a)}}\notin\{d_1, d_2, \ldots, d_k\}-\{d_h, d_1, d_2, \ldots, d_{|H|}\}$ for some $1\leq a\leq k-|H|$. Since $d_{h,F(d_h)}=0$ and $d_{i,F(d_i)}=1$ for all $1\leq i\leq|H|$, we have $h\notin\{1, 2, \ldots, |H|\}$. Hence, $d_h\in\{d_{|H|+1}, d_{|H|+2}, \ldots, d_k\}$, which means $|\{d_1, d_2, \ldots, d_k\}-\{d_h, d_1, d_2, \ldots, d_{|H|}\}|=k-|H|-1$. Since $\overline{d_h^{(t_1)}}$, $\overline{d_h^{(t_2)}}$, …, $\overline{d_h^{(t_{k-|H|})}}$ are all distinct, there exists $\overline{d_h^{(t_a)}}\notin\{d_1, d_2, \ldots, d_k\}-\{d_h, d_1, d_2, \ldots, d_{|H|}\}$ for some $1\leq a\leq k-|H|$. □

Recall that we obtained a set $D=\{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ in Case 5 of *Paths2*(s, d_1, d_2, …, d_{k+1}) (refer to the paragraph enclosed with /* and */ at the beginning of Case 5), where $0\leq r'\leq u'\leq k$. Suppose G: $D\rightarrow\{1, 2, \ldots, k\}$ is a minimal routing function. The following lemma shows $q_{i,G(q_i)}=1$ for all $1\leq i\leq r'$.

Lemma 16. Suppose that $D=\{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ results finally in Case 5 of *Paths2*(s, d_1, d_2, …, d_{k+1}) and G: $D\rightarrow\{1, 2, \ldots, k\}$ is a minimal routing function, where $0\leq r'\leq u'\leq k$. Then $q_{i,G(q_i)}=1$ for all $1\leq i\leq r'$.

Proof. Recall that in Case 5 of *Paths2*(s, d_1, d_2, …, d_{k+1}), we let $D=\{\overline{d_1}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ initially, then modify D iteratively, and obtain $D=\{q_1, \ldots, q_{r'}, \overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ finally. In each iteration, $\overline{d_h}$ in D was replaced with q_h for some $1\leq h\leq r'$, and a new minimal routing function W was determined. Without loss of generality, we assume $D=\{q_1, \ldots, q_t, \overline{d_{t+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ at the end of the tth iteration, and we use W_t: $\{q_1, \ldots, q_t, \overline{d_{t+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\} \rightarrow \{1, 2, \ldots, k\}$ to denote the W obtained in the tth iteration, where $0 \le t \le r'$ and W_0 denotes the initial W. Clearly, $V_{W_{r'}} = V_G$. Suppose $V_{W_0} = (v_1, v_2, \ldots, v_k)$ and $V_G = (v_1^G, v_2^G, \ldots, v_k^G)$. We show below that if $v_c^G = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, $v_{\lfloor k/2 \rfloor - 2}^G = v_{\lfloor k/2 \rfloor - 2} - r'$, and $v_{\lfloor k/2 \rfloor - 1}^G = v_{\lfloor k/2 \rfloor - 1}$, then $q_{i, G(q_i)} = 1$ for all $1 \le i \le r'$.

Define a routing function P: $\{\overline{d_1}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\} \rightarrow \{1, 2, \ldots, k\}$ as follows: $P(\overline{d_i}) = G(q_i)$ for all $1 \le i \le r'$, $P(\overline{d_j}) = G(\overline{d_j})$ for all $r'+1 \le j \le u'$, and $P(d_p) = G(d_p)$ for all $u'+1 \le p \le k$. Suppose $V_P = (v_1^P, v_2^P, \ldots, v_k^P)$. We have $v_c^P = v_c^G$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, because $|\overline{d_i}| = \lfloor k/2 \rfloor - 2$, $|q_i| = \lfloor k/2 \rfloor - 1$ for all $1 \le i \le r'$, and $x_{P(x)} = x_{G(x)}$ for all $x \in \{\overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$. Moreover, we have $v_{\lfloor k/2 \rfloor - 2}^P = v_{\lfloor k/2 \rfloor - 2}^G + a$ and $v_{\lfloor k/2 \rfloor - 1}^P = v_{\lfloor k/2 \rfloor - 1}^G - b$, where $0 \le a = \left|\{\overline{d_i} \mid \overline{d}_{i, P(\overline{d_i})} = 0 \text{ and } 1 \le i \le r'\}\right| \le r'$ and $0 \le b = \left|\{q_i \mid q_{i, G(q_i)} = 0 \text{ and } 1 \le i \le r'\}\right| \le r'$. If $v_c^G = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, $v_{\lfloor k/2 \rfloor - 2}^G = v_{\lfloor k/2 \rfloor - 2} - r'$, and $v_{\lfloor k/2 \rfloor - 1}^G = v_{\lfloor k/2 \rfloor - 1}$, then $v_c^P = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, $v_{\lfloor k/2 \rfloor - 2}^P = v_{\lfloor k/2 \rfloor - 2} - r' + a$, and $v_{\lfloor k/2 \rfloor - 1}^P = v_{\lfloor k/2 \rfloor - 1} - b$. Since W_0 is minimal, either $v_{\lfloor k/2 \rfloor - 2} < v_{\lfloor k/2 \rfloor - 2}^P$ or $v_{\lfloor k/2 \rfloor - 2} = v_{\lfloor k/2 \rfloor - 2}^P$ and $v_{\lfloor k/2 \rfloor - 1} \le v_{\lfloor k/2 \rfloor - 1}^P$ holds, which implies $a = r'$ and $b = 0$. Hence we have $q_{i, G(q_i)} = 1$ for all $1 \le i \le r'$, as a consequence of $b = 0$.

In the rest of the proof, we show $v_c^G = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, $v_{\lfloor k/2 \rfloor - 2}^G = v_{\lfloor k/2 \rfloor - 2} - r'$, and $v_{\lfloor k/2 \rfloor - 1}^G = v_{\lfloor k/2 \rfloor - 1}$. Suppose $V_{W_i} = (v_1^i, v_2^i, \ldots, v_k^i)$, where $1 \le i \le r'$. We have $V_{W_0} = (v_1^0, v_2^0, \ldots, v_k^0) = (v_1, v_2, \ldots, v_k)$, and $V_{W_{r'}} = (v_1^{r'}, v_2^{r'}, \ldots, v_k^{r'}) = (v_1^G, v_2^G, \ldots, v_k^G) = V_G$. It suffices to show that if $v_c^{t-1} = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, $v_{\lfloor k/2 \rfloor - 2}^{t-1} = v_{\lfloor k/2 \rfloor - 2} - (t-1)$, and $v_{\lfloor k/2 \rfloor - 1}^{t-1} = v_{\lfloor k/2 \rfloor - 1}$, then $v_c^t = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, $v_{\lfloor k/2 \rfloor - 2}^t = v_{\lfloor k/2 \rfloor - 2} - t$, and $v_{\lfloor k/2 \rfloor - 1}^t = v_{\lfloor k/2 \rfloor - 1}$, where $1 \le t \le r'$. Suppose $v_c^{t-1} = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, $v_{\lfloor k/2 \rfloor - 2}^{t-1} = v_{\lfloor k/2 \rfloor - 2} - (t-1)$, and $v_{\lfloor k/2 \rfloor - 1}^{t-1} = v_{\lfloor k/2 \rfloor - 1}$. The execution of the tth iteration is considered below.

We have $D = \{q_1, \ldots, q_{t-1}, \overline{d_t}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ at the beginning, and $|\overline{d_t}| = \lfloor k/2 \rfloor - 2$, $\overline{d_t} = d_l$ for some $u'+1 \le l \le k$, and $\overline{d}_{t, W_{t-1}(\overline{d_t})} = 0$ or $\overline{d}_{t, W_{t-1}(d_l)} = 0$. If $\overline{d}_{t, W_{t-1}(\overline{d_t})} = 1$, then swap $W_{t-1}(\overline{d_t})$ and $W_{t-1}(d_l)$ (hence, we have $\overline{d}_{t, W_{t-1}(\overline{d_t})} = 0$, eventually). Since $\overline{d_t} = d_l$, the resulting W_{t-1} remains minimal. There exists an adjacent node q_t of $\overline{d_t}$ so that $q_t \notin D$, $\overline{q_t} \notin D$, and for some $w \in \{1, 2, \ldots, k\} - \{W_{t-1}(z) \mid z \in D,\ z_{W_{t-1}(z)} = 1,$ and $|z| \le \lfloor k/2 \rfloor - 1\}$, $q_{t,w} = 1$, $\overline{d}_{t,w} = 0$, and $q_{t,i} = \overline{d}_{t,i}$ for all $1 \le i \le k$ and $i \ne w$ (hence, $|q_t| = |\overline{d_t}| + 1$), according to

Lemma 15 (substituting W_{t-1}, $\overline{d_t}$, and q_t for F, d_h, and x, respectively.).

Suppose $w=W_{t-1}(g)$, where $g=g_1g_2\ldots g_k \in\{q_1, \ldots, q_{t-1}, \overline{d_t}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$. Define a routing function Y: $\{q_1, \ldots, q_t, \overline{d_{t+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}\to\{1, 2, \ldots, k\}$ as follows: $Y(q_t)=w$ and $Y(y)=W_{t-1}(y)$ for all $y\in\{q_1, \ldots, q_{t-1}, \overline{d_{t+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$ if $W_{t-1}(\overline{d_t})=W_{t-1}(g)$, and $Y(q_t)=w$, $Y(g)=W_{t-1}(\overline{d_t})$, and $Y(y)=W_{t-1}(y)$ for all $y\in\{q_1, \ldots, q_{t-1}, \overline{d_{t+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}-\{g\}$ if $W_{t-1}(\overline{d_t})\neq W_{t-1}(g)$. Suppose $V_Y=(v_1^Y, v_2^Y, \ldots, v_k^Y)$. We show $v_c^Y=v_c$ for all $1\le c\le\lfloor k/2\rfloor-3$, $v_{\lfloor k/2\rfloor-2}^Y=v_{\lfloor k/2\rfloor-2}-t$, and $v_{\lfloor k/2\rfloor-1}^Y=v_{\lfloor k/2\rfloor-1}$ as follows.

If $W_{t-1}(\overline{d_t})=W_{t-1}(g)$, then we have $v_c^Y=v_c^{t-1}$ for all $1\le c\le\lfloor k/2\rfloor-3$, because $|\overline{d_t}|=\lfloor k/2\rfloor-2$ and $|q_t|=\lfloor k/2\rfloor-1$. Moreover, we have $v_{\lfloor k/2\rfloor-2}^Y=v_{\lfloor k/2\rfloor-2}^{t-1}-1$ and $v_{\lfloor k/2\rfloor-1}^Y=v_{\lfloor k/2\rfloor-1}^{t-1}$, because $q_{t,Y(q_t)}=q_{t,w}=1$ and $\overline{d}_{t,W_{t-1}(\overline{d_t})}=0$. On the other hand, if $W_{t-1}(\overline{d_t})\neq W_{t-1}(g)$, then $g\in\{q_1, \ldots, q_{t-1}, \overline{d_{t+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$. Since $W_{t-1}(g)=w$, either $|g|>\lfloor k/2\rfloor-1$ or $|g|\le\lfloor k/2\rfloor-1$ and $g_{W_{t-1}(g)}=0$ holds. When $|g|>\lfloor k/2\rfloor-1$, we have $v_c^Y=v_c^{t-1}$ for all $1\le c\le\lfloor k/2\rfloor-3$, $v_{\lfloor k/2\rfloor-2}^Y=v_{\lfloor k/2\rfloor-2}^{t-1}-1$, and $v_{\lfloor k/2\rfloor-1}^Y=v_{\lfloor k/2\rfloor-1}^{t-1}$, similarly. When $|g|\le\lfloor k/2\rfloor-1$ and $g_{W_{t-1}(g)}=0$, we have $v_c^Y=v_c^{t-1}$ for all $1\le c\le\lfloor k/2\rfloor-3$, $v_{\lfloor k/2\rfloor-2}^Y=v_{\lfloor k/2\rfloor-2}^{t-1}-1$, and $v_{\lfloor k/2\rfloor-1}^Y=v_{\lfloor k/2\rfloor-1}^{t-1}$, provided $g_{Y(g)}=g_{W_{t-1}(g)}$. In what follows, we show $g_{Y(g)}=g_{W_{t-1}(g)}$.

For convenience, we use $f_1, \ldots, f_{t-1}, f_t, \ldots, f_{u'}, f_{u'+1}, \ldots, f_k$ to represent $q_1, \ldots, q_{t-1}, \overline{d_t}, \ldots, \overline{d_{u'}}$, $d_{u'+1}, \ldots, d_k$, respectively, where $f_m=f_{m,1}f_{m,2}\ldots f_{m,k}$ for all $1\le m\le k$. We assume $W_{t-1}(g)=W_{t-1}(f_h)$, where $1\le h\le k$ and $h\neq t$ (note that $W_{t-1}(g)=W_{t-1}(f_h)$ assures $g=f_h$, but, $g=f_h$ does not imply $W_{t-1}(g)=W_{t-1}(f_h)$ because $\{f_1, f_2, \ldots, f_k\}$ is a multiset). Since $f_{h,W_{t-1}(f_h)}=g_{W_{t-1}(f_h)}=g_{W_{t-1}(g)}=0$ and $f_{t,W_{t-1}(f_t)}=\overline{d}_{t,W_{t-1}(\overline{d_t})}=0$, we have $f_{h,W_{t-1}(f_t)}=0$, for otherwise W_{t-1} is not minimal by Lemma 3 ($f_{h,W_{t-1}(f_h)}=0$, $f_{t,W_{t-1}(f_t)}=0$, and $f_{h,W_{t-1}(f_t)}=1$), which is a contradiction. Hence, $g_{Y(g)}=g_{W_{t-1}(\overline{d_t})}=f_{h,W_{t-1}(\overline{d_t})}=f_{h,W_{t-1}(f_t)}=0=g_{W_{t-1}(g)}$. Consequently, we have $v_c^Y=v_c$ for all $1\le c\le\lfloor k/2\rfloor-3$, $v_{\lfloor k/2\rfloor-2}^Y=v_{\lfloor k/2\rfloor-2}-t$, and $v_{\lfloor k/2\rfloor-1}^Y=v_{\lfloor k/2\rfloor-1}$.

It is time to show $v_c^t=v_c$ for all $1\le c\le\lfloor k/2\rfloor-3$, $v_{\lfloor k/2\rfloor-2}^t=v_{\lfloor k/2\rfloor-2}-t$, and $v_{\lfloor k/2\rfloor-1}^t=v_{\lfloor k/2\rfloor-1}$. Define a routing function Q: $\{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}\to\{1, 2, \ldots, k\}$ as follows: $Q(\overline{d_i})=W_t(q_i)$ for all $1\le i\le t$, $Q(\overline{d_j})=W_t(\overline{d_j})$ for all $t+1\le j\le u'$, and $Q(d_p)=W_t(d_p)$ for all $u'+1\le p\le k$. Suppose $V_Q=(v_1^Q, v_2^Q, \ldots, v_k^Q)$. Since $|\overline{d_i}|=\lfloor k/2\rfloor-2$ and $|q_i|=\lfloor k/2\rfloor-1$ for all $1\le i\le t$, we have $v_c^Q=v_c^t$ for all $1\le c\le\lfloor k/2\rfloor-3$,

$v^{Q}_{\lfloor k/2 \rfloor-2} = v^{t}_{\lfloor k/2 \rfloor-2} + j$, and $v^{Q}_{\lfloor k/2 \rfloor-1} = v^{t}_{\lfloor k/2 \rfloor-1} - d$, where $j = \left|\{\, \overline{d_i} \mid \overline{d}_{i,Q(\overline{d_i})} = 0 \text{ and } 1 \le i \le t\}\right|$ and $d = \left|\{q_i \mid q_{i,W_t(q_i)} = 0 \text{ and } 1 \le i \le t\}\right|$. If $v^t_1 = v_1$, $v^t_2 = v_2$, …, $v^t_{h-1} = v_{h-1}$, $v^t_h < v_h$ for some $1 \le h \le \lfloor k/2 \rfloor - 3$, then $V_Q < V_{W_0}$, which is a contradiction because W_0 is minimal. On the other hand, if $v^t_1 = v_1$, $v^t_2 = v_2$, …, $v^t_{l-1} = v_{l-1}$, $v^t_l > v_l$ for some $1 \le l \le \lfloor k/2 \rfloor - 3$, then we have $v^t_1 = v^Y_1$, $v^t_2 = v^Y_2$, …, $v^t_{l-1} = v^Y_{l-1}$, $v^t_l > v^Y_l$, which implies $V_Y < V_{W_t}$. This is also a contradiction because W_t is minimal. Hence we have $v^t_c = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$.

Similarly, if $v^t_{\lfloor k/2 \rfloor-2} < v_{\lfloor k/2 \rfloor-2} - t$, then $v^Q_{\lfloor k/2 \rfloor-2} = v^t_{\lfloor k/2 \rfloor-2} + j < (v_{\lfloor k/2 \rfloor-2} - t) + j \le v_{\lfloor k/2 \rfloor-2}$, which implies $V_Q < V_{W_0}$, a contradiction, because $v^Q_c = v^t_c = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$. If $v^t_{\lfloor k/2 \rfloor-2} > v_{\lfloor k/2 \rfloor-2} - t$, then $v^Y_{\lfloor k/2 \rfloor-2} = v_{\lfloor k/2 \rfloor-2} - t < v^t_{\lfloor k/2 \rfloor-2}$, which implies $V_Y < V_{W_t}$, a contradiction, because $v^Y_c = v_c = v^t_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$. Hence we have $v^t_{\lfloor k/2 \rfloor-2} = v_{\lfloor k/2 \rfloor-2} - t$.

Finally, if $v^t_{\lfloor k/2 \rfloor-1} < v_{\lfloor k/2 \rfloor-1}$, then $v^Q_{\lfloor k/2 \rfloor-1} = v^t_{\lfloor k/2 \rfloor-1} - d < v_{\lfloor k/2 \rfloor-1}$. Since $v^Q_{\lfloor k/2 \rfloor-2} = v^t_{\lfloor k/2 \rfloor-2} + j = v_{\lfloor k/2 \rfloor-2} - t + j$, we have $j = t$ because W_0 is minimal. Then, $v^Q_{\lfloor k/2 \rfloor-2} = v^t_{\lfloor k/2 \rfloor-2} + j = v_{\lfloor k/2 \rfloor-2} - t + j = v_{\lfloor k/2 \rfloor-2}$. Since $v^Q_c = v^t_c = v_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$, we have $V_Q < V_{W_0}$, a contradiction. If $v^t_{\lfloor k/2 \rfloor-1} > v_{\lfloor k/2 \rfloor-1}$, then $v^Y_{\lfloor k/2 \rfloor-1} = v_{\lfloor k/2 \rfloor-1} < v^t_{\lfloor k/2 \rfloor-1}$, which implies $V_Y < V_{W_t}$, a contradiction, because $v^Y_{\lfloor k/2 \rfloor-2} = v_{\lfloor k/2 \rfloor-2} - t = v^t_{\lfloor k/2 \rfloor-2}$ and $v^Y_c = v_c = v^t_c$ for all $1 \le c \le \lfloor k/2 \rfloor - 3$. Hence we have $v^t_{\lfloor k/2 \rfloor-1} = v_{\lfloor k/2 \rfloor-1}$. □

The following lemma will be proved in Section 3.2.2.

Lemma 17. Suppose that s, d_1, d_2, …, d_{k+1} are k+2 distinct nodes of a k-fcube. Procedure *Paths2*(s, d_1, d_2, …, d_{k+1}) can produce k+1 disjoint paths from s to d_1, d_2, …, d_{k+1} in the k-fcube whose lengths do not exceed $\lceil k/2 \rceil + 1$.

3.2.2 The proof of Lemma 17

There are two steps, i.e., Step 1 and Step 2, in *Paths2*(s, d_1, d_2, …, d_{k+1}). Step 1 is easy and no further explanation is needed. Step 2 constructs k+1 disjoint paths, denoted by Q_1, Q_2, …, Q_{k+1}, according to five cases. In the rest of this section, we show that Q_1, Q_2, …, Q_{k+1} are disjoint and have maximal length not greater than $\lceil k/2 \rceil + 1$.

3.2.2.1 Case 1: $|d_{k+1}| \le \lceil k/2 \rceil - 2$

$Q_1, Q_2, \ldots, Q_k$ were first obtained by executing *Paths1*$(F, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$. Lemma 8 assures their disjoint property. Besides, since $|d_i|\le|d_{k+1}|\le\lceil k/2\rceil-2$ for all $1\le i\le k$ as a consequence of Step 1, each Q_i has length at most $|d_i|+2\le\lceil k/2\rceil$. Then Q_{k+1} was constructed and Q_l was changed, depending on whether $d_{k+1}\in Q_l$ for some $1\le l\le k$ or not.

When $d_{k+1}\in Q_l$, Q_{k+1} was constructed as the subpath of Q_l from s to d_{k+1}, and then Q_l was reconstructed as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$. Q_{k+1} has length at most $\lceil k/2\rceil-1$, and (new) Q_l has length $(k-|\overline{d_l}|)+2\le\lceil k/2\rceil$. Q_{k+1} is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$. Suppose that a is a node in $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_{k+1}$ and $b\notin\{s, d_l\}$ is a node in (new) Q_l. (New) Q_l is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_{k+1}$, provided $a\ne b$. We have $a\ne b$ because $|a|\le|d_{k+1}|+1\le\lceil k/2\rceil-1<|\overline{d_l}|\le|b|$.

When $d_{k+1}\notin Q_i$ for all $1\le i\le k$, Q_{k+1} was constructed as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_{k+1}}$ in a k-cube, and a link $(\overline{d_{k+1}}, d_{k+1})$. Q_{k+1} has length $(k-|\overline{d_{k+1}}|)+2\le\lceil k/2\rceil$. Suppose that g is a node in $Q_1, Q_2, \ldots, Q_k$ and $h\notin\{s, d_{k+1}\}$ is a node in Q_{k+1}. Q_{k+1} is disjoint with $Q_1, Q_2, \ldots, Q_k$ because $|g|\le|d_{k+1}|+1\le\lceil k/2\rceil-1<|\overline{d_{k+1}}|\le|h|$.

3.2.2.2 Case 2: $|d_{k+1}|=\lceil k/2\rceil-1$

$Q_1, Q_2, \ldots, Q_{k+1}$ were obtained, depending on whether k is even or odd. If k is even, then $Q_1, Q_2, \ldots, Q_{k+1}$ were constructed all the same as in Case 1. Similarly, they are disjoint to each other, and their maximal length is at most $|d_{k+1}|+2=\lceil k/2\rceil+1$. If k is odd, then $Q_1, Q_2, \ldots, Q_k$ were obtained by executing *Paths1*$(F, k, k, \{d_1, d_2, \ldots, d_k\}, \{1, 2, \ldots, k\})$. By Lemma 8, they are disjoint to each other, and their maximal length is at most $|d_{k+1}|+2=\lceil k/2\rceil+1$. Q_{k+1} was constructed, depending on whether $d_{k+1}\in Q_l$ for some $1\le l\le k$ or not.

When $d_{k+1}\in Q_l$, Q_{k+1} was constructed as a subpath of Q_l from s to d_{k+1}. Q_{k+1} has length at most $\lceil k/2\rceil$ and is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$. Then Q_l needs to be changed, depending on whether d_{k+1} is the immediate predecessor of d_l in Q_l or not. If d_{k+1} is the immediate predecessor of d_l in Q_l, then Q_l was reconstructed as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$. Since d_l and d_{k+1} are adjacent and $|d_l|\le|d_{k+1}|$, we have $|d_l|=|d_{k+1}|-1=\lceil k/2\rceil-2$. Hence, (new) Q_l has length $(k-|\overline{d_l}|)+2=\lceil k/2\rceil$. Define a and b all the same as in Section 3.2.2.1. (New) Q_l is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_{k+1}$ because $|a|\le|d_{k+1}|+1=\lceil k/2\rceil<\lfloor k/2\rfloor+2=|\overline{d_l}|\le|b|$.

On the other hand, if d_{k+1} is not the immediate predecessor of d_l in Q_l, then Q_l was reconstructed as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to x in a k-cube, and a link (x, d_l), where x is the immediate predecessor of d_l in (old) Q_l. (New) Q_l has length $\lceil k/2 \rceil$+1, as explained below. Since $d_{k+1} \in$ (old) Q_l, (old) Q_l has length at least $|d_{k+1}|+2=\lceil k/2 \rceil+1$. Also, (old) Q_l has length at most $|d_l|+2\leq|d_{k+1}|+2=\lceil k/2 \rceil+1$ by Lemma 8. Hence, (old) Q_l has length exactly $\lceil k/2 \rceil+1$. Again, by Lemma 8, $|d_l|=\lceil k/2 \rceil-1$ and $|x|=|d_l|+1=\lceil k/2 \rceil$. The length of (new) Q_l is equal to $(k-|x|)+2=\lceil k/2 \rceil+1$. Suppose that a has the same meaning as in Section 3.2.2.1 and $t \notin \{s, x, d_l\}$ is a node in (new) Q_l. Clearly, we have $a\neq x$. Since $|a|\leq|d_{k+1}|+1=\lceil k/2 \rceil=|x|<|t|$, we have $a\neq t$. Hence (new) Q_l is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_{k+1}$.

When $d_{k+1} \notin Q_i$ for all $1\leq i\leq k$, Q_{k+1} was constructed as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube, which has length $(k-|d_{k+1}|)+1=\lceil k/2 \rceil+1$. In case Q_{k+1} intersects with Q_h for some $1\leq h\leq k$, they need to be changed as follows. Q_{k+1} was reconstructed as the combination of the subpath of Q_h from s to y and a shortest path from y to d_{k+1}, where y $(\neq s)$ is the intersection of (old) Q_{k+1} and Q_h. Q_h was reconstructed as the combination of a link $(s, \bar{s})$, a shortest path from $\bar{s}$ to z in a k-cube, and a link (z, d_h), where z is an adjacent node of d_h so that $|z|=|d_h|+1$ and $z \notin$ (old) Q_i for all $1\leq i\leq k$. In the following, we show the existence of z.

We first show $|d_h|=\lceil k/2 \rceil-1$ as follows. Since $y \in$ (old) Q_h, we have $|y|\leq|d_h|+1\leq|d_{k+1}|+1$. Further, we have $y\neq d_{k+1}$ because $d_{k+1} \notin$ (old) Q_h. On the other hand, since $y \in$ (old) Q_{k+1} and the subpath of (old) Q_{k+1} from $\bar{s}$ to d_{k+1} is shortest, we have $|y|\geq|d_{k+1}|+1$. Consequently, we have $|y|=|d_{k+1}|+1$, and hence $|d_h|=|d_{k+1}|=\lceil k/2 \rceil-1$. Suppose that q_t's, $1\leq t\leq k-|d_h|+1$, are the $k-|d_h|+1$ distinct adjacent nodes of d_h in a k-fcube with $|q_t|=|d_h|+1$ (one of q_t's is $\overline{d_h}$). A node $q_w \notin$ (old) Q_i for all $1\leq i\leq k$ can serve as z, where $1\leq w\leq k-|d_h|+1$. Since $|d_v|\leq|d_{k+1}|=|d_h|<|d_h|+1=|q_t|$, each (old) Q_v with $d_{h,F(d_v)}=0$ contains at most one q_t, where $1\leq v\leq k$. On the other hand, as explained below, each (old) Q_u with $d_{h,F(d_u)}=1$ does not contain any q_t, where $1\leq u\leq k$. Since $|y|=|d_h|+1$, we have $d_{h,F(d_h)}=0$ as a consequence of Lemma 8. Lemma 11 assures $|d_u|\leq|d_h|$ and $d_{u,F(d_u)}=1$. By Lemma 8, (old) Q_u is shortest with length $|d_u|$. Since $|d_u|\leq|d_h|<|d_h|+1=|q_t|$, (old) Q_u does not contain q_t. Since there are only $k-|d_h|$ (old) Q_v's with $d_{h,F(d_v)}=0$, a q_w as required can be found.

Since y is adjacent to d_{k+1} and $y\neq d_h$, (new) Q_{k+1} has length at most $1+(|d_h|+1)=\lceil k/2 \rceil+1$. (New) Q_h has length $(k-|z|)+2=\lceil k/2 \rceil+1$. (New) Q_{k+1} is disjoint with $Q_1, Q_2, \ldots, Q_{h-1}, Q_{h+1}, \ldots, Q_k$. Suppose that l is a node in $Q_1, \ldots, Q_{h-1}, Q_{h+1}, \ldots, Q_k$, (new) Q_{k+1} and $r \notin \{s, z, d_h\}$ is a node in (new) Q_h. (New) Q_h is disjoint with $Q_1, Q_2, \ldots, Q_{h-1}, Q_{h+1}, \ldots, Q_k$, (new) Q_{k+1} because z is not contained in $Q_1, Q_2, \ldots, Q_{h-1}, Q_{h+1}, \ldots, Q_k$, (new) Q_{k+1} and $|l| \leq |d_{k+1}|+1=\lceil k/2 \rceil=|z|<|r|$.

3.2.2.3 Case 3: $|d_{k+1}|=\lceil k/2 \rceil$

$Q'_1, Q'_2, \ldots, Q'_r, Q_{r+1}, \ldots, Q_k$ were first obtained by executing *Paths1*(W, k, k, $\{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_r}, d_{r+1}, \ldots, d_k\}$, $\{1, 2, \ldots, k\}$), where Q'_i ends at $\overline{d_i}$ for all $1 \leq i \leq r$. That W is minimal was assured by Lemma 12 (substituting r, $\overline{d_i}$, and W for h, d'_i, and Y, respectively). By Lemma 8, $Q'_1, Q'_2, \ldots, Q'_r, Q_{r+1}, \ldots, Q_k$ are disjoint to each other. Besides, each Q'_i is shortest with length $|\overline{d_i}|=\lfloor k/2 \rfloor$ because $\overline{d}_{i,\, W(\overline{d_i})}=\overline{d}_{i,\, F(d_i)}=1$. For all $r+1 \leq j \leq k$, if $|d_j| \leq \lceil k/2 \rceil-1$, then Q_j has length at most $|d_j|+2 \leq \lceil k/2 \rceil+1$. If $|d_j|=\lceil k/2 \rceil$, then $d_{j,W(d_j)}=1$ (by our assumption), which implies by Lemma 8 that Q_j is shortest with length $|d_j|=\lceil k/2 \rceil$. We note $|d_j| \leq |d_{k+1}|=\lceil k/2 \rceil$.

For all $1 \leq i \leq r$, Q_i was constructed as the combination of the subpath of Q'_i from s to f_i and two links (f_i, $\overline{f_i}$) and ($\overline{f_i}$, d_i) if $\overline{d_i}=d_h$ for some $r+1 \leq h \leq k+1$, and the combination of Q'_i and a link ($\overline{d_i}$, d_i) else (refer to Figure 3), where f_i is the immediate predecessor of $\overline{d_i}$ in Q'_i. Q_i has length $\lfloor k/2 \rfloor+1$. $Q_1, Q_2, \ldots, Q_k$ are disjoint to each other, provided that Q_i is disjoint with $Q_{r+1}, Q_{r+2}, \ldots, Q_k$ and $Q_1, Q_2, \ldots, Q_r$ are disjoint to each other. We first show that Q_i is disjoint with $Q_{r+1}, Q_{r+2}, \ldots, Q_k$. It suffices to show that d_i is not contained internally in $Q_{r+1}, Q_{r+2}, \ldots, Q_k$ and $\overline{f_i}$ is not contained in $Q_{r+1}, Q_{r+2}, \ldots, Q_k$. The former is assured by Lemma 13 (substituting r, $\overline{d_i}$, and W for h, d'_i, and Y, respectively). The latter can be shown as follows. Suppose that a is a node in Q_j, where $r+1 \leq j \leq k$. When $|d_j|=\lceil k/2 \rceil$, we have $|a| \leq \lceil k/2 \rceil$ because Q_j is shortest with length $\lceil k/2 \rceil$. When $|d_j| \leq \lceil k/2 \rceil-1$, we have $|a| \leq |d_j|+1 \leq \lceil k/2 \rceil$. Since $|\overline{f_i}|=k-(|\overline{d_i}|-1)=\lceil k/2 \rceil+1>|a|$, $\overline{f_i}$ is not contained in $Q_{r+1}, Q_{r+2}, \ldots, Q_k$.

We then show that $Q_1, Q_2, \ldots, Q_r$ are disjoint to each other. It suffices to show that $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}$ are all distinct, $\{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}\} \cap \{d_1, d_2, \ldots, d_r\}$ is empty, and each of $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}, d_1, d_2, \ldots, d_r$ is not contained in $Q'_1, Q'_2, \ldots, Q'_r$. Clearly, $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}$ are all distinct. Since $|\overline{f_i}|=\lceil k/2 \rceil+1$ and $|d_i|=\lceil k/2 \rceil$ for all $1 \leq i \leq r$, we have $\{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}\} \cap \{d_1, d_2, \ldots, d_r\}$ empty. Besides, each of $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}$ is not contained in $Q'_1, Q'_2, \ldots, Q'_r$ because $Q'_1, Q'_2, \ldots, Q'_r$ are all shortest with length $\lfloor k/2 \rfloor$. Since Lemma 13 assures that each of $d_1, d_2, \ldots, d_r$ is not contained internally in $Q'_1, Q'_2, \ldots, Q'_r$, we only need to show $\{d_1, d_2, \ldots, d_r\} \cap \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_r}\}$ empty. Suppose conversely $d_i = \overline{d_j}$, where $1 \le i \le r$, $1 \le j \le r$, and $i \ne j$. It follows by Lemma 3 ($d_{i,F(d_i)} = 0$, $d_{j,F(d_j)} = 0$, and $d_{i,F(d_j)} = \overline{d}_{j,F(d_j)} = 1$) that F is not minimal, which is a contradiction.

Q_{k+1} was constructed as the combination of a link $(s, \bar{s})$ and a shortest path from $\bar{s}$ to d_{k+1} in a k-cube whose length is $(k-|d_{k+1}|)+1 \le \lceil k/2 \rceil +1$. Q_{k+1} is not necessarily disjoint with $Q_1, Q_2, \ldots, Q_k$. As stated below, there are two situations in which some of $Q_1, Q_2, \ldots, Q_{k+1}$ need to be changed, in order to satisfy the disjoint property. The two situations have to be treated sequentially.

The first situation is that Q_{k+1} intersects with Q_t for some $1 \le t \le r$ at a node x $(\ne s)$. Q_t and Q_m need to be changed as follows: Q_t was reconstructed as the combination of the subpath of Q_m from s to g and two links $(g, \bar{g})$ and $(\bar{g}, d_t)$, and Q_m was reconstructed as the combination of the subpath of (old) Q_t from s to $\bar{x}$ and a shortest path from $\bar{x}$ to d_m, where $\overline{d_t} = d_m$ for some $r+1 \le m \le k$ and g is the immediate predecessor of d_m in (old) Q_m. That $\bar{x}$ $(=f_t)$ is contained in (old) Q_t and d_m surely exists is explained below.

According to the construction of (old) Q_t, we have $x \ne d_{k+1}$ and $f_t \in$ (old) Q_t. Besides, $|x| \le \lceil k/2 \rceil +1 = |\overline{f_t}|$ if $\overline{d_t} = d_h$ for some $r+1 \le h \le k+1$, and $|x| \le \lceil k/2 \rceil$ else. On the other hand, since $x \in Q_{k+1}$, we have $|x| \ge |d_{k+1}|+1 = \lceil k/2 \rceil +1$. Hence $|x| = \lceil k/2 \rceil +1$, which implies that $x = \overline{f_t}$ (hence $\bar{x} = f_t$) and x is the immediate predecessor of d_{k+1} in Q_{k+1}. Since $\overline{f_t} = x \in$ (old) Q_t, we have $\overline{d_t} = d_h$. We have $\overline{d_t} \ne d_{k+1}$, for otherwise ($\overline{d_t} = d_{k+1}$) both $\overline{f_t}$ and f_t are adjacent to d_{k+1}, which is impossible because $k \ge 4$. Hence $\overline{d_t} = d_m$ for some $r+1 \le m \le k$.

Since $\bar{x} = f_t$ and $d_m = \overline{d_t}$, (new) Q_m is identical with Q'_t. Hence (new) Q_m is shortest with length $\lfloor k/2 \rfloor$. (New) Q_t has length greater by one than that of (old) Q_m. (Old) Q_m is shortest with length $\lfloor k/2 \rfloor$, as explained below. By Lemma 9 (substituting d_t and d_m for d_h and d_l, respectively), we have $d_{m,F(d_m)} = 1$. Hence, $d_{m,W(d_m)} = d_{m,F(d_m)} = 1$ and by Lemma 8 (old) Q_m is shortest with length $|d_m| = \lfloor k/2 \rfloor$. Besides, $|g| = |d_m| - 1 = \lfloor k/2 \rfloor - 1$. That $Q_1, \ldots, Q_{t-1}$, (new) Q_t, $Q_{t+1}, \ldots, Q_{m-1}$, (new) Q_m, $Q_{m+1}, \ldots, Q_k$ are disjoint to each other is mainly due to the construction method. More reasons are supplemented as follows. (New) Q_m is disjoint with $Q_1, \ldots, Q_{t-1}, Q_{t+1}, \ldots, Q_{m-1}, Q_{m+1}, \ldots, Q_k$, as a consequence of (new) Q_m identical with Q'_t. Since (old) Q_m is disjoint with $Q'_1, Q'_2, \ldots, Q'_r$, we have $g \ne f_i$ (hence $\bar{g} \ne \overline{f_i}$) for all $1 \le i \le r$. (New) Q_t is disjoint with $Q_1, \ldots, Q_{t-1}, Q_{t+1}, \ldots, Q_{m-1}, Q_{m+1}, \ldots, Q_k$, as a consequence of $\bar{g} \ne \overline{f_i}$ and $|\bar{g}| = \lceil k/2 \rceil +1 > |a|$.

(New) Q_t and (new) Q_m are disjoint because $\overline{g} \notin$ (new) Q_m.

Q_{k+1} is disjoint with $Q_1, \ldots, Q_{t-1}$, (new) Q_t, $Q_{t+1}, \ldots, Q_r$, as explained below. Suppose that $b \notin \{s, x, d_{k+1}\}$ is a node in Q_{k+1} and g is a node in $Q_1, \ldots, Q_{t-1}$, (new) Q_t, $Q_{t+1}, \ldots, Q_r$. The construction method assures $|b|>|x|=\lceil k/2\rceil+1$ (recall that x is the immediate predecessor of d_{k+1} in Q_{k+1}) and $|g|\leq\lceil k/2\rceil+1$. Hence we have $b\neq g$. The construction method also assures that x and d_{k+1} are not contained in $Q_1, \ldots, Q_{t-1}$, (new) Q_t, $Q_{t+1}, \ldots, Q_r$ (recall $x=\overline{f_t} \neq \overline{g}$).

To sum up, $Q_1, Q_2, \ldots, Q_k$ that we obtained with the first situation taken into consideration are disjoint to each other. Besides, Q_{k+1} is disjoint with $Q_1, Q_2, \ldots, Q_r$. It should be noted that Q_{k+1} intersects with at most one of (old) Q_1, (old) Q_2, …, (old) Q_r. If Q_{k+1} intersects with another (old) Q_w ($1\leq w\leq r$ and $w\neq t$) at a node x' ($\notin \{s, x\}$), in addition to (old) Q_t, then we have $|x'|=\lceil k/2\rceil+1=|x|$. Since both x and x' are contained in Q_{k+1}, we have $x=x'$, which is a contradiction.

The second situation is that Q_{k+1} intersects with Q_l for some $r+1\leq l\leq k$. Since $|a|\leq\lceil k/2\rceil=|d_{k+1}|$, the intersection of Q_{k+1} and Q_l must be d_{k+1}. Q_{k+1} was reconstructed as the subpath of Q_l from s to d_{k+1} whose length is at most $\lceil k/2\rceil$. (New) Q_{k+1} is disjoint with $Q_1, Q_2, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$. Q_l needs to be reconstructed, depending on whether k is even or odd. If k is even, then Q_l was reconstructed as the combination of a link $(s, \overline{s})$, a shortest path from $\overline{s}$ to $\overline{d_l}$ in a k-cube, and a link $(\overline{d_l}, d_l)$, which has length $(k-|\overline{d_l}|)+2=|d_l|+2$. We have $|d_l|=\lceil k/2\rceil-1$, as explained below. It was shown that (old) Q_l has length at most $\lceil k/2\rceil+1$. Also, (old) Q_l has length at least $|d_{k+1}|+1=\lceil k/2\rceil+1$ because $d_{k+1} \in$ (old) Q_l. Hence (old) Q_l has length exactly $\lceil k/2\rceil+1$. Since $|d_l|\leq\lceil k/2\rceil$, we have $|d_l|=\lceil k/2\rceil-1$ as a consequence of Lemma 8.

(New) Q_l is disjoint with $Q_1, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1}, as explained below. Suppose that $h \notin \{s, \overline{d_l}, d_l\}$ is a node in (new) Q_l and t is a node in $Q_1, Q_2, \ldots, Q_r$. We have $|h|>|\overline{d_l}|=\lfloor k/2\rfloor+1=\lceil k/2\rceil+1$. According to the construction method, we have $|t|=\lceil k/2\rceil+1$ if $t \in \{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}, \overline{g}\}$, and $|t|<\lceil k/2\rceil+1$ else. Besides, we have $d_l \notin \{f_1, f_2, \ldots, f_r, g\}$ (equivalently, $\overline{d_l} \notin \{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}, \overline{g}\}$). Hence (new) Q_l is disjoint with $Q_1, Q_2, \ldots, Q_r$. Since $|a|\leq\lceil k/2\rceil$, (new) Q_l is disjoint with $Q_{r+1}, \ldots, Q_{l-1}$, (old) Q_l, $Q_{l+1}, \ldots, Q_k$, which also implies that (new) Q_l and (new) Q_{k+1} are disjoint.

If k is odd, then Q_l was reconstructed as the combination of a link $(s, \overline{s})$, a shortest path from $\overline{s}$ to y in a k-cube, and a link (y, d_l), where y is an adjacent node of d_l so that $|y|=|d_l|+1$ and $y \notin$ (old) Q_i for all $1\leq i\leq k$. We have $|d_l|=\lceil k/2\rceil-1$ similarly. (New) Q_l has length $(k-|y|)+2=\lceil k/2\rceil+1$. Suppose that q_c's,

$1 \le c \le k-|d_l|+1$, are $k-|d_l|+1$ distinct adjacent nodes of d_l in a k-fcube with $|q_c|=|d_l|+1$ (one of q_c's is $\overline{d_l}$). In the following, we show the existence of y by showing the existence of a $q_w \notin$ (old) Q_i for all $1 \le i \le k$, where $1 \le w \le k-|d_l|+1$.

We first show that each (old) Q_u with $d_{l,F(d_u)} = 1$ does not contain any q_c, where $1 \le u \le k$. Since $d_{k+1} \in$ (old) Q_l and $|d_{k+1}|=|d_l|+1$, we have $d_{l,W(d_l)} = 0$ by Lemma 8. By Lemma 11 ($d_{l,F(d_l)} = d_{l,W(d_l)} = 0$), we have $|d_u| \le |d_l|$ and $d_{u,F(d_u)} = 1$. By our assumption, we have $r+1 \le u \le k$, and hence $d_{u,W(d_u)} = d_{u,F(d_u)} = 1$. By Lemma 8, (old) Q_u is shortest with length $|d_u|$. Since $|d_u| \le |d_l| < |d_l|+1 = |q_c|$, (old) Q_u does not contain q_c. Then we show that each (old) Q_v with $d_{l,F(d_v)} = 0$ contains at most one q_c, where $1 \le v \le k$. It suffices to show that each (old) Q_v with $d_{l,F(d_v)} = 0$ contains at most one node, say r, with $|r| = \lceil k/2 \rceil = |q_c|$. When $1 \le v \le r$, each (old) Q_v contains exactly one r according to the construction method. When $r+1 \le v \le k$, it was shown that if $|d_v| = \lceil k/2 \rceil$, then (old) Q_v is shortest with length $\lceil k/2 \rceil$. Hence (old) Q_v contains exactly one r. If $|d_v| \le \lceil k/2 \rceil - 1$, then Lemma 8 assures that (old) Q_v contains at most one r. Since there are $k-|d_l|+1$ distinct q_c's but only $k-|d_l|$ (old) Q_v's with $d_{l,F(d_v)} = 0$, a q_w as required can be found.

Suppose that $l \notin \{s, y, d_l\}$ is a node in (new) Q_l. Since $|l| > |y| = \lceil k/2 \rceil \ge |a|$ and y is not contained in $Q_1, \ldots, Q_{l-1}$, (old) Q_l, $Q_{l+1}, \ldots, Q_k$, we have (new) Q_l disjoint with $Q_{r+1}, \ldots, Q_{l-1}, Q_{l+1}, \ldots, Q_k$, (new) Q_{k+1} (recall that (new) Q_{k+1} is a subpath of (old) Q_l). If (new) Q_l intersects with Q_v for some $1 \le v \le r$ at a node z ($\ne s$), then Q_v and Q_w need to be reconstructed in order that the final $Q_1, Q_2, \ldots, Q_{k+1}$ are disjoint, where $\overline{d_v} = d_w$ for some $r+1 \le w \le k$ and $w \ne l$. We have $z \ne d_l$ because (old) Q_l is disjoint with $Q_1, Q_2, \ldots, Q_r$. That is, z is contained in the subpath of (new) Q_l from s to y. Since $|y| = \lceil k/2 \rceil = |d_{k+1}|$, the reconstruction of Q_v and Q_w can be made all the same as the reconstruction of Q_t and Q_m in the first situation that Q_{k+1} intersects with Q_t for some $1 \le t \le r$ (substituting y, v, w, p, z, and the subpath of (new) Q_l from s to y for d_{k+1}, t, m, g, x, and Q_{k+1}, respectively).

Consequently, (new) Q_v and (new) Q_w have lengths $\lfloor k/2 \rfloor + 1$ and $\lfloor k/2 \rfloor$, respectively. Besides, $Q_1, \ldots$, (new) $Q_v, \ldots, Q_{l-1}, Q_{l+1}, \ldots$, (new) $Q_w, \ldots, Q_k$, (new) Q_{k+1} are disjoint ($l < w$ is assumed), (new) Q_l is disjoint with $Q_1, \ldots$, (new) $Q_v, \ldots, Q_r$, and (new) Q_l is disjoint with (new) Q_w (because (new) Q_w has length $\lfloor k/2 \rfloor < \lceil k/2 \rceil = |y| < |l|$). We have $\overline{d_v} = d_w$ for some $r+1 \le w \le k$ with the same arguments as $\overline{d_t} = d_m$ for some $r+1 \le m \le k$ in the first situation that Q_{k+1} intersects with Q_t for some $1 \le t \le r$. Further, $\overline{d_v} \ne d_l$ (i.e., $w \ne l$) was shown below.

Suppose conversely $\overline{d_v} = d_l$. Since y is adjacent to d_l, either $y = \overline{d_l}$ or y is adjacent to d_l in a k-cube. If

$y=\overline{d_l}$, then $y=d_v$, which contradicts to $y \notin Q_v$. If y is adjacent to d_l in a k-cube, then d_l and $\overline{d_l}$ are at a distance of three at most in a k-cube, as explained below, which contradicts to our assumption of $k\geq 4$. It suffices to show a path (d_l, y, z, $\overline{d_l}$) in a k-cube. We have shown $x=\overline{f_t}$ above (for the situation that Q_{k+1} intersects with Q_t). Similarly, we have $z=\overline{f_v}$. Nodes y and z are adjacent in a k-cube because $z\in$ (new) Q_l and $|z|=|\overline{f_v}|=\lceil k/2\rceil+1=|y|+1$. Nodes z and $\overline{d_l}$ are adjacent in a k-cube because $\overline{d_l}=d_v$, $z=\overline{f_v}$, and d_v is adjacent to $\overline{f_v}$ in a k-cube.

3.2.2.4 Case 4: $|d_{k+1}|$=ék/2ù+1

Q'_1, …, Q'_v, Q_{v+1}, …, Q_k were first obtained by invoking *Paths1*(Y, k, k, D={$\overline{d_1}$, …, $\overline{d_u}$, p_{u+1}, …, p_v, d_{v+1}, …, d_k}, I={1, 2, …, k}), where Q'_j ends at $\overline{d_j}$ for all $1\leq j\leq u$ and Q'_t ends at p_t for all $u+1\leq t\leq v$. We first show by induction that $Y: D\rightarrow I$ is minimal. Let $D^{(m)}$ and $Y^{(m)}$ represent the D and Y, respectively, that were obtained after the mth modification, where $0\leq m\leq v-u$ and $D^{(0)}$ and $Y^{(0)}$ denote the initial D and Y, respectively. By Lemma 12 (substituting u, $\overline{d_i}$, and $Y^{(0)}$ for h, d'_i, and Y, respectively), $Y^{(0)}$ is minimal. Suppose that $Y^{(m)}$ is minimal, and assume, without loss of generality, $D^{(m)}$={$\overline{d_1}$, …, $\overline{d_u}$, p_{u+1}, …, p_{u+m}, d_{u+m+1}, …, d_k}. If there exists a $d_l\in D^{(m)}$ so that $u+m+1\leq l\leq k$, $d_{l,Y^{(m)}(d_l)}=0$, $|d_l|=\lceil k/2\rceil$, and $\overline{d_l}\in D^{(m)}$, then $D^{(m+1)}$ and $Y^{(m+1)}$ are obtained by modifying $D^{(m)}$ and $Y^{(m)}$. We show below that $Y^{(m+1)}$ is minimal.

By Lemma 14 (substituting d_l, p_l, $D^{(m)}$, and $Y^{(m)}$ for d_h, x, {d_1, d_2, …, d_k}, and F, respectively), there exists an adjacent node p_l of $\overline{d_l}$ so that $|p_l|=|\overline{d_l}|-1$, $p_l\notin D^{(m)}$, and $\overline{p_l}\notin D^{(m)}\cup\{d_{k+1}\}$. Besides, we have $p_{l,Y^{(m)}(d_l)}=1$ or $p_{l,Y^{(m)}(\overline{d_l})}=1$. Since $|\overline{d_l}|=\lfloor k/2\rfloor\neq\lfloor k/2\rfloor-1=|p_w|$ for all $u+1\leq w\leq u+m$, we have $\overline{d_l}=d_h$ for some $u+m+1\leq h\leq k$ and $h\neq l$. We would like to have $p_{l,Y^{(m)}(d_l)}=1$. If $p_{l,Y^{(m)}(d_l)}=0$, then swap $Y^{(m)}(d_l)$ and $Y^{(m)}(d_h)$, and hence $p_{l,(\text{new})Y^{(m)}(d_l)}=p_{l,(\text{old})Y^{(m)}(d_h)}=p_{l,(\text{old})Y^{(m)}(\overline{d_l})}=1$, where (new) $Y^{(m)}$ is minimal by Lemma 10 (substituting l, h, (old) $Y^{(m)}$, and (new) $Y^{(m)}$ for h, l, F, and Y, respectively). $D^{(m+1)}$ was obtained by replacing d_l with p_l in $D^{(m)}$, and $Y^{(m+1)}$ results by assigning $Y^{(m+1)}(p_l)$ with $Y^{(m)}(d_l)$. We have $p_{l,Y^{(m+1)}(p_l)}=p_{l,Y^{(m)}(d_l)}=1$. By Lemma 12 (letting h=1 and substituting d_l, p_l, $Y^{(m)}$, and $Y^{(m+1)}$ for d_1, d'_1, F, and Y, respectively), $Y^{(m+1)}$ is minimal. Hence, the final Y is minimal.

If $Y^{(m)}(d_l)$ and $Y^{(m)}(d_h)$ were not swapped, then it is easy to see that $Y^{(m+1)}(p_l)=Y^{(m)}(d_l)=F(d_l)$ and

$Y^{(m+1)}(d_h)=Y^{(m)}(d_h)=F(d_h)$. On the other hand, if $Y^{(m)}(d_l)$ and $Y^{(m)}(d_h)$ were swapped, then $Y^{(m+1)}(p_l)=F(d_l)$ and $Y^{(m+1)}(d_h)=F(d_h)$ remain valid after a slight modification of F, as explained below. We simply swap $F(d_l)$ and $F(d_h)$, and then we have $Y^{(m+1)}(p_l) =$ (new) $Y^{(m)}(d_l) =$ (old) $Y^{(m)}(d_h) =$ (old) $F(d_h) =$ (new) $F(d_l)$ and $Y^{(m+1)}(d_h) =$ (new) $Y^{(m)}(d_h) =$ (old) $Y^{(m)}(d_l) =$ (old) $F(d_l) =$ (new) $F(d_h)$. Since $d_{l,(\text{old})F(d_l)} = d_{l,(\text{old})Y^{(m)}(d_l)} = 0$, (new) F is minimal and $d_{i,(\text{new})F(d_i)} = d_{i,(\text{old})F(d_i)}$ for all $1 \le i \le k$ by Lemma 10 (substituting l, h, (old) F, and (new) F for h, l, F, and Y, respectively). Without loss of generality, we simply assume $Y^{(m+1)}(p_l)=F(d_l)$ and $Y^{(m+1)}(d_h)=F(d_h)$. For the final $D=\{\overline{d_1}, \ldots, \overline{d_u}, p_{u+1}, \ldots, p_v, d_{v+1}, \ldots, d_k\}$, we have $Y(p_t)=F(d_t)$ for all $u+1 \le t \le v$ and $Y(d_c)=F(d_c)$ for all $v+1 \le c \le k$.

By Lemma 8, $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$ are disjoint to each other. Besides, Q'_i is shortest with length $|\overline{d_i}|=\lfloor k/2 \rfloor-1$ for all $1 \le i \le r$, Q'_j is shortest with length $|\overline{d_j}|=\lfloor k/2 \rfloor$ for all $r+1 \le j \le u$, and Q'_t is shortest with length $|p_t|=\lfloor k/2 \rfloor-1$ for all $u+1 \le t \le v$, because $\overline{d}_{i,Y(\overline{d_i})} = \overline{d}_{i,F(d_i)} = 1$ for all $1 \le i \le r$, $\overline{d}_{j,Y(\overline{d_j})} = \overline{d}_{j,F(d_j)} = 1$ for all $r+1 \le j \le u$, and $p_{t,Y(p_t)} = 1$ for all $u+1 \le t \le v$. The lengths of Q_c's for all $v+1 \le c \le k$ are computed as follows. If $|d_c| \le \lceil k/2 \rceil-1$, then Q_c has length not greater than $|d_c|+2 \le \lceil k/2 \rceil+1$. If $|d_c|=\lceil k/2 \rceil+1$, then $d_{c,Y(d_c)} = d_{c,F(d_c)} = 1$, and hence Q_c is shortest with length $|d_c|=\lceil k/2 \rceil+1$. If $|d_c|=\lceil k/2 \rceil$, then Q_c is shortest with length $|d_c|=\lceil k/2 \rceil$ because $d_{c,Y(d_c)} = 1$, as shown below.

Suppose conversely $d_{c,Y(d_c)} = 0$. We have $\overline{d_c} \notin \{d_1, d_2, \ldots, d_r\}$ because $|\overline{d_c}|=\lfloor k/2 \rfloor \neq \lceil k/2 \rceil+1=|d_i|$ for all $1 \le i \le r$. We have $\overline{d_c} \notin \{d_{r+1}, d_{r+2}, \ldots, d_u\}$ because $\overline{d_c} \in \{d_{r+1}, d_{r+2}, \ldots, d_u\}$ implies $\overline{\overline{d_c}}=d_c \notin \{d_1, d_2, \ldots, d_k\}$, which is a contradiction. We have $\overline{d_c} \notin \{d_{u+1}, d_{u+2}, \ldots, d_v\}$ because $\overline{d_c} \in \{d_{u+1}, d_{u+2}, \ldots, d_v\}$ implies $\overline{d}_{c,Y(\overline{d_c})} = 0$ and then by Lemma 9 we have $d_{c,Y(d_c)} = 1$, which is a contradiction. On the other hand, we have $\overline{d_c} \in \{d_1, d_2, \ldots, d_k\}$ because $\overline{d_c} \notin \{d_1, d_2, \ldots, d_k\}$ along with $d_{c,F(d_c)} = d_{c,Y(d_c)} = 0$ and $|d_c|=\lceil k/2 \rceil$ implies $r+1 \le c \le u$, which is a contradiction. Consequently, $\overline{d_c} \in \{d_{v+1}, d_{v+2}, \ldots, d_k\}$, i.e., $\overline{d_c} \in D$, is concluded. Since $u+1 \le c \le k$, $d_{c,Y(d_c)} = 0$, $|d_c|=\lceil k/2 \rceil$, and $\overline{d_c} \in D$, a further modification of D and Y is required, which is a contradiction.

Q_{k+1} was constructed as the combination of a link $(s, \overline{s})$ and a shortest path from $\overline{s}$ to d_{k+1} in a k-cube, which has length $\lfloor k/2 \rfloor$. Q_{k+1} is disjoint with $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$, as explained below. Suppose that $a \notin \{s, d_{k+1}\}$ is a node in Q_{k+1}, b is a node in $Q'_1, Q'_2, \ldots, Q'_v$, and $g \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\}$ is a node in Q_{v+1},

Q_{v+2}, …, Q_k. Since $|a|>|d_{k+1}|=\lceil k/2\rceil+1$, $|b|\le\lfloor k/2\rfloor$, and $|g|<\lceil k/2\rceil+1$, we have $a\ne b$, $a\ne g$, $d_{k+1}\ne b$, and $d_{k+1}\ne g$.

For all $1\le i\le r$, Q_i was constructed, depending on whether $\overline{d_i}=d_w$ for some $v+1\le w\le k$ or not. If $\overline{d_i}=d_w$, then Q_i was constructed as the combination of the subpath of Q'_i from s to f_i and two links (f_i, $\overline{f_i}$) and ($\overline{f_i}$, d_i) (refer to Figure 4(a)), where f_i denotes the immediate predecessor of $\overline{d_i}$ in Q'_i. If $\overline{f_i}\in Q_{k+1}$, then Q'_i and Q_w need to be swapped prior to the construction of Q_i. Q'_i is shortest with length $\lfloor k/2\rfloor-1$. By Lemma 9 ($d_{i,F(d_i)}=0$ and $\overline{d_i}=d_w$), we have $d_{w,F(d_w)}=1$. Since $d_{w,Y(d_w)}=d_{w,F(d_w)}=1$, Q_w is shortest with length $|d_w|=|\overline{d_i}|=\lfloor k/2\rfloor-1$ by Lemma 8. Hence Q_i has length $\lfloor k/2\rfloor$. Q_i is disjoint with Q'_{r+1}, …, Q'_v, Q_{v+1}, …, Q_{k+1}, as explained below.

We first assume $\overline{f_i}\notin Q_{k+1}$. For all $u+1\le t\le v$, since $F(d_t)=Y(p_t)$ and $p_{t,Y(p_t)}=1$, we have $p_{t,F(d_t)}=1$. By Lemma 13 (substituting v and $\overline{d_i}$ (or p_i) for h and d'_i, respectively), each of d_1, d_2, …, d_v is not contained internally in Q'_1, …, Q'_v, Q_{v+1}, …, Q_k. Since $|\overline{f_i}|=\lceil k/2\rceil+2$ is greater than $|b|$, $|g|$, and $|d_c|$ for all $v+1\le c\le k$, $\overline{f_i}$ is not contained in Q'_{r+1}, …, Q'_v, Q_{v+1}, …, Q_k. Hence Q_i is disjoint with Q'_{r+1}, …, Q'_v, Q_{v+1}, …, Q_k. Suppose that $t\notin\{\overline{f_i}, d_i\}$ is a node in Q_i. Since $|t|\le|f_i|=\lfloor k/2\rfloor-2<|a|$ and $|t|<|d_{k+1}|$, each node in $Q_i-\{\overline{f_i}, d_i\}$ is not contained in Q_{k+1}. We have $d_i\notin Q_{k+1}$ because $|d_i|\le|d_{k+1}|<|a|$. Hence Q_i and Q_{k+1} are disjoint. If $\overline{f_i}\in Q_{k+1}$, then Q'_i and Q_w were swapped and f_i was changed to be the immediate predecessor of $\overline{d_i}$ in (new) Q'_i ($\overline{f_i}$ was changed accordingly). We have (new) $\overline{f_i}\notin Q_{k+1}$, because (new) $\overline{f_i}\ne$ (old) $\overline{f_i}$, |(new) $\overline{f_i}$|=|(old) $\overline{f_i}$|, and the subpath of Q_{k+1} from $\bar{s}$ to d_{k+1} is shortest. With the same arguments, Q_i is disjoint with Q'_{r+1}, …, Q'_v, Q_{v+1}, …, Q_{k+1}.

On the other hand, if $\overline{d_i}\ne d_j$ for all $v+1\le j\le k$, then Q_i was constructed as the combination of Q'_i and a link ($\overline{d_i}$, d_i) (refer to Figure 4(b)), which has length $\lfloor k/2\rfloor$. Q_i is disjoint with Q'_{r+1}, …, Q'_v, Q_{v+1}, …, Q_{k+1}, provided $\overline{d_i}\notin\{\overline{d_{r+1}}, \ldots, \overline{d_u}, p_{u+1}, \ldots, p_v, d_{k+1}\}$ and d_i is not contained internally in Q'_{r+1}, …, Q'_v, Q_{v+1}, …, Q_{k+1}. We have $\overline{d_i}\notin\{\overline{d_{r+1}}, \overline{d_{r+2}}, \ldots, \overline{d_u}\}$ because $d_i\notin\{d_{r+1}, d_{r+2}, \ldots, d_u\}$, $\overline{d_i}\notin\{p_{u+1}, p_{u+2}, \ldots, p_v\}$ because $\{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_u}\}\cap\{p_{u+1}, p_{u+2}, \ldots, p_v\}$ is empty, and $\overline{d_i}\ne d_{k+1}$ because $|\overline{d_i}|=\lfloor k/2\rfloor-1\ne|d_{k+1}|$. We have shown above that d_i is not contained internally in Q'_{r+1}, …, Q'_v, Q_{v+1}, …, Q_k. Since $|d_i|=\lceil k/2\rceil+1=|d_{k+1}|<|a|$, d_i is not contained internally in Q_{k+1}.

Next we show that Q_1, Q_2, …, Q_r are disjoint to each other. Without loss of generality, we assume

$\overline{d_c} \in \{d_{v+1}, d_{v+2}, \ldots, d_k\}$ for all $1 \le c \le m$ and $\overline{d_c} \notin \{d_{v+1}, d_{v+2}, \ldots, d_k\}$ for all $m+1 \le c \le r$, where $0 \le m \le r$. It suffices to show that $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}$ are all distinct, $\{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}\} \cap \{d_1, d_2, \ldots, d_r\}$ is empty, and each of $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_m}, d_1, d_2, \ldots, d_r$ is not contained in $Q'_1, Q'_2, \ldots, Q'_r$. The first two hold because $f_1, f_2, \ldots, f_m$ are all distinct, $|\overline{f_c}| = \lceil k/2 \rceil + 2$ for all $1 \le c \le m$, and $|d_i| = \lceil k/2 \rceil + 1$ for all $1 \le i \le r$. The last holds because $Q'_1, Q'_2, \ldots, Q'_r$ are all shortest with length $\lfloor k/2 \rfloor - 1$.

For all $r+1 \le j \le u$, Q_j was constructed as the combination of Q'_j and a link $(\overline{d_j}, d_j)$, which has length $\lfloor k/2 \rfloor + 1$. Q_j is disjoint with $Q_1, Q_2, \ldots, Q_r, Q'_{u+1}, \ldots, Q'_v, Q_{v+1}, \ldots, Q_{k+1}$, provided $\overline{d_j} \notin \{d_1, d_2, \ldots, d_r, p_{u+1}, \ldots, p_v, d_{v+1}, \ldots, d_{k+1}\}$ and d_j is not contained internally in $Q_1, Q_2, \ldots, Q_r, Q'_{u+1}, \ldots, Q'_v, Q_{v+1}, \ldots, Q_{k+1}$. By our assumption (refer to the beginning paragraph enclosed by /* and */ in Case 4), we have $\overline{d_j} \notin \{d_1, d_2, \ldots, d_k\}$. Since $|\overline{d_j}| = \lfloor k/2 \rfloor \ne \lfloor k/2 \rfloor - 1 = |p_t|$ for all $u+1 \le t \le v$, we have $\overline{d_j} \notin \{p_{u+1}, p_{u+2}, \ldots, p_v\}$. Also we have $\overline{d_j} \ne d_{k+1}$ because $|\overline{d_j}| \ne |d_{k+1}|$. On the other hand, since d_j is not contained internally in $Q'_1, \ldots, Q'_v, Q_{v+1}, \ldots, Q_k$, we only need to show $d_j \notin \{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}, \overline{d_1}, \overline{d_2}, \ldots, \overline{d_r}\}$ and d_j is not contained internally in Q_{k+1}. The former holds because $|d_j| = \lceil k/2 \rceil \ne \lceil k/2 \rceil + 2 = |\overline{f_i}|$ and $|d_j| = \lceil k/2 \rceil \ne \lfloor k/2 \rfloor - 1 = |\overline{d_i}|$ for all $1 \le i \le r$. The latter holds because $|d_j| < |a|$. Further, $Q_{r+1}, Q_{r+2}, \ldots, Q_u$ are disjoint to each other because $\overline{d_j} \notin \{d_1, d_2, \ldots, d_k\}$ and d_j is not contained internally in $Q'_{r+1}, Q'_{r+2}, \ldots, Q'_u$.

For all $u+1 \le t \le v$, Q_t was constructed as the combination of Q'_t and two links $(p_t, \overline{p_t})$ and $(\overline{p_t}, d_t)$, which has length $\lfloor k/2 \rfloor + 1$ (refer to Figure 5). Q_t is disjoint with $Q_1, Q_2, \ldots, Q_u, Q_{v+1}, Q_{v+2}, \ldots, Q_{k+1}$, provided $p_t \notin \{d_1, d_2, \ldots, d_u, d_{v+1}, d_{v+2}, \ldots, d_{k+1}\}$, $\overline{p_t}$ is not contained in $Q_1, Q_2, \ldots, Q_u, Q_{v+1}, Q_{v+2}, \ldots, Q_{k+1}$, and d_t is not contained internally in $Q_1, Q_2, \ldots, Q_u, Q_{v+1}, Q_{v+2}, \ldots, Q_{k+1}$. By our assumption, we have $p_t \notin \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_u}, d_{v+1}, d_{v+2}, \ldots, d_k\}$ and $\overline{p_t} \notin \{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_u}, d_{v+1}, d_{v+2}, \ldots, d_k, d_{k+1}\}$, from which $\overline{p_t} \notin \{d_1, d_2, \ldots, d_u\}$ and $p_t \notin \{d_1, d_2, \ldots, d_u\}$ are implied. Since $|p_t| = \lfloor k/2 \rfloor - 1 \ne |d_{k+1}|$, we have $p_t \ne d_{k+1}$. Suppose that $h \notin \{d_1, d_2, \ldots, d_u, d_{v+1}, d_{v+2}, \ldots, d_{k+1}\}$ is a node in $Q_1, Q_2, \ldots, Q_u, Q_{v+1}, Q_{v+2}, \ldots, Q_{k+1}$. We have $\overline{p_t} \ne h$ because $|\overline{p_t}| = \lceil k/2 \rceil + 1 \ne |h|$ (as a consequence of our construction method). Since d_t is not contained internally in $Q'_1, Q'_2, \ldots, Q'_u, Q_{v+1}, Q_{v+2}, \ldots, Q_k$, d_t is not contained internally in $Q_1, Q_2, \ldots, Q_u, Q_{v+1}, \ldots, Q_{k+1}$, provided $d_t \notin \{\overline{f_1}, \overline{f_2}, \ldots, \overline{f_r}, \overline{d_1}, \overline{d_2}, \ldots, \overline{d_u}\}$ and d_t is not contained internally in Q_{k+1}. The former holds because $|d_t|=\lceil k/2\rceil$, $|\overline{f_i}|=\lceil k/2\rceil+2$ and $|\overline{d_i}|=\lfloor k/2\rfloor-1$ for all $1\le i\le r$, and $\overline{d_j}\notin\{d_1, d_2, \ldots, d_k\}$ for all $r+1\le j\le u$ (by our assumption). The latter holds because $|d_t|<\lceil k/2\rceil+1<|a|$.

$Q_{u+1}, Q_{u+2}, \ldots, Q_v$ are disjoint to each other for the following reason. By our assumption, we have $p_{u+1}, p_{u+2}, \ldots, p_v$ all distinct (hence, $\overline{p_{u+1}}, \overline{p_{u+2}}, \ldots, \overline{p_v}$ all distinct). For all $u+1\le t\le v$, since $|\overline{p_t}|=\lceil k/2\rceil+1$, $|d_t|=\lceil k/2\rceil$, and Q'_t is shortest with length $\lfloor k/2\rfloor-1$, neither of $\overline{p_t}$ and d_t is contained in $Q'_{u+1}, Q'_{u+2}, \ldots, Q'_v$. Besides, since $|\overline{p_t}|=\lceil k/2\rceil+1$ and $|d_t|=\lceil k/2\rceil$ for all $u+1\le t\le v$, we have $\{\overline{p_{u+1}}, \overline{p_{u+2}}, \ldots, \overline{p_v}\}\cap\{d_{u+1}, d_{u+2}, \ldots, d_v\}$ empty.

3.2.2.5 Case 5: $|d_{k+1}|$³ é$k/2$ù+2

By carefully checking Section 3.2.2.4, it can be found that the execution of Case 4 remains correct even if there are the following two extensions: (1) $|d_m|\le\lceil k/2\rceil+1$ for all $1\le m\le k$ and $|d_{k+1}|\ge\lceil k/2\rceil+1$; (2) $d_i=d_j$ is allowed if $|d_i|=|d_j|<\lceil k/2\rceil$, where $1\le i\le k+1$ and $1\le j\le k+1$. Since $|q_i|=\lfloor k/2\rfloor-1$ for all $1\le i\le r'$, $|\overline{d_j}|\le\lfloor k/2\rfloor-2$ for all $r'+1\le j\le u'$, and $|d_c|\le\lceil k/2\rceil+1$ for all $u'+1\le c\le k$, $Q'_1, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$ can be obtained by the construction method of Case 4, where Q'_i ends at q_i for all $1\le i\le r'$ and Q'_j ends at $\overline{d_j}$ for all $r'+1\le j\le u'$. Besides, $Q'_1, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$ are disjoint to each other and their maximal length is not greater than $\lceil k/2\rceil+1$. We also note that there is no complement link in $Q'_1, Q'_2, \ldots, Q'_{u'}$. Actually, $Q'_1, Q'_2, \ldots, Q'_{u'}$ have lengths smaller than $\lceil k/2\rceil+1$. Since W is minimal, Lemma 16 assures $q_{i,W(q_i)}=1$ for all $1\le i\le r'$. According to Lemma 8, Q'_i with $1\le i\le r'$ is shortest with length $|q_i|=\lfloor k/2\rfloor-1$. Besides, Q'_j with $r'+1\le j\le u'$ has length not greater than $|\overline{d_j}|+2\le\lfloor k/2\rfloor$.

For all $1\le i\le r'$, Q_i was constructed as the combination of Q'_i and two links $(q_i, \overline{q_i})$ and $(\overline{q_i}, d_i)$, which has length $\lfloor k/2\rfloor+1$ (refer to Figure 6). Q_i is disjoint with $Q'_{r'+1}, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$, provided $q_i\notin\{\overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k, d_{k+1}\}$, $\overline{q_i}$ is not contained in $Q'_{r'+1}, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$, and d_i is not contained internally in $Q'_{r'+1}, \ldots, Q'_{u'}, Q_{u'+1}, \ldots, Q_{k+1}$. By our assumption (refer to the beginning paragraph enclosed by /* and */ in Case 5), we have $q_i\notin\{\overline{d_{r'+1}}, \ldots, \overline{d_{u'}}, d_{u'+1}, \ldots, d_k\}$. Also, $q_i\ne d_{k+1}$ because $|q_i|<|d_{k+1}|$. Since $|\overline{q_i}|=\lceil k/2\rceil+1$ and $Q'_{r'+1}, Q'_{r'+2}, \ldots, Q'_{u'}$ have lengths at most $\lfloor k/2\rfloor$, $\overline{q_i}$ is not contained in $Q'_{r'+1}, Q'_{r'+2}, \ldots, Q'_{u'}$. $\overline{q_i}$ is also not contained in $Q_{u'+1}, Q_{u'+2}, \ldots, Q_k$, as explained below.

We assume, without loss of generality, that $Q_{m+1}, Q_{m+2}, \ldots, Q_{m+c}$ contain $\overline{p_{m+1}}, \overline{p_{m+2}}, \ldots, \overline{p_{m+c}}$, respectively (refer to Case 4 where Q_t was constructed as the combination of Q'_t and two links (p_t, $\overline{p_t}$) and ($\overline{p_t}$, d_t) for all $u+1 \le t \le v$), where $u'+1 \le m+1 \le m+c \le k$. Suppose that $a \notin \{d_{u'+1}, d_{u'+2}, \ldots, d_k\} \cup \{\overline{p_{m+1}}, \overline{p_{m+2}}, \ldots, \overline{p_{m+c}}\}$ is a node in $Q_{u'+1}$, $Q_{u'+2}$, …, Q_k. We have $\overline{q_i} \neq a$ because $|\overline{q_i}| = \lceil k/2 \rceil + 1 \neq |a|$ (as a consequence of Case 4). By our assumption (refer to the beginning paragraphs enclosed by /* and */ in Case 4 and Case 5), we have $q_i \notin \{p_{m+1}, p_{m+2}, \ldots, p_{m+c}\}$, i.e., $\overline{q_i} \notin \{\overline{p_{m+1}}, \overline{p_{m+2}}, \ldots, \overline{p_{m+c}}\}$, and $\overline{q_i} \notin \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$.

Then we show that d_i is not contained internally in $Q'_{r'+1}$, …, $Q'_{u'}$, $Q_{u'+1}$, …, Q_{k+1}. Since $|d_i| = \lceil k/2 \rceil + 2$ and $Q'_{r'+1}$, $Q'_{r'+2}$, …, $Q'_{u'}$ have lengths at most $\lfloor k/2 \rfloor$, d_i is not contained internally in $Q'_{r'+1}$, $Q'_{r'+2}$, …, $Q'_{u'}$. We assume, without loss of generality, that $Q_{m'+1}$, $Q_{m'+2}$, …, $Q_{m'+c'}$ contain $\overline{f_{m'+1}}$, $\overline{f_{m'+2}}$, …, $\overline{f_{m'+c'}}$, respectively, where $u'+1 \le m'+1 \le m'+c' \le k$. Suppose that $b \notin \{\overline{f_{m'+1}}, \overline{f_{m'+2}}, \ldots, \overline{f_{m'+c'}}\}$ is a node in $Q_{u'+1}$, $Q_{u'+2}$, …, Q_k. We have $|b| \le \lceil k/2 \rceil + 1$ as a consequence of the execution of Case 4. Since $|d_i| = \lceil k/2 \rceil + 2$, we have $d_i \neq b$. By our assumption, we have $\overline{d_i} = d_l$ ($\notin \{f_{m'+1}, f_{m'+2}, \ldots, f_{m'+c'}\}$) for some $u'+1 \le l \le k$, i.e., $d_i \notin \{\overline{f_{m'+1}}, \overline{f_{m'+2}}, \ldots, \overline{f_{m'+c'}}\}$. Since $|d_i| \le |d_{k+1}|$ and the subpath of Q_{k+1} from $\overline{s}$ to d_{k+1} is shortest, d_i is not contained internally in Q_{k+1}.

Q_1, Q_2, …, $Q_{r'}$ are disjoint to each other, provided that q_1, q_2, …, $q_{r'}$ are all distinct, $\overline{q_1}$, $\overline{q_2}$, …, $\overline{q_{r'}}$ are all distinct, each of $\overline{q_1}$, $\overline{q_2}$, …, $\overline{q_{r'}}$, d_1, d_2, …, $d_{r'}$ is not contained in Q'_1, Q'_2, …, $Q'_{r'}$, and $\{\overline{q_1}, \overline{q_2}, \ldots, \overline{q_{r'}}\} \cap \{d_1, d_2, \ldots, d_{r'}\}$ is empty. By our assumption, we have q_1, q_2, …, $q_{r'}$ all distinct (i.e., $\overline{q_1}$, $\overline{q_2}$, …, $\overline{q_{r'}}$ all distinct). Since $|\overline{q_i}| = \lceil k/2 \rceil + 1$ and $|d_i| = \lceil k/2 \rceil + 2$ for all $1 \le i \le r'$, we have $\{\overline{q_1}, \overline{q_2}, \ldots, \overline{q_{r'}}\} \cap \{d_1, d_2, \ldots, d_{r'}\}$ empty. Besides, since Q'_i is shortest with length $|q_i| = \lfloor k/2 \rfloor - 1$ for all $1 \le i \le r'$, each of $\overline{q_1}$, $\overline{q_2}$, …, $\overline{q_{r'}}$, d_1, d_2, …, $d_{r'}$ is not contained in Q'_1, Q'_2, …, $Q'_{r'}$.

For all $r'+1 \le j \le u'$, Q_j was constructed depending on whether $\overline{d_j} = d_t$ for some $u'+1 \le t \le k$ or not. If $\overline{d_j} = d_t$, then Q_j was constructed as the combination of the subpath of Q'_j from s to g_j and two links (g_j, $\overline{g_j}$) and ($\overline{g_j}$, d_j), where g_j denotes the immediate predecessor of $\overline{d_j}$ in Q'_j. If $\overline{g_j} \in Q_{k+1}$, then Q'_j and Q_t need to be swapped prior to the construction of Q_j. Since Q'_j has length at most $\lfloor k/2 \rfloor$, Q_j has length at most $(\lfloor k/2 \rfloor - 1) + 2 = \lfloor k/2 \rfloor + 1$. Q_j is disjoint with Q_1, Q_2, …, $Q_{r'}$, $Q_{u'+1}$, $Q_{u'+2}$, …, Q_{k+1}, as explained below.

If $\overline{g_j} \notin Q_{k+1}$, then it suffices to show that $\overline{g_j}$ is not contained in Q_1, Q_2, …, $Q_{r'}$, $Q_{u'+1}$, $Q_{u'+2}$, …, Q_k

and d_j is not contained internally in $Q_1, Q_2, \ldots, Q_{r'}, Q_{u'+1}, Q_{u'+2}, \ldots, Q_{k+1}$. We first show $|g_j|\le\lfloor k/2\rfloor-2$, i.e., $|\overline{g_j}|\ge\lceil k/2\rceil+2$. Since there is no complement link in Q'_j, we have $|g_j|=|\overline{d_j}|-1$ or $|\overline{d_j}|+1$. Since $|\overline{d_j}|\le\lfloor k/2\rfloor-2$, we have $|g_j|\le\lfloor k/2\rfloor-1$. If $|g_j|=\lfloor k/2\rfloor-1$, then $|\overline{d_j}|=\lfloor k/2\rfloor-2$. By Lemma 8 ($|g_j|=|\overline{d_j}|+1$), we have $\overline{d}_{j,W(\overline{d_j})}=0$. Consequently ($|\overline{d_j}|=\lfloor k/2\rfloor-2$, $\overline{d_j}=d_t$, and $\overline{d}_{j,W(\overline{d_j})}=0$), a further modification of D and W is still required, which is a contradiction.

We show that $\overline{g_j}$ is not contained in $Q_1, Q_2, \ldots, Q_{r'}, Q_{u'+1}, Q_{u'+2}, \ldots, Q_k$ as follows. Since $|\overline{g_j}|\ge\lceil k/2\rceil+2$ and Q'_i is shortest with length $|q_i|=\lfloor k/2\rfloor-1$ for all $1\le i\le r'$, $\overline{g_j}$ is not contained in $Q'_1, Q'_2, \ldots, Q'_{r'}$. Further, $\overline{g_j}$ is not contained in $Q_1, Q_2, \ldots, Q_{r'}$, provided $\overline{g_j}\notin\{\overline{q_1}, \overline{q_2}, \ldots, \overline{q_{r'}}, d_1, d_2, \ldots, d_{r'}\}$. Clearly, we have $g_j\notin\{q_1, q_2, \ldots, q_{r'}\}$, i.e., $\overline{g_j}\notin\{\overline{q_1}, \overline{q_2}, \ldots, \overline{q_{r'}}\}$. Since $\overline{d_i}\in\{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ for all $1\le i\le r'$ and $g_j\notin\{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$, we have $g_j\notin\{\overline{d_1}, \overline{d_2}, \ldots, \overline{d_{r'}}\}$, i.e., $\overline{g_j}\notin\{d_1, d_2, \ldots, d_{r'}\}$. On the other hand, $\overline{g_j}$ is not contained in $Q_{u'+1}, Q_{u'+2}, \ldots, Q_k$ because $|\overline{g_j}|>|b|$ and $g_j\notin\{f_{m'+1}, f_{m'+2}, \ldots, f_{m'+c'}\}$ (i.e., $\overline{g_j}\notin\{\overline{f_{m'+1}}, \overline{f_{m'+2}}, \ldots, \overline{f_{m'+c'}}\}$).

Then we show that d_j is not contained internally in $Q_1, Q_2, \ldots, Q_{r'}, Q_{u'+1}, Q_{u'+2}, \ldots, Q_{k+1}$. Suppose that $g\notin\{d_1, d_2, \ldots, d_{r'}\}$ is a node in $Q_1, Q_2, \ldots, Q_{r'}$. Since $|d_j|\ge\lceil k/2\rceil+2$ (by our assumption) and $|g|\le\lceil k/2\rceil+1$ (refer to the construction method of Case 5), d_j is not contained internally in $Q_1, Q_2, \ldots, Q_{r'}$. Besides, d_j is not contained internally in $Q_{u'+1}, Q_{u'+2}, \ldots, Q_k$ because $|d_j|>|b|$ (i.e., $d_j\ne b$) and $\overline{d_j}\notin\{f_{m'+1}, f_{m'+2}, \ldots, f_{m'+c'}\}$ (i.e., $d_j\notin\{\overline{f_{m'+1}}, \overline{f_{m'+2}}, \ldots, \overline{f_{m'+c'}}\}$). Since $|d_j|\le|d_{k+1}|$ and the subpath of Q_{k+1} from $\overline{s}$ to d_{k+1} is shortest, d_j is not contained internally in Q_{k+1}.

On the other hand, if $\overline{g_j}\in Q_{k+1}$, then Q'_j and Q_t were swapped and g_j was changed to be the immediate predecessor of $\overline{d_j}$ in (new) Q'_j. It suffices to show (new) $\overline{g_j}\notin Q_{k+1}$, in order for Q_j to be disjoint with $Q_1, Q_2, \ldots, Q_{r'}, Q_{u'+1}, Q_{u'+2}, \ldots, Q_{k+1}$. If $d_{t,W(d_t)}=1$ and $\overline{d}_{j,W(\overline{d_j})}=1$, then by Lemma 8 (old) Q_t and (old) Q'_j are shortest with lengths $|d_t|$ ($=|\overline{d_j}|$), which implies $|$(new) $g_j|=|$(old) $g_j|$, i.e., $|$(new) $\overline{g_j}|=|$(old) $\overline{g_j}|$. Since (old) $\overline{g_j}\in Q_{k+1}$, $|$(new) $\overline{g_j}|=|$(old) $\overline{g_j}|$, (new) $\overline{g_j}\ne$ (old) $\overline{g_j}$, and the subpath of Q_{k+1} from $\overline{s}$ to d_{k+1} is shortest, we have (new) $\overline{g_j}\notin Q_{k+1}$. If $d_{t,W(d_t)}=0$, then Lemma 8 assures $|$(new)

g_j|=|d_t|+1=|$\overline{d_j}$|+1, i.e., |(new) $\overline{g_j}$|= |d_j|–1<|d_{k+1}|. It follows that we have (new) $\overline{g_j} \notin Q_{k+1}$. If $\overline{d}_{j,\,W(\overline{d_j})} = 0$, then we have |(old) g_j|=|$\overline{d_j}$|+1, i.e., |(old) $\overline{g_j}$|=|d_j|–1<|d_{k+1}|. Hence, (old) $\overline{g_j} \notin Q_{k+1}$, which is a contradiction.

If $\overline{d_j} \neq d_v$ for all $u'+1 \leq v \leq k$, then Q_j was constructed as the combination of Q'_j and a link ($\overline{d_j}$, d_j), which has length at most $\lfloor k/2 \rfloor+1$. Q_j is disjoint with $Q_1, Q_2, \ldots, Q_{r'}, Q_{u'+1}, Q_{u'+2}, \ldots, Q_{k+1}$, provided d_j is not contained internally in $Q_1, Q_2, \ldots, Q_{r'}, Q_{u'+1}, Q_{u'+2}, \ldots, Q_{k+1}$ and $\overline{d_j} \notin \{d_1, d_2, \ldots, d_{r'}, d_{k+1}\}$. The former has been shown above. The latter holds because $|\overline{d_j}| \leq \lfloor k/2 \rfloor - 2$ and $|d_{k+1}| \geq |d_i| \geq \lceil k/2 \rceil + 2$ for all $1 \leq i \leq r'$.

Finally we show that $Q_{r'+1}, Q_{r'+2}, \ldots, Q_{u'}$ are disjoint to each other. Without loss of generality, we assume $\overline{d_c} \in \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ for all $r'+1 \leq c \leq l'$ and $\overline{d_c} \notin \{d_{u'+1}, d_{u'+2}, \ldots, d_k\}$ for all $l'+1 \leq c \leq u'$, where $r' \leq l' \leq u'$. It suffices to show that $\overline{g_{r'+1}}, \overline{g_{r'+2}}, \ldots, \overline{g_{l'}}$ are all distinct, $\{\overline{g_{r'+1}}, \overline{g_{r'+2}}, \ldots, \overline{g_{l'}}\} \cap \{d_{r'+1}, d_{r'+2}, \ldots, d_{u'}\}$ is empty, and each of $\overline{g_{r'+1}}, \overline{g_{r'+2}}, \ldots, \overline{g_{l'}}, d_{r'+1}, d_{r'+2}, \ldots, d_{u'}$ is not contained in $Q'_{r'+1}, Q'_{r'+2}, \ldots, Q'_{u'}$. The first two hold because $g_{r'+1}, g_{r'+2}, \ldots, g_{l'}$ are all distinct and $\{g_{r'+1}, g_{r'+2}, \ldots, g_{l'}\} \cap \{\overline{d_{r'+1}}, \overline{d_{r'+2}}, \ldots, \overline{d_{u'}}\}$ is empty. The last holds because $|\overline{g_c}| \geq \lceil k/2 \rceil + 2$ for all $r'+1 \leq c \leq l'$, $|d_j| \geq \lceil k/2 \rceil + 2$ for all $r'+1 \leq j \leq u'$, and $Q'_{r'+1}, Q'_{r'+2}, \ldots, Q'_{u'}$ have lengths at most $\lfloor k/2 \rfloor$ (recall that none of $Q'_{r'+1}, Q'_{r'+2}, \ldots, Q'_{u'}$ contain complement links).

3.3 The main result

Since the folded hypercube is node symmetric, the following theorem is an immediate consequence of Lemma 17.

Theorem 1. Given arbitrary $k+2$ distinct nodes $s, d_1, d_2, \ldots, d_{k+1}$ of a k-dimensional folded hypercube, there are $k+1$ node-disjoint paths from s to $d_1, d_2, \ldots, d_{k+1}$, respectively, whose maximal length is at most $\lceil k/2 \rceil+1$, where $k \geq 4$.

Theorem 1 provides an upper bound of $\lceil k/2 \rceil+1$ on the Rabin number of a k-dimensional folded hypercube. Since a lower bound of $\lceil k/2 \rceil+1$ was proposed in [10, 18], the Rabin number of a k-dimensional folded hypercube is equal to $\lceil k/2 \rceil+1$, as stated below.

Corollary 1. The Rabin number of a k-dimensional folded hypercube is equal to $\lceil k/2 \rceil+1$, where $k \geq 4$.

It is not difficult to see that the Rabin number of a k-dimensional folded hypercube is equal to $\lceil k/2 \rceil$

for k=2 or 3.

4 Discussion and Conclusion

The main contribution of this paper is to show the effectiveness of routing functions in constructing one-to-many disjoint paths in networks. The hypercube and the folded hypercube were taken as two illustrative examples. For both, any minimal routing function can be used to produce a maximal number of disjoint paths whose maximal length is minimized. As a byproduct, the strong Rabin number of the hypercube and the Rabin number of the folded hypercube were computed. The latter has been left as an open problem in [18]. Besides minimal routing functions, there are some other routing functions with different characteristics which can be used to produce disjoint paths with different properties. For example, the disjoint paths produced for the hypercube will have minimum total length if an implicit routing function in [14] is used. Some existing methods (e.g., [14, 20]) for constructing disjoint paths involved the use of an implicit routing function.

In [14], n non-empty subsets $X_1, X_2, \ldots, X_n$ of $\{1, 2, \ldots, n\}$ were used to represent n nodes $d_1, d_2, \ldots, d_n$ so that for all $1 \le u \le n$ and $1 \le w \le n$, $w \in X_u$ if and only if $d_{u,w}=1$. A set of m distinct integers $t_1 \in X_{k_1}$, $t_2 \in X_{k_2}$, …, $t_m \in X_{k_m}$ is called a *partial System of Distinct Representatives* (*SDR* for short) for $\{X_1, X_2, \ldots, X_n\}$ if $k_1, k_2, \ldots, k_m$ are all distinct, where $m \le n$. Further, $\{t_1, t_2, \ldots, t_m\}$ is *maximum* if m is maximized, and is called an *SDR* if $m=n$. A maximum partial SDR $\{t_1, t_2, \ldots, t_m\}$ can be used to construct a maximal number of disjoint paths whose total length is minimized, if there is no $j \in \{1, 2, \ldots, n\} - \{k_1, k_2, \ldots, k_m\}$ satisfying the following two: (C1) $X_j \subset X_{k_i}$ for some $1 \le i \le m$ and (C2) there exists an SDR for $\{X_{k_1}, X_{k_2}, \ldots, X_{k_{i-1}}, X_j, X_{k_{i+1}}, \ldots, X_{k_m}\}$. Such a maximum partial SDR can be determined in $O(n^{2.5})$ time (see [14]).

Actually, a maximum partial SDR $\{t_1, t_2, \ldots, t_m\}$ can be regarded as a routing function from $\{X_1, X_2, \ldots, X_n\}$ to $\{1, 2, \ldots, n\}$ which maps X_{k_i} to t_i for all $1 \le i \le m$. Suppose $L_F = \{u \mid d_{u,F(d_u)} = 1 \text{ and } 1 \le u \le n\}$. It follows that *Paths1*$(F, n, n, \{d_1, d_2, \ldots, d_n\}, \{1, 2, \ldots, n\})$ will produce a maximal number of disjoint paths whose total length is minimized, if $|L_F|$ is maximized and there is no $j \in \{1, 2, \ldots, n\} - L_F$ satisfying the following two: (C1′) $d_j \ne d_l$ and $d_{j,w} \le d_{l,w}$ for some $l \in L_F$ and all $1 \le w \le n$ and (C2′) there exists a routing function Y with $d_{j,Y(d_j)} = d_{v,Y(d_v)} = 1$ for all $v \in L_F - \{l\}$. Here, F: $\{d_1, d_2, \ldots, d_n\} \to \{1, 2, \ldots, n\}$ with maximum $|L_F|$ corresponds to a maximum partial SDR (L_F corresponds to $\{k_1, k_2, \ldots, k_m\}$ and $\{F(d_u) \mid u \in L_F\}$ corresponds to $\{t_1, t_2, \ldots, t_m\}$). Besides, (C1′) and (C2′) correspond to (C1) and (C2), respectively. A routing function F with maximum $|L_F|$ so that there is no $j \in \{1, 2, \ldots, n\} - L_F$ satisfying (C1′) and (C2′) can be determined with the same time complexity as a maximum partial SDR so that there is no $j \in \{1, 2, \ldots, n\} - \{k_1, k_2, \ldots, k_m\}$ satisfying (C1) and (C2).

There exist other routing functions that can result in m disjoint paths from s to $d_1, d_2, \ldots, d_m$ whose total length is minimized. For example, if (C1′) is changed to "$|d_j| < |d_l|$ for some $l \in L_F$", then the resulting F also serves the purpose. The method of [20] for constructing disjoint paths also involved the use of an implicit routing function. It is worth while exploring more relations between the characteristics of routing functions and the properties of disjoint paths.

Although both [20] and [14] also constructed disjoint paths in the hypercube, their construction methods cannot be used to obtain Theorem 1. The reason is that the correctness of Lemma 16 cannot be assured by their construction methods.

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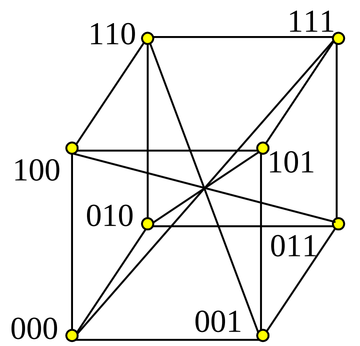

Figure 1. The structure of a 3 - fcube.

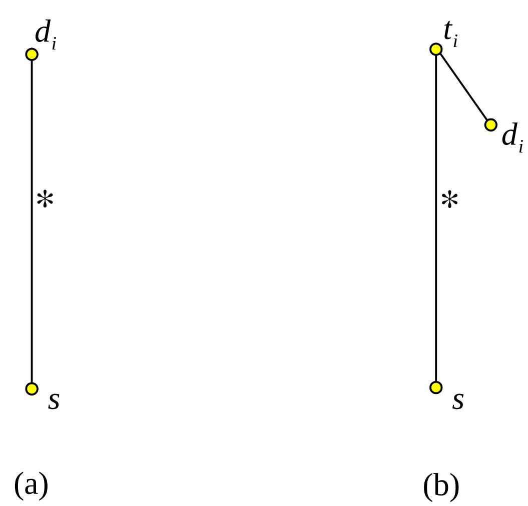

Figure 2. The path from s to d_i (* denotes a shortest path).
(a) When $d_{i,F(d_i)} = 1$. (b) When $d_{i,F(d_i)} = 0$.

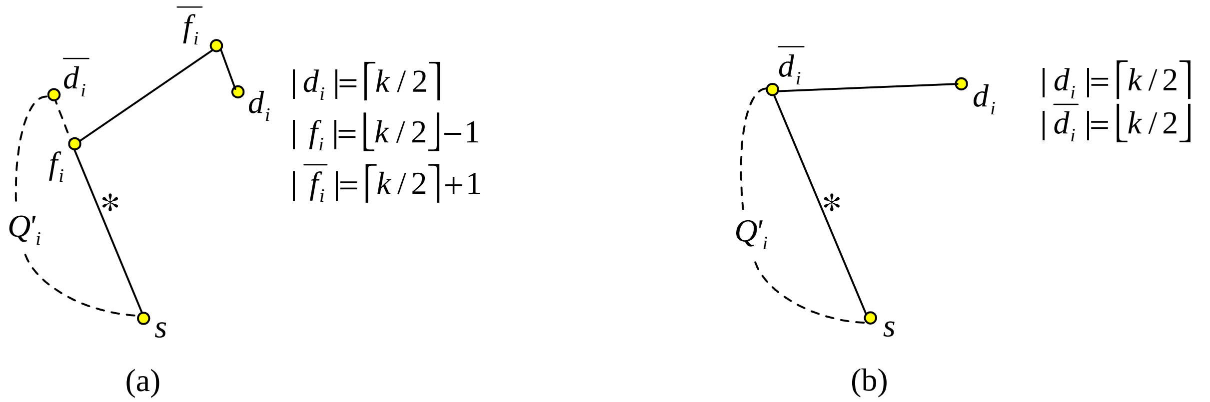

Figure 3. The construction of Q_i $(1 \le i \le r)$ from Q'_i in Case 3. (a) When $\overline{d_i} = d_h$ for some $r+1 \le h \le k+1$. (b) When $\overline{d_i} \ne d_j$ for all $r+1 \le j \le k+1$.

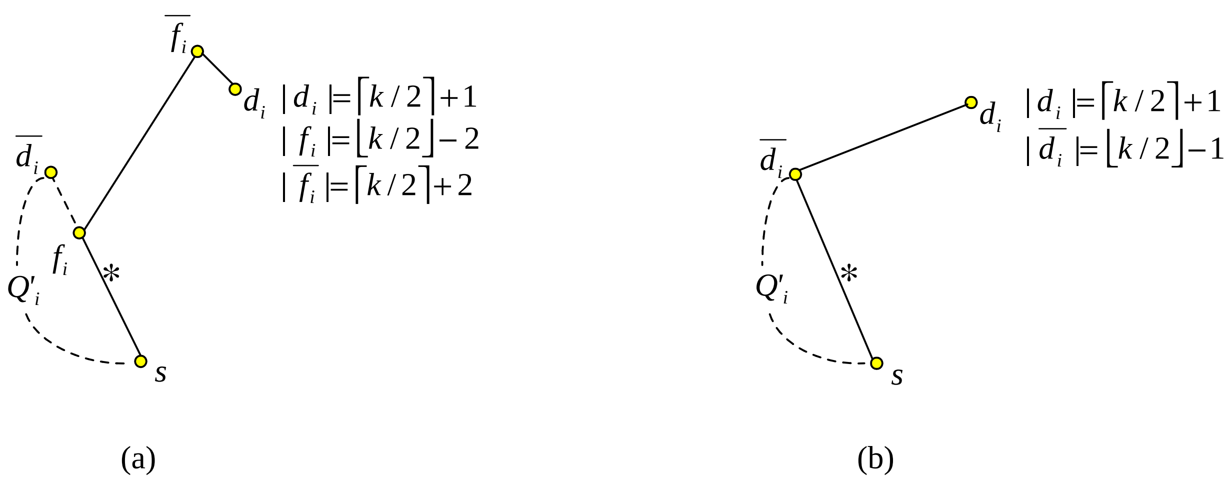

Figure 4. The construction of Q_i $(1 \le i \le r)$ from Q'_i in Case 4. (a) When $\overline{d_i} = d_w$ for some $v+1 \le w \le k$. (b) When $\overline{d_i} \ne d_j$ for all $v+1 \le j \le k$.

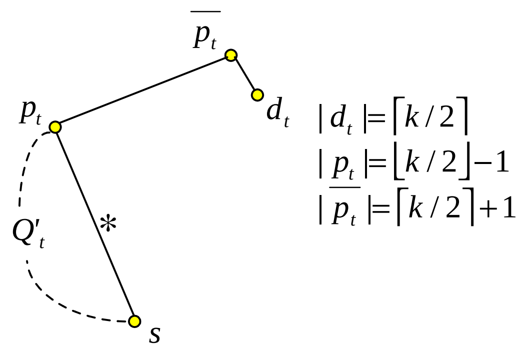

Figure 5. The construction of Q_t $(u+1 \le t \le v)$ from Q'_t in Case 4.

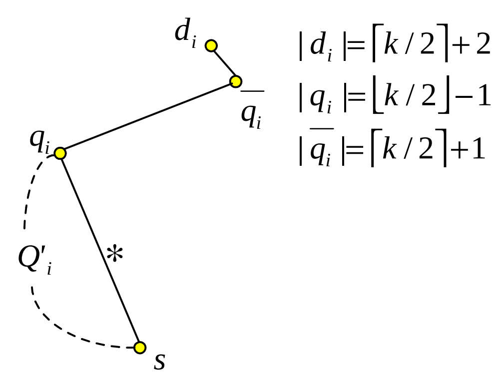

Figure 6. The construction of Q_i $(1 \le i \le r')$ from Q'_i in Case 5.