

THERMOELASTIC PROPERTIES OF SHORT-COATED FIBER COMPOSITES: EFFECTS OF LENGTH AND ORIENTATION DISTRIBUTIONS

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Abstract

The effective thermoelastic properties of composites reinforced with short-coated fibers whose orientations and aspect ratios are varied have been formulated. Under the assumption of thin coating, the stress field of the coated layer remains uniform across the thickness of the layer but otherwise possesses variation along other directions and can be found in terms of the stress field of the fiber and the direction cosines through the use of the interface jump condition between the coating and the fiber. The effective thermoelastic properties are then derived based on the Mori–Tanaka scheme and the modified Walpole method. Numerical examples including some parametric studies are also included.

Keywords: effective thermoelastic property, coating, random orientation, short fiber

1 INTRODUCTION

Composite materials have been extensively used in many applications because of their high specific stiffness and strength. Discontinuous fiber-reinforced plastics are attractive in their versatility of properties and relatively low fabrication costs (e.g. injection molding). They consist of relatively short variable length, and imperfectly aligned fibers distributed in a continuous matrix material (usually isotropic). The orientation of short fibers depends on the processing conditions (e.g. hot pressing, extrusion or rolling) employed and may vary from random to nearly aligned. The length distribution of short fibers depends on the shear forces of screw and ram which break the fibers during extrusion compounding or injection molding. Thus it is imperative to take into consideration the effects of the bias in variation of fiber orientation and fiber length on the thermoelastic properties of the composites.

In most of the applications fibers are not coated. Recently, there has been a growing demand for coated fibers as a reinforcement in some new application areas such as electrical composites, metal-matrix

composites (MMC) and ceramic-matrix composites (CMC) intended for high-temperature applications. In MMC and CMC, coating prevents the fiber from debonding owing to the chemical reaction between the matrix and the fiber. In electrical composites, conductive metals are coated on nonconductive fibers to obtain the desired level of conductivity of the composite at reduced material cost. Coating can also act as a protective barrier for the fiber and prevent oxidation, thus enabling it to be used for extended periods of time in an oxidizing environment. Coating of different materials and of varying amounts have also been applied to the fibers to prevent or control the degree of bonding at the fiber/matrix interface, which leads to a substantial improvement in the strength and damage tolerance of composites.

A basic problem in composites with coated fibers is the calculation of stress field and thermoelastic properties. Walpole¹ proposed a simple method, avoiding actually solving the elastic field, to calculate the stress field within a thin coating if the solution to the stress field is known *a priori* for a single noncoated fiber embedded in an infinite matrix. Hatta and Taya² calculated the thermal stress field within the coating for a coated fiber composite by modifying the Walpole method together with Eshelby's equivalent inclusion method. The above-mentioned studies limited themselves only to the stress field and did not discuss thermoelastic properties. Pagano and Tandon^{3,4} gave their bounds to the thermoelastic properties for multidirectional coated fiber composites, in which the fibers were continuous. Their work holds for both thin and thick coatings but they limited themselves to composites with long continuous fibers, partially due to the difficulty in solving the stress field of short-coated fiber composites. Recently the work of Hatta and Taya² has been extended⁵ to calculate the thermoelastic properties of composites reinforced with aligned short-coated fibers. Most of the existing models to predict the thermoelastic properties of a short-coated fiber composite were based on the assumption that all short fibers have the same length

(or aspect ratio). Since the thermoelastic properties of a composite are not a linear function of fiber length, a uniform fiber length assumption may result in greater prediction error than in the actual case. Thus, a further extension of previous work to predict the effective thermoelastic properties of composites reinforced with randomly oriented short-coated fibers is aimed at in this work. Again it is assumed that the coated layer is thin and hence the stress field within the coating is reasonably assumed to be uniform through the thickness but otherwise may vary along other directions. Numerical examples including several parametric studies are conducted and the results presented.

2 EFFECTIVE ELASTIC MODULI

Consider an infinite elastic body (without inclusions, i.e. the matrix only) subjected to a uniform stress field, σ_0 . The uniform strain, ϵ_0 , produced is:

$$\epsilon_0 = \mathbf{C}^0{}^{-1} \sigma_0 \quad (1)$$

in which $\mathbf{C}^0$ is the elastic modulus tensor of the matrix (which is assumed to be isotropic). In this paper boldface upper-case English letters and lower-case Greek letters denote fourth- and second-order tensors, respectively. When there are ellipsoidal inclusions (coated fibers) present in the matrix, a perturbed stress field is induced and is denoted as $\tilde{\sigma}(x)$. The total stress field, $\sigma(x)$, is now the sum of two stress fields: $\sigma_0 + \tilde{\sigma}(x)$. The volumetric average of the perturbed stress and strain fields of the matrix, $\bar{\sigma}$ and $\bar{\epsilon}$, are defined according to:

$$\bar{\sigma} = \langle \tilde{\sigma}(x) \rangle_{D-\Omega} = \frac{1}{V_{D-\Omega}} \int_{D-\Omega} \tilde{\sigma}(x) dV = \mathbf{C}^0 \bar{\epsilon} \quad (2)$$

in which D , Ω , and $D - \Omega$ are the domains of the whole elastic body (composite), all the coated fibers and the matrix respectively; $V_{D-\Omega}$ is the volume of the matrix; and $\langle \rangle_{D-\Omega}$ is the volumetric average for the domain $D - \Omega$. Thus the average stress field in the matrix is:

$$\sigma_m = \sigma_0 + \bar{\sigma} = \mathbf{C}^0 \cdot (\epsilon_0 + \bar{\epsilon}) \quad (3)$$

The elastic modulus tensors of the fiber and the coating are denoted by $\mathbf{C}^f$ and $\mathbf{C}^c$.

Consider now the fibers being oriented in a distributional way. Let $x'_1 - x'_2 - x'_3$ be the local coordinate system for the inclusions and $x_1 - x_2 - x_3$ be the global coordinate system of the whole elastic body. Let $\mathbf{Q}$ be a coordinate transformation matrix, such that:

$$X' = \mathbf{Q} \cdot X \quad (4)$$

In the following the symbols with and without primes denote respectively quantities in local and global

coordinate systems. Short fibers with different aspect ratios are modelled by spheroidal inhomogeneities (i.e. the cases of $l/d > 1$, $l/d = 1$, and $l/d < 1$ for prolate spheroid, sphere and oblate spheroid, respectively). The domain Ω consists of N subdomains, $\Omega_{(i)}$, $i = 1, 2, \dots, N$. The subscript i denotes the i th group of fibers which happen to share the same fiber aspect ratio $(l/d)_{(i)}$. Thus the i th group of fibers and the j th group of fibers are viewed as different kinds of fibers if their aspect ratios are different from each other no matter whether they possess the same material moduli or not. Volume fractions of fiber and coating of the i th group of fibers are denoted $V_{f(i)}$ and $V_{c(i)}$, respectively. The volume fractions of whole fibers and whole coatings are hence the sum of those of all the groups, namely, $V_f = \sum_{i=1}^N V_{f(i)}$ and $V_c = \sum_{i=1}^N V_{c(i)}$. Then by the use of Eshelby's equivalent inclusion method,^{6,7} one has, in the domain of a typical fiber, $\Omega_{f(i)}$:

$$\begin{aligned} \sigma'_{f(i)} &= \mathbf{C}^f \cdot \epsilon'_{f(i)} = \mathbf{C}^f \cdot (\epsilon'_0 + \bar{\epsilon}' + \epsilon'^{11'}_{(i)} + \epsilon'^{12'}_{(i)}) \\ &= \mathbf{C}^{m'} \cdot (\epsilon'_0 + \bar{\epsilon}' + \epsilon'^{11'}_{(i)} + \epsilon'^{12'}_{(i)} - \epsilon'^{*1'}_{(i)}) \end{aligned} \quad (5)$$

and, in the domain of coating of a typical fiber, $\Omega_{c(i)}$:

$$\begin{aligned} \sigma'_{c(i)} &= \mathbf{C}^c \cdot \epsilon'_{c(i)} = \mathbf{C}^c \cdot (\epsilon'_0 + \bar{\epsilon}' + \epsilon'^{22'}_{(i)} + \epsilon'^{21'}_{(i)}) \\ &= \mathbf{C}^{m'} \cdot (\epsilon'_0 + \bar{\epsilon}' + \epsilon'^{22'}_{(i)} + \epsilon'^{21'}_{(i)} - \epsilon'^{*2'}_{(i)}) \end{aligned} \quad (6)$$

in which $\epsilon'^{*1'}$ and $\epsilon'^{*2'}$ are 'eigenstrains'⁸ defined in $\Omega_{f(i)}$ and $\Omega_{c(i)}$, respectively, and $\epsilon'^{kl'}_{(i)}$ is the disturbance of the strain field in the k th domain due to the existence of the l th phase with k (or l) = 1 (fiber) and 2 (coating).

From previous work,⁶ we know that:

$$\epsilon'^{11'}_{(i)} = \mathbf{S}'_{(i)} \cdot \epsilon'^{*1'}_{(i)} \quad (7)$$

where $\mathbf{S}'_{(i)}$ is Eshelby's tensor.^{6,8}

In the present model the disturbance of the strain field in $\Omega_{f(i)}$ due to the coating is averaged over the fiber domain $\Omega_{f(i)}$. The average of disturbed strain field is denoted as $\epsilon'^{12'}_{(i)}$. Thus we have $\epsilon'^{12'}_{(i)} = \langle \tilde{\epsilon}^{12'}_{(i)}(x) \rangle_1$. Then according to Hatta and Taya,² under the assumption of thin coating layer, one has:

$$\epsilon'^{12'} = \frac{V_{c(i)}}{V_{f(i)}} \langle \mathbf{S}^{0'}_{(i)} \cdot \epsilon'^{*2'}_{(i)} \rangle_2 \quad (8)$$

in which $\frac{V_{c(i)}}{V_{f(i)}}$ is assumed to be a constant for every i and $\langle \rangle_1$ and $\langle \rangle_2$ denote the averaged quantity over Ω_1 and $\Omega_2 - \Omega_1$, respectively, and further, the fourth-rank tensor $\mathbf{S}^{0'}_{(i)}$ is related to the Eshelby tensor $\mathbf{S}'_{(i)}$ as well as the surface directional cosine $n'_{k(i)}$ of a fiber surface outward unit normal $\mathbf{n}'_{(i)}$, namely:

$$S^{0'}_{klmn(i)} = S'_{klmn(i)} - C^{m'}_{pqmn} n'_{q(i)} n'_{l(i)} K^{m-1'}_{kp(i)} \quad (9)$$

in which

$$K^{m'}_{ij(i)} \equiv C^{m'}_{ikjl} n'_{k(i)} n'_{l(i)} \quad (10)$$

Under the assumption of thin coating and hence the constant variation of stresses and strains through the thickness of the coating, the volume average of a stress or strain function, being a function of surface direction, i.e. $F(\mathbf{n}')$, over the domain of coating, $\Omega_{c(i)}$, can be calculated in a simple way.⁵

Now since the total volumetric average of the stress field of the whole composite must be equal to σ_0 , it follows that:

$$\sigma_0 = f_0 \sigma_m + \left\langle \left\langle \sum V_{f(i)} \sigma_{f(i)}^f \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} \sigma_{c(i)}^c \right\rangle \right\rangle_2 \quad (11)$$

in which f_0 is the volume fraction of the matrix and

$$\langle \langle A_{(i)}(\theta, \phi) \rangle \rangle \equiv \frac{\int_0^\Theta \int_0^{2\pi} A_{(i)}(\theta, \phi) \rho_{(i)}(\theta, \phi) \sin \theta d\theta d\phi}{\int_0^\Theta \int_0^{2\pi} \rho_{(i)}(\theta, \phi) \sin \theta d\theta d\phi} \quad (11a)$$

and $\rho_{(i)}(\theta, \phi)$ and Θ are respectively the fiber distribution density (i.e. number of the i th kind of inclusions per unit solid angle) and the fiber distribution limit angle ($\Theta \leq 90^\circ$). If $\rho(\theta, \phi) = \text{constant}$ and $\Theta = 90^\circ$, then the composite is isotropic. Substituting eqns (3), (5) and (6) into eqn (11) gives:

$$\bar{\epsilon} + \left\langle \left\langle \sum V_{f(i)} (\epsilon_{(i)}^{11} + \epsilon_{(i)}^{12} - \epsilon_{(i)}^{*1}) \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} (\epsilon_{(i)}^{21} + \epsilon_{(i)}^{22} - \epsilon_{(i)}^{*2}) \right\rangle \right\rangle_2 = 0 \quad (12)$$

In addition, the total volumetric average of the strain field of the whole composite, ϵ^T , can be expressed as

$$\begin{aligned} \epsilon^T &= f_0(\epsilon_0 + \bar{\epsilon}) + \left\langle \left\langle \sum V_{f(i)} (\epsilon_{0(i)} + \bar{\epsilon}_{(i)} + \epsilon_{(i)}^{11} + \epsilon_{(i)}^{12}) \right\rangle \right\rangle \\ &+ \left\langle \left\langle \sum V_{c(i)} (\epsilon_{0(i)} + \bar{\epsilon}_{(i)} + \epsilon_{(i)}^{21} + \epsilon_{(i)}^{22}) \right\rangle \right\rangle_2 \\ &= \epsilon_0 + \bar{\epsilon} \\ &+ \left\langle \left\langle \sum v_{f(i)} (\epsilon^{11} + \epsilon^{12}) \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} (\epsilon_{(i)}^{21} + \epsilon_{(i)}^{22}) 1F_2 \right\rangle \right\rangle \\ &= \epsilon_0 + \left\langle \left\langle \sum V_{f(i)} \epsilon_{(i)}^{*1} \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} \epsilon_{(i)}^{*2} \right\rangle \right\rangle_2 \quad (13) \end{aligned}$$

in which the last equality is obtained through the use of eqn (12).

Let us now focus on the interface between a short fiber and a thin coating. By continuity of traction and displacement vectors at the interface, it requires that:^{9,10}

$$\mathbf{u}_{(i)}^s - \mathbf{u}_{(i)}^f = 0 \quad (14)$$

and

$$(\sigma_{(i)}^c - \sigma_{(i)}^f) \cdot \mathbf{n}_{(i)}' = 0 \quad (15)$$

where $\mathbf{n}'$ is a unit vector outward normal to the fiber surface. The displacement gradient tensor $u_{k,l(i)}'$ is discontinuous across the interface and the jump of $u_{k,l(i)}'$ across the interface can be expressed as:

$$u_{k,l(i)}^{c'} - u_{k,l(i)}^{f'} = \lambda_{k(i)}' n_{l(i)}'$$

or

$$\epsilon_{(i)}^{c'} - \epsilon_{(i)}^{f'} = \frac{1}{2} (\gamma_{(i)}' \mathbf{n}_{(i)}' + \mathbf{n}_{(i)}' \gamma_{(i)}') \quad (16)$$

Substituting eqns (5), (6) and (16) into eqn (15) yields:

$$\lambda_{p(i)}' = K_{pi(i)}^{c-1'} (C_{ijkl}^{m'} - C_{ijkl}^{c'}) \epsilon_{kl(i)}^{f'} n_{j(i)}' - K_{pi(i)}^{c-1'} C_{ijkl}^{m'} \epsilon_{kl(i)}^{*1'} n_{j(i)}' \quad (17)$$

in which

$$K_{ij(i)}^{c'} \equiv C_{ijkl}^{c'} n_{k(i)}' n_{l(i)}' \quad (17a)$$

Substituting eqns (16) and (17) into eqn (6) gives:

$$\sigma_{(i)}^{c'} = \mathbf{D}_{(i)}^{1'} \cdot \epsilon_{(i)}^{f'} - \mathbf{D}_{(i)}^{2'} \cdot \epsilon_{(i)}^{*1'} \quad (18)$$

in which

$$\mathbf{D}_{(i)}^{1'} = \mathbf{C}^{c'} \cdot \mathbf{P}_{(i)}^{c'} \cdot (\mathbf{C}^{m'} - \mathbf{C}^{c'})$$

$$\mathbf{D}_{(i)}^{2'} = \mathbf{C}^{c'} \cdot \mathbf{P}_{(i)}^{c'} \cdot \mathbf{C}^{m'}$$

$$\begin{aligned} P_{ijkl(i)}^{c'} &= \frac{1}{4} (K_{ik(i)}^{c-1'} n_{j(i)}' n_{l(i)}' + K_{il(i)}^{c-1'} n_{j(i)}' n_{k(i)}' + K_{jk(i)}^{c-1'} n_{i(i)}' n_{l(i)}' \\ &+ K_{jl(i)}^{c-1'} n_{i(i)}' n_{k(i)}') \end{aligned} \quad (18a)$$

From eqns (6) and (18), we obtain:

$$\begin{aligned} \epsilon_{(i)}^{*2'} &= (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \sigma_{(i)}^{c'} \\ &= (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}_{(i)}^{1'} \cdot (\epsilon_0' + \bar{\epsilon}' + \epsilon_{(i)}^{11'} + \epsilon_{(i)}^{12'}) \\ &- (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}_{(i)}^{2'} \cdot \epsilon_{(i)}^{*1'} \end{aligned} \quad (19)$$

substituting the above into eqn (8) leads to:

$$\epsilon_{(i)}^{12'} = \mathbf{R}_{(i)}^{1'} \cdot (\epsilon_0' + \bar{\epsilon}' + \epsilon_{(i)}^{11'}) - \mathbf{R}_{(i)}^{2'} \cdot \epsilon_{(i)}^{*1'} \quad (20)$$

in which

$$\mathbf{R}_{(i)}^{1'} = (\mathbf{I} - \mathbf{D}_{(i)}^{3'})^{-1} \cdot \mathbf{D}_{(i)}^{3'}$$

$$\mathbf{R}_{(i)}^{2'} = (\mathbf{I} - \mathbf{D}_{(i)}^{3'})^{-1} \cdot \mathbf{D}_{(i)}^{4'}$$

$$\mathbf{D}_{(i)}^{3'} = \left(\frac{V_{c(i)}}{V_{f(i)}} \right) \langle \mathbf{S}_{(i)}^{0'} \cdot (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}_{(i)}^{1'} \rangle_2$$

$$\mathbf{D}_{(i)}^{4'} = \left(\frac{V_{c(i)}}{V_{f(i)}} \right) \langle \mathbf{S}_{(i)}^{0'} \cdot (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}_{(i)}^{2'} \rangle_2 \quad (20a)$$

and $\mathbf{I}$ is a fourth-rank identity tensor defined as:

$$I_{pqrs} = \frac{1}{2} (\delta_{pr} \delta_{qs} + \delta_{ps} \delta_{qr}) \quad (20b)$$

From eqns (5) and (7) one can find:

$$\epsilon_{(i)}^{*1'} = \mathbf{L}_{(i)}^{1'} \cdot (\epsilon_0' + \bar{\epsilon}') \quad (21)$$

where

$$\mathbf{L}_{(i)}^{1'} = (\mathbf{I} + \mathbf{A}' \cdot \mathbf{T}_{(i)}' \cdot \mathbf{R}_{(i)}^{2'} - \mathbf{A}' \cdot \mathbf{T}_{(i)}' \cdot \mathbf{R}_{(i)}^{1'} \cdot \mathbf{S}_{(i)}')^{-1} \cdot (\mathbf{A}' \cdot \mathbf{T}_{(i)}' + \mathbf{A}' \cdot \mathbf{T}_{(i)}' \cdot \mathbf{R}_{(i)}^{1'}) \quad (22a)$$

$$\mathbf{A}' = \mathbf{I} - \mathbf{C}^{m-1'} \cdot \mathbf{C}^r \quad (22b)$$

$$\mathbf{T}_{(i)}' = (\mathbf{I} + \mathbf{S}_{(i)}' \cdot \mathbf{C}^{m-1'} \cdot \mathbf{C}^r - \mathbf{S}_{(i)}')^{-1} \quad (22c)$$

Substituting eqns (7) and (21) into eqn (20) yields:

$$\begin{aligned} \epsilon_{(i)}^{12'} &= [\mathbf{R}_{(i)}^{1'} \cdot (\mathbf{I} + \mathbf{S}_{(i)}' \cdot \mathbf{L}_{(i)}^{1'}) - \mathbf{R}_{(i)}^{2'} \cdot \mathbf{L}_{(i)}^{1'}] \cdot (\epsilon_0' + \bar{\epsilon}') \\ &\equiv \mathbf{L}_{(i)}^{2'} \cdot (\epsilon_0' + \bar{\epsilon}') \end{aligned} \quad (23)$$

From eqns (7), (21) and (23) one has:

$$\begin{aligned} \epsilon_{(i)}^{11'} + \epsilon_{(i)}^{12'} - \epsilon_{(i)}^{*1'} &= [(\mathbf{S}_{(i)}' - \mathbf{I}) \cdot \mathbf{L}_{(i)}^{1'} + \mathbf{L}_{(i)}^{2'}] \cdot (\epsilon_0' + \bar{\epsilon}') \\ &\equiv \mathbf{L}_{(i)}^{3'} \cdot (\epsilon_0' + \bar{\epsilon}') \end{aligned} \quad (24)$$

Transforming the above equation under a local coordinate system into one under a global coordinate system gives:

$$\epsilon_{(i)}^{11} + \epsilon_{(i)}^{12} - \epsilon_{(i)}^{*1} = \mathbf{U}_{(i)}^1 \cdot (\epsilon_0 + \bar{\epsilon}) \quad (25)$$

where

$$\mathbf{U}_{klmn(i)}^1 = L_{pqrs(i)}^{3'} Q_{pk(i)} Q_{ql(i)} Q_{rm(i)} Q_{sn(i)}$$

Also, from eqn (6) one has:

$$\epsilon_{(i)}^{21'} + \epsilon_{(i)}^{22'} - \epsilon_{(i)}^{*2'} = \mathbf{C}_m^{-1'} \cdot \sigma_{(i)}' - \epsilon_0' - \bar{\epsilon}' \quad (26)$$

Substituting eqns (7), (21) and (23) into eqn (26) yields:

$$\epsilon_{(i)}^{21} + \epsilon_{(i)}^{22} - \epsilon_{(i)}^{*2'} = \mathbf{L}_{(i)}^{4'} \cdot (\epsilon_0' + \bar{\epsilon}')$$

where

$$\begin{aligned} \mathbf{L}_{(i)}^{4'} &= \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{1'} - \mathbf{I} + \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{1'} \cdot (\mathbf{S}_{(i)}' \cdot \mathbf{L}_{(i)}^{1'} \\ &\quad + \mathbf{L}_{(i)}^{2'}) - \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{2'} \cdot \mathbf{L}_{(i)}^{1'} \end{aligned}$$

Again, in a global system one has:

$$\epsilon_{(i)}^{21} + \epsilon_{(i)}^{22} - \epsilon_{(i)}^{*2'} = \mathbf{U}_{(i)}^2 \cdot (\epsilon_0 + \bar{\epsilon}) \quad (27)$$

where

$$\mathbf{U}_{klmn(i)}^2 = L_{pqrs(i)}^{4'} Q_{pk(i)} Q_{ql(i)} Q_{rm(i)} Q_{sn(i)}$$

Substituting eqns (25) and (27) into eqn (11) yields:

$$\begin{aligned} \bar{\epsilon} + \left\langle \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^1 \right\rangle \right\rangle \cdot (\epsilon_0 + \bar{\epsilon}) \\ + \left\langle \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^3 \right\rangle \right\rangle F \cdot (\epsilon_0 + \bar{\epsilon}) = 0 \end{aligned} \quad (28)$$

in which

$$U_{klmn(i)}^3 = \langle U_{klmn(i)}^2 \rangle_2 = \langle L_{pqrs(i)}^{4'} \rangle_2 Q_{pk(i)} Q_{ql(i)} Q_{rm(i)} Q_{sn(i)}$$

Rearranging eqn (28) yields:

$$\begin{aligned} \epsilon_0 + \bar{\epsilon} &= (\mathbf{I} + \left\langle \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^1 \right\rangle \right\rangle \\ &\quad + \left\langle \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^3 \right\rangle \right\rangle)^{-1} \cdot \epsilon_0 \end{aligned} \quad (29)$$

Substituting eqns (7), (21) and (23) into eqn (19) gives:

$$\epsilon_{(i)}^{*2'} = \mathbf{L}_{(i)}^{6'} \cdot (\epsilon_0' + \bar{\epsilon}') \quad (30)$$

where

$$\begin{aligned} \mathbf{L}_{(i)}^{6'} &= (\mathbf{C}_c^{-1'} - \mathbf{C}_m^{-1'}) \cdot \mathbf{D}_{(i)}^{1'} \cdot (\mathbf{I} + \mathbf{S}_{(i)}' \cdot \mathbf{L}_{(i)}^{1'} + \mathbf{L}_{(i)}^{2'}) \\ &\quad - (\mathbf{C}_c^{-1'} - \mathbf{C}_m^{-1'}) \cdot \mathbf{D}_{(i)}^{2'} \cdot \mathbf{L}_{(i)}^{1'} \end{aligned}$$

Now rewriting eqns (21) and (30) in a global coordinate system gives:

$$\epsilon_{(i)}^{*1} = \mathbf{U}_{(i)}^4 \cdot (\epsilon_0 + \bar{\epsilon}) \quad (31)$$

and

$$\epsilon_{(i)}^{*2} = \mathbf{U}_{(i)}^5 \cdot (\epsilon_0 + \bar{\epsilon}) \quad (32)$$

where

$$U_{klmn(i)}^4 = L_{pqrs(i)}^{1'} Q_{pk(i)} Q_{ql(i)} Q_{rm(i)} Q_{sn(i)}$$

$$U_{klmn(i)}^5 = L_{pqrs(i)}^{6'} Q_{pk(i)} Q_{ql(i)} Q_{rm(i)} Q_{sn(i)}$$

Substituting eqns (29), (31) and (32) into eqn (12) yields:

$$\begin{aligned} \epsilon^T &= \left[\mathbf{I} + \left(\left\langle \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^4 \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^6 \right\rangle \right\rangle \right) \right. \\ &\quad \cdot \left(\mathbf{I} + \left\langle \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^1 \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^3 \right\rangle \right\rangle \right)^{-1} \left. \right] \cdot \epsilon_0 \end{aligned} \quad (33)$$

where

$$U_{klmn(i)} = \langle U_{klmn(i)} \rangle_2 = \langle L_{pqrs(i)} \rangle_2 Q_{pk(i)} Q_{ql(i)} Q_{rm(i)} Q_{sn(i)}$$

Finally one can define the effective elastic modulus tensor $\mathbf{C}^*$ according to:

$$\sigma_0 = \mathbf{C}^* \cdot \epsilon^T \quad (34)$$

Thus from eqns (1), (34) and (33), the effect elastic moduli $\mathbf{C}^*$ can be obtained as:

$$\begin{aligned} \mathbf{C}^* &= \mathbf{C}_m \cdot \left[\mathbf{I} + \left(\left\langle \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^4 \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^6 \right\rangle \right\rangle \right) \right. \\ &\quad \cdot \left(\mathbf{I} + \left\langle \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^1 \right\rangle \right\rangle + \left\langle \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^3 \right\rangle \right\rangle \right)^{-1} \left. \right]^{-1} \end{aligned} \quad (35)$$

3 EFFECTIVE COEFFICIENTS OF THERMAL EXPANSION

As a parallel to the derivation above, to evaluate the effective coefficients of thermal expansion (CTE) of the composites with thin coated fibers, let us first subject the matrix alone to a uniform temperature field ΔT . The uniform thermal strain, ϵ_0 , produced due to such temperature field is:

$$\epsilon_0 = \Delta T \alpha_m \quad (36)$$

in which α_m is the second-order tensor of the

coefficients of thermal expansion of the matrix. It is noted that due to free expansion and free external loading, there are no stresses in the matrix, i.e. $\sigma_0 = 0$. When there are ellipsoidal coated fibers present in the matrix, a perturbed stress field $\tilde{\sigma}(x)$ is induced and the volumetric average of the perturbed stress field of the matrix, $\bar{\sigma}$, is again given by eqn (2). By Eshelby's principle of equivalent inclusion, one now has, in the local domain of a typical fiber, $\Omega'_{f(i)}$:

$$\begin{aligned}\sigma'_{(i)} &= \mathbf{C}' \cdot (\bar{\epsilon}' + \epsilon^{11'}_{(i)} + \epsilon^{12'}_{(i)} - \epsilon^{T1'}) \\ &= \mathbf{C}^{m'} \cdot (\bar{\epsilon}' + \epsilon^{11'}_{(i)} + \epsilon^{12'}_{(i)} - \epsilon^{**1'})\end{aligned}\quad (37)$$

and in the local domain of coating of a typical fiber, $\Omega'_{c(i)}$:

$$\begin{aligned}\sigma'^{c'}_{(i)} &= \mathbf{C}^{c'} \cdot (\bar{\epsilon}' + \epsilon^{22'}_{(i)} + \epsilon^{21'}_{(i)} - \epsilon^{T2'}) \\ &= \mathbf{C}^{m'} \cdot (\bar{\epsilon}' + \epsilon^{22'}_{(i)} + \epsilon^{21'}_{(i)} - \epsilon^{**2'})\end{aligned}\quad (38)$$

in which

$$\begin{aligned}\epsilon^{T1'} &= (\alpha'_f - \alpha'_m)\Delta T \\ \epsilon^{T2'} &= (\alpha'_c - \alpha'_m)\Delta T\end{aligned}\quad (39)$$

and α'_f and α'_c are the tensors of the coefficients of thermal expansion of the fiber and the coating, respectively. It is noted that the relationships for $\epsilon^{11'}_{(i)}$ and $\epsilon^{12'}_{(i)}$, eqns (7)–(10), are again valid. Further, the traction continuity and jump condition of displacement gradient given by eqns (15) and (16), hold here. Substituting eqns (37), (38) and (16) into eqn (15) yields:

$$\begin{aligned}\lambda'_{p(i)} &= K_{pi(i)}^{-1} [(C^{m'}_{ijkl} - C^{c'}_{ijkl})(\bar{\epsilon}'_{kl} + \epsilon^{11'}_{kl(i)} + \epsilon^{12'}_{kl(i)})n'_{j(i)} \\ &\quad - C^{m'}_{ijkl}\epsilon^{**1'}_{j(i)}n'_{j(i)} + C^{c'}_{ijkl}\epsilon^{T2'}_{kl}n'_{j(i)}]\end{aligned}\quad (40)$$

in which $K_{pi(i)}$ is defined as in eqn (16a). Substituting eqns (15) and (40) into eqn (38) gives

$$\sigma'^{c'}_{(i)} = \mathbf{D}^{1'}_{(i)} \cdot (\bar{\epsilon}' + \epsilon^{11'}_{(i)} + \epsilon^{12'}_{(i)}) - \mathbf{D}^{2'}_{(i)} \cdot \epsilon^{**1'}_{(i)} + \mathbf{D}^{5'}_{(i)} \cdot \epsilon^{T2'}\quad (41)$$

in which

$$\mathbf{D}^5_{(i)} = \mathbf{C}^{c'} \cdot \mathbf{P}^{c'}_{(i)} \cdot \mathbf{C}^{c'} - \mathbf{C}^{c'}\quad (41a)$$

and $\mathbf{D}^{1'}_{(i)}$, $\mathbf{D}^{2'}_{(i)}$ and $\mathbf{P}^{c'}_{(i)}$ are defined by eqn (18a).

The total average stress over the whole domain is again given by eqn (12); however the total average strain becomes:

$$\begin{aligned}\epsilon^T &= \Delta T \alpha_m + \left\langle \sum V_{f(i)} \epsilon^{**1}_{(i)} \right\rangle \\ &\quad + \left\langle \sum V_{c(i)} \langle \epsilon^{**2}_{(i)} \rangle_2 \right\rangle\end{aligned}\quad (42)$$

From eqns (38) and (41), we get:

$$\begin{aligned}\epsilon^{**2'}_{(i)} &= (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \sigma_{c(i)} + \epsilon^{T2'} \\ &= (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}^{1'}_{(i)} \cdot (\bar{\epsilon}' + \epsilon^{11'}_{(i)} + \epsilon^{12'}_{(i)}) \\ &\quad - (\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}^{2'}_{(i)} \cdot \epsilon^{**1'}_{(i)} \\ &\quad + [(\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}^{5'}_{(i)} + \mathbf{I}] \cdot \epsilon^{T2'}\end{aligned}\quad (43)$$

Thus eqn (8) can be rewritten as:

$$\epsilon^{12'}_{(i)} = \mathbf{R}^{1'}_{(i)} \cdot (\bar{\epsilon}' + \epsilon^{11'}_{(i)}) - \mathbf{R}^{2'}_{(i)} \cdot \epsilon^{**1'}_{(i)} + \mathbf{R}^{7'}_{(i)} \cdot \epsilon^{T2'}\quad (44)$$

in which $\mathbf{R}^{1'}_{(i)}$ and $\mathbf{R}^{2'}_{(i)}$ are given by eqn (20a) and:

$$\begin{aligned}\mathbf{R}^{7'}_{(i)} &= (\mathbf{I} - \mathbf{D}^{3'}_{(i)})^{-1} \cdot \mathbf{D}^{6'}_{(i)} \\ \mathbf{D}^{6'}_{(i)} &= \left(\frac{V_{c(i)}}{V_{f(i)}} \right) \langle \mathbf{S}^{0'}_{(i)} \cdot [(\mathbf{C}^{c-1'} - \mathbf{C}^{m-1'}) \cdot \mathbf{D}^{5'}_{(i)} + \mathbf{I}] \rangle_2\end{aligned}\quad (44a)$$

Now from eqn (37) we can find:

$$\epsilon^{**1'}_{(i)} = \mathbf{L}^{1'}_{(i)} \cdot \bar{\epsilon}' + \mathbf{L}^{7'}_{(i)} \cdot \epsilon^{T2} + \mathbf{L}^{8'}_{(i)} \cdot \epsilon^{T1}\quad (45)$$

where

$$\begin{aligned}\mathbf{L}^{7'}_{(i)} &= (\mathbf{I} + \mathbf{A}' \cdot \mathbf{T}'_{(i)} \cdot \mathbf{R}^{2'}_{(i)} - \mathbf{A}' \cdot \mathbf{T}'_{(i)} \cdot \mathbf{R}^{1'}_{(i)} \cdot \mathbf{S}'_{(i)})^{-1} \\ &\quad \cdot (\mathbf{A}' \cdot \mathbf{T}'_{(i)} \cdot \mathbf{R}^{7'}_{(i)}) \\ \mathbf{L}^{8'}_{(i)} &= (\mathbf{I} + \mathbf{A}' \cdot \mathbf{T}'_{(i)} \cdot \mathbf{R}^{2'}_{(i)} - \mathbf{A}' \cdot \mathbf{T}'_{(i)} \cdot \mathbf{R}^{1'}_{(i)} \cdot \mathbf{S}'_{(i)})^{-1} \\ &\quad \cdot (\mathbf{A}' \cdot \mathbf{T}'_{(i)} \cdot \mathbf{S}'_{(i)} + \mathbf{I}) \cdot \mathbf{C}_f^{-1'} \cdot \mathbf{C}'_f\end{aligned}\quad (45a)$$

Substituting eqn (45) into eqn (44) yields:

$$\epsilon^{12'}_{(i)} = \mathbf{L}^{2'}_{(i)} \cdot \bar{\epsilon}' + \mathbf{L}^{9'}_{(i)} \cdot \epsilon^{T2} + \mathbf{L}^{10'}_{(i)} \cdot \epsilon^{T1}\quad (46)$$

where

$$\begin{aligned}\mathbf{L}^{9'}_{(i)} &= (\mathbf{R}_{(i)} \cdot \mathbf{S}'_{(i)} - \mathbf{R}^{2'}_{(i)}) \cdot \mathbf{L}^{7'}_{(i)} + \mathbf{R}^{7'}_{(i)} + \mathbf{R}^{7'}_{(i)} \\ \mathbf{L}^{10'}_{(i)} &= (\mathbf{R}_{(i)} \cdot \mathbf{S}'_{(i)} - \mathbf{R}^{2'}_{(i)}) \cdot \mathbf{L}^{8'}_{(i)}\end{aligned}\quad (46a)$$

Thus:

$$\epsilon^{11'}_{(i)} + \epsilon^{12'}_{(i)} - \epsilon^{**1'}_{(i)} = \mathbf{L}^{3'}_{(i)} \cdot \bar{\epsilon}' + \mathbf{L}^{11'}_{(i)} \cdot \epsilon^{T2} + \mathbf{L}^{12'}_{(i)} \cdot \epsilon^{T1}\quad (47)$$

where

$$\begin{aligned}\mathbf{L}^{11'}_{(i)} &= (\mathbf{S}'_{(i)} - \mathbf{I}) \cdot \mathbf{L}^{7'}_{(i)} + \mathbf{L}^{9'}_{(i)} \\ \mathbf{L}^{12'}_{(i)} &= (\mathbf{S}'_{(i)} - \mathbf{I}) \cdot \mathbf{L}^{8'}_{(i)} + \mathbf{L}^{10'}_{(i)}\end{aligned}\quad (47a)$$

Substituting eqns (7), (41), (45) and (46) into eqn (6) yields:

$$\epsilon^{21'}_{(i)} + \epsilon^{22'}_{(i)} - \epsilon^{*2'}_{(i)} = \mathbf{L}^{4'}_{(i)} \cdot \bar{\epsilon}' + \mathbf{L}^{13'}_{(i)} \cdot \epsilon^{T2} + \mathbf{L}^{14'}_{(i)} \cdot \epsilon^{T1}\quad (48)$$

where

$$\begin{aligned}\mathbf{L}_{(i)}^{13'} &= (\mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{1'} \cdot \mathbf{S}_{(i)}' - \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{2'}) \cdot \mathbf{L}_{(i)}^{7'} \\ &\quad + \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{1'} \cdot \mathbf{L}_{(i)}^{9'} + \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{5'} \\ \mathbf{L}_{(i)}^{14'} &= (\mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{1'} \cdot \mathbf{S}_{(i)}' - \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{2'}) \cdot \mathbf{L}_{(i)}^{8'} \\ &\quad + \mathbf{C}_m^{-1'} \cdot \mathbf{D}_{(i)}^{1'} \cdot \mathbf{L}_{(i)}^{10'}\end{aligned}\quad (48a)$$

Rewriting eqns (47) and (48) from a local coordinate system to a global coordinate system yields:

$$\epsilon_{(i)}^{11} + \epsilon_{(i)}^{12} - \epsilon_{(i)}^{*1} = \mathbf{U}_{(i)}^1 \cdot \bar{\epsilon} + \mathbf{U}_{(i)}^7 \cdot \epsilon^{T_2} + \mathbf{U}_{(i)}^8 \cdot \epsilon^{T_1} \quad (49)$$

$$\epsilon_{(i)}^{21} + \epsilon_{(i)}^{11} - \epsilon_{(i)}^{*2} = \mathbf{U}_{(i)}^2 \cdot \bar{\epsilon} + \mathbf{U}_{(i)}^9 \cdot \epsilon^{T_2} + \mathbf{U}_{(i)}^{10} \cdot \epsilon^{T_1} \quad (50)$$

where the components of $\mathbf{U}_{(i)}^j$ are given as follows:

$$\begin{aligned}U_{klmn(i)}^7 &= L_{pqmn(i)}^{11'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)} \\ U_{klmn(i)}^8 &= L_{pqmn(i)}^{12'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)} \\ U_{klmn(i)}^9 &= L_{pqmn(i)}^{13'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)} \\ U_{klmn(i)}^{10} &= L_{pqmn(i)}^{14'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)}\end{aligned}\quad (51)$$

Substituting eqns (49) and (50) into eqn (12) yields:

$$\begin{aligned}\bar{\epsilon} + \left(\left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^1 \right\rangle \cdot \bar{\epsilon} + \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^7 \right\rangle \cdot \epsilon^{T_2} \right. \\ \left. + \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^8 \right\rangle \cdot \epsilon^{T_1} \right) + \left\langle V_{c(i)} \mathbf{U}_{(i)}^3 \right\rangle \cdot \bar{\epsilon} \\ + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{11} \right\rangle \cdot \epsilon^{T_2} + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{12} \right\rangle \cdot \epsilon^{T_1} = 0\end{aligned}\quad (52)$$

where

$$\begin{aligned}U_{klmn(i)}^{11} &= \langle U_{klmn(i)}^2 \rangle_2 = \langle L_{pqmn(i)}^{13'} \rangle_2 \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)} \\ U_{klmn(i)}^{12} &= \langle U_{klmn(i)}^{10} \rangle_2 = \langle L_{klmn(i)}^{14} \rangle_2 \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)}\end{aligned}\quad (52a)$$

Solving $\bar{\epsilon}$ from eqn (52) yields:

$$\bar{\epsilon} = \mathbf{X}_1 \cdot \epsilon^{T_2} + \mathbf{X}_2 \cdot \epsilon^{T_1} \quad (53)$$

where

$$\begin{aligned}\mathbf{X}_1 &= - \left(\mathbf{I} + \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^1 \right\rangle + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^2 \right\rangle \right)^{-1} \\ &\quad \cdot \left(\left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^7 \right\rangle + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{11} \right\rangle \right) \\ \mathbf{X}_2 &= - \left(\mathbf{I} + \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^1 \right\rangle + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^2 \right\rangle \right)^{-1} \\ &\quad \cdot \left(\left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^8 \right\rangle + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{12} \right\rangle \right)\end{aligned}\quad (53a)$$

Similarly, transformed to a global system, eqn (45) becomes:

$$\epsilon_{(i)}^{*1} = \mathbf{U}_{(i)}^4 \cdot \bar{\epsilon} + \mathbf{U}_{(i)}^{13} \cdot \epsilon^{T_2} + \mathbf{U}_{(i)}^{14} \cdot \epsilon^{T_1} \quad (54)$$

where

$$\begin{aligned}U_{klmn(i)}^{13} &= L_{pqmn(i)}^{7'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)} \\ U_{klmn(i)}^{14} &= L_{pqmn(i)}^{8'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)}\end{aligned}\quad (54a)$$

From eqn (43) one has:

$$\epsilon_{(i)}^{*2'} = \mathbf{L}_{(i)}^{6'} \cdot \bar{\epsilon}' + \mathbf{L}_{(i)}^{17'} \cdot \epsilon^{T_2} + \mathbf{L}_{(i)}^{18'} \cdot \epsilon^{T_1} \quad (55)$$

where

$$\begin{aligned}\mathbf{L}_{(i)}^{17'} &= (\mathbf{C}_c^{-1'} - \mathbf{C}_m^{-1'}) \cdot \mathbf{D}_{(i)}^{1'} \cdot (\mathbf{S}_{(i)}' \cdot \mathbf{L}_{(i)}^{7'} + \mathbf{L}_{(i)}^{9'}) \\ &\quad - (\mathbf{C}_c^{-1} - \mathbf{C}_m^{-1'}) \cdot \mathbf{D}_{(i)}^{2'} \cdot \mathbf{L}_{(i)}^{7'} \\ &\quad + [(\mathbf{C}_c^{-1'} - \mathbf{C}_m^{-1'}) \cdot \mathbf{D}_{(i)}^{5'} + \mathbf{I}] \\ \mathbf{L}_{(i)}^{18'} &= (\mathbf{C}_c^{-1'} - \mathbf{C}_m^{-1'}) \cdot \mathbf{D}_{(i)}^{1'} \cdot (\mathbf{S}_{(i)}' \cdot \mathbf{L}_{(i)}^{8'} + \mathbf{L}_{(i)}^{10'}) \\ &\quad - (\mathbf{C}_c^{-1'} - \mathbf{C}_m^{-1'}) \cdot \mathbf{D}_{(i)}^{2'} \cdot \mathbf{L}_{(i)}^{8'}\end{aligned}\quad (55a)$$

Again, in a global system, eqn (55) becomes:

$$\epsilon^{*2} = \mathbf{U}_{(i)}^5 \cdot \bar{\epsilon} + \mathbf{U}_{(i)}^{15} \cdot \epsilon^{T_2} + \mathbf{U}_{(i)}^{16} \cdot \epsilon^{T_1} \quad (56)$$

where

$$\begin{aligned}U_{klmn(i)}^{15} &= L_{pqmn(i)}^{17'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)} \\ U_{klmn(i)}^{16} &= L_{pqmn(i)}^{18'} \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)}\end{aligned}\quad (56a)$$

Finally substituting eqns (54) and (56) into eqn (42) yields:

$$\begin{aligned}\epsilon^T &= \alpha_m \Delta T + \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^4 \right\rangle \cdot \bar{\epsilon} + \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^{13} \right\rangle \cdot \epsilon^{T_2} \\ &\quad + \left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^{14} \right\rangle \cdot \epsilon^{T_1} + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^6 \right\rangle \cdot \bar{\epsilon} \\ &\quad + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{17} \right\rangle \cdot \epsilon^{T_2} + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{18} \right\rangle \cdot \epsilon^{T_1}\end{aligned}\quad (57)$$

where

$$\begin{aligned}U_{klmn(i)}^{17} &= \langle U_{klmn(i)}^{17} \rangle_2 = \langle L_{pqmn(i)}^{15'} \rangle_2 \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)} \\ U_{klmn(i)}^{18} &= \langle U_{klmn(i)}^{16} \rangle_2 = \langle L_{pqmn(i)}^{18'} \rangle_2 \mathcal{Q}_{pk(i)} \mathcal{Q}_{ql(i)}\end{aligned}\quad (57a)$$

Thus the effective coefficient tensor, $\bar{\alpha}$, of thermal expansion can be obtained as:

$$\begin{aligned}\bar{\alpha} &= \frac{1}{\Delta T} \epsilon^T \\ &= \alpha_m + \left[\left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^4 \right\rangle + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^6 \right\rangle \right] \\ &\quad \cdot [\mathbf{X}_1 \cdot (\alpha'_c - \alpha_m) + \mathbf{X}_2 \cdot (\alpha'_f - \alpha_m)] \\ &\quad + \left[\left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^{13} \right\rangle + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{17} \right\rangle \right] \\ &\quad \cdot (\alpha'_c - \alpha_m) + \left[\left\langle \sum V_{f(i)} \mathbf{U}_{(i)}^{14} \right\rangle \right. \\ &\quad \left. + \left\langle \sum V_{c(i)} \mathbf{U}_{(i)}^{18} \right\rangle \right] \cdot (\alpha'_f - \alpha_m)\end{aligned}\quad (58)$$

4 FIBER LENGTH DISTRIBUTION (FLD)

When preparing a specimen by extrusion compounding or injection molding, the shear forces exerted by the screw or ram may further break the original chopped strand fibers, which results in a length distribution. In the literature^{11,12} the two-parameter Weibull distribution has been used frequently. In the numerical illustration that follows, such a distribution is also adopted. Since dimensionless quantities are used in this work, we use the parameter (aspect ratio) $t \equiv l/d$ (where l and d are the long and short axis of the ellipsoidal fibers, respectively) instead of the dimensional parameter fiber length in the Weibull distribution.

For the Weibull distribution function,¹² the probability density function and the accumulated probability function are, respectively:

$$f(t) = \frac{c}{b} \left(\frac{t}{b} \right)^{c-1} \exp \left[- \left(\frac{t}{b} \right)^c \right] \quad (59)$$

$$F(t) = \int_0^t f(x) dx = 1 - e^{-(t/b)^c} \quad (60)$$

where b is the scale parameter and c is the shape parameter. The expected fiber aspect ratio is:

$$\bar{t} = \int_0^\infty t f(x) dx = b \Gamma \left(\frac{1}{c} + 1 \right) \quad (61)$$

where $\Gamma(x)$ is the gamma function. Consider now an aligned short-coated fiber composite with variable aspect ratio (all of the fibers are along the x_1 axis). Equation (35) reduces to:

$$\mathbf{C}^* = \mathbf{C}_m \cdot \left[\mathbf{I} + \left(\sum V_{f(i)} \mathbf{U}_{(i)}^4 + \sum V_{c(i)} \mathbf{U}_{(i)}^6 \right) \cdot \left(\mathbf{I} + \sum V_{f(i)} \mathbf{U}_{(i)}^1 + \sum V_{c(i)} \mathbf{U}_{(i)}^3 \right)^{-1} \right]^{-1} \quad (35')$$

and eqn (58) reduces to:

$$\begin{aligned} \bar{\alpha} &= \frac{\epsilon^T}{\Delta T} \\ &= \alpha_m + \left[\sum V_{f(i)} \mathbf{U}_{(i)}^4 + \sum V_{c(i)} \mathbf{U}_{(i)}^6 \right] \cdot [\mathbf{X}_1 \cdot (\alpha'_c - \alpha_m) \\ &\quad + \mathbf{X}_2 \cdot (\alpha'_f - \alpha_m)] + \left[\sum V_{f(i)} \mathbf{U}_{(i)}^{13} + \sum V_{c(i)} \mathbf{U}_{(i)}^{17} \right] \\ &\quad \cdot (\alpha'_c - \alpha_m) + \left[\sum V_{f(i)} \mathbf{U}_{(i)}^{14} + \sum V_{c(i)} \mathbf{U}_{(i)}^{18} \right] \cdot (\alpha'_f - \alpha_m) \end{aligned} \quad (58')$$

where

$$\begin{aligned} \mathbf{X}_1 &= - \left(\mathbf{I} + \sum V_{f(i)} \mathbf{U}_{(i)}^1 + \sum V_{c(i)} \mathbf{U}_{(i)}^2 \right)^{-1} \\ &\quad \cdot \left(\sum V_{f(i)} \mathbf{U}_{(i)}^7 + \sum V_{c(i)} \mathbf{U}_{(i)}^{11} \right) \\ \mathbf{X}_2 &= - \left(\mathbf{I} + \sum V_{f(i)} \mathbf{U}_{(i)}^1 + \sum V_{c(i)} \mathbf{U}_{(i)}^2 \right)^{-1} \\ &\quad \cdot \left(\sum V_{f(i)} \mathbf{U}_{(i)}^8 + \sum V_{c(i)} \mathbf{U}_{(i)}^{12} \right) \end{aligned} \quad (53a')$$

When the length distribution follows the statistics of eqn (59), the number of kinds of fibers, N , becomes infinity. Thus eqn (35') becomes:

$$\mathbf{C}^* = \mathbf{C}_m \cdot [\mathbf{I} + (V_f \langle \mathbf{U}^4 \rangle + V_c \langle \mathbf{U}^6 \rangle) \cdot (\mathbf{I} + V_f \langle \mathbf{U}^1 \rangle + V_c \langle \mathbf{U}^3 \rangle)^{-1}]^{-1} \quad (35'')$$

and eqn (58') becomes:

$$\begin{aligned} \bar{\alpha} &= \frac{\epsilon^T}{\Delta T} \\ &= \alpha_m + [V_f \langle \mathbf{U}^4 \rangle + V_c \langle \mathbf{U}^6 \rangle] \cdot [\mathbf{X}_1 \cdot (\alpha'_c - \alpha_m) \\ &\quad + \mathbf{X}_2 \cdot (\alpha'_f - \alpha_m)] + [V_f \langle \mathbf{U}^{13} \rangle + V_c \langle \mathbf{U}^{17} \rangle] \\ &\quad \cdot (\alpha'_c - \alpha_m) + [V_f \langle \mathbf{U}^{14} \rangle + V_c \langle \mathbf{U}^{18} \rangle] \cdot (\alpha'_f - \alpha_m) \end{aligned} \quad (58'')$$

where

$$\langle A \rangle = \frac{\int_0^\infty (t) A(t) dt}{\int_0^\infty f(t) dt}$$

and

$$\begin{aligned} \mathbf{X}_1 &= - (\mathbf{I} + V_f \langle \mathbf{U}^1 \rangle + V_c \langle \mathbf{U}^2 \rangle)^{-1} \\ &\quad \cdot (V_f \langle \mathbf{U}^7 \rangle + V_c \langle \mathbf{U}^{11} \rangle) \\ \mathbf{X}_2 &= - (\mathbf{I} + V_f \langle \mathbf{U}^1 \rangle + V_c \langle \mathbf{U}^2 \rangle)^{-1} \\ &\quad \cdot (V_f \langle \mathbf{U}^8 \rangle + V_c \langle \mathbf{U}^{12} \rangle) \end{aligned} \quad (53a'')$$

5 FIBER ORIENTATION DISTRIBUTION (FOD)

A description of the orientation distribution function ρ of a specimen made by extrusion compounding and injection molding is usually three dimensional. However, two special cases¹² can be considered. The first assumes that the specimen possesses a transverse isotropy, say in the 2–3 plane. Thus the distribution function ρ must be axisymmetric with respect to the x_1 axis, i.e. ρ is a function of θ only. The second assumes a planar fiber distribution. Then angle $\phi = 0$ and again

one has $\rho = \rho(\theta)$. For both cases, Kacir's^{13,14} one parameter fiber orientation distribution can be used:

$$\rho(\theta) = \frac{\lambda e^{-\lambda\theta}}{1 - e^{-\lambda\pi/2}} \quad (62)$$

in which $\rho(\theta)$ is the density function of the fibers oriented at $\pm\theta$, λ is the shape parameter and θ is the orientation angle ranging from 0 to $\pi/2$. A larger value of λ corresponds to a more aligned fiber distribution.

Consider now an oriented short fiber composite with the same fiber aspect ratio, i.e. the shape of fibers is the same. Then the equivalent elastic moduli, eqn (35), reduces to:

$$\mathbf{C}^* = \mathbf{C}_m \cdot [\mathbf{I} + (V_f \langle \mathbf{U}^4 \rangle + V_c \langle \mathbf{U}^6 \rangle) \cdot (\mathbf{I} + V_f \langle \mathbf{U}^1 \rangle + V_c \langle \mathbf{U}^3 \rangle)^{-1}]^{-1} \quad (35''')$$

and the equivalent coefficients of thermal expansion, eqn (58), become:

$$\begin{aligned} \bar{\alpha} &= \frac{\epsilon^T}{\Delta T} \\ &= \alpha_m + [V_f \langle \mathbf{U}^4 \rangle + V_c \langle \mathbf{U}^6 \rangle] \cdot [\mathbf{X}_1 \cdot (\alpha'_c - \alpha_m) \\ &\quad + \mathbf{X}_2 \cdot (\alpha'_f - \alpha_m)] + [V_f \langle \mathbf{U}^{13} \rangle + V_c \langle \mathbf{U}^{17} \rangle] \\ &\quad \cdot (\alpha'_c - \alpha_m) + [V_f \langle \mathbf{U}^{14} \rangle + V_c \langle \mathbf{U}^{18} \rangle] \cdot (\alpha'_f - \alpha_m) \end{aligned} \quad (58''')$$

where the forms of $\mathbf{X}_1$ and $\mathbf{X}_2$ are the same as eqn (53a'') except now:

$$\langle \mathbf{A}(\theta, \phi) \rangle = \frac{\int_0^{2\pi} \int_0^{\pi/2} \mathbf{A}(\theta, \phi) \rho(\theta) \sin\theta d\theta d\phi}{\int_0^{2\pi} \int_0^{\pi/2} \rho(\theta) \sin\theta d\theta d\phi} d\phi \quad (63)$$

6 NUMERICAL EXAMPLES AND DISCUSSION

In this section, numerical examples including some parametric studies based on the formulation developed in this work are conducted and the results presented. The materials of matrix, fiber and coating used are all isotropic. The following notation is used:

- E_m, E_f, E_c = Young's moduli of matrix, fiber and coating
- ν_m, ν_f, ν_c = Poisson's ratio of matrix, fiber and coating
- $\alpha_m, \alpha_f, \alpha_c$ = coefficients of thermal expansion of matrix, fiber and coating
- V_m, V_f, V_c = volume fraction of matrix, fiber and coating
- l, d = long and short axis of the ellipsoidal fiber.

The first three examples below are comparisons of our theoretical work with the existing theoretical and

Table 1. Comparison of experimental and analytical data for the longitudinal CTE at 75°F

Material system	α_1 ($10^{-6}/^\circ\text{F}$)		
	Exp.	RH	Present
T300/5208	-0.063	-0.051	-0.051
T300/934	-0.001	-0.089	0.089
P75/934	-0.584	-0.512	-0.512
P75/930	-0.598	-0.627	-0.627
P75/CE339	-0.567	-0.477	-0.477
C6000/Pi	-0.118	-0.104	-0.104
HMS/glass	-0.230	-0.180	-0.180
P100/Al	-0.800	-0.907	-0.908

experimental work. The last two examples illustrate some parametric effects based on the present work.

6.1 Example 1

In this example, the present formulae were reduced to the case of unidirectional non-coated long fibers. The material systems were used from previous work.¹⁵ Only CTE were computed and compared with experimental data and other theoretical schemes¹⁵ (only the RH method was chosen since it was in better agreement with experimental data). The results are given in Tables 1 and 2. It follows that the present EIAS method was as accurate as the RH method and both were satisfactory when compared with the experimental data.

6.2 Example 2

The second example is a comparison of our solution with those of the bound solutions of Pagano and Tandon^{3,4} on the thermoelastic moduli for a composite with long-coated fibers, an extreme case of our work. The material properties of matrices, fibers and coatings are given elsewhere.⁴ The coated fibers were randomly distributed along each direction in the matrix, which results in a three-dimensional isotropic composite. The comparisons are given in Table 3. A good consistency of our prediction with the two bounds based on Pagano and Tandon's displacement formulation and traction formulation was reached.

Table 2. Comparison of experimental and analytical data for the transverse CTE at 75°F

Material system	α_2 ($10^{-6}/^\circ\text{F}$)		
	Exp.	RH	Present
T300/5208	14.02	13.60	13.58
T300/934	16.13	16.50	16.48
P75/934	19.18	18.90	18.92
P75/930	17.62	13.90	13.90
P75/CE339	26.34	23.70	23.74
C6000/Pi	12.46	12.40	12.54
HMS/glass	2.10	2.49	2.49
P100/Al	16.06	15.00	15.00

Table 3. Comparisons of present model with that of Pagano and Tandon^{3,4} on the effective thermoelastic moduli of a composite with uniformly distributed long-coated fibers

Composite system	Thickness parameter ^a	Formulation	E (GPa)	G (GPa)	ν	α ($10^{-6}/^{\circ}\text{C}$)
Nicalon/BMAS	0.00	Displacement ^b	128.16	51.25	0.250	2.8896
		Traction ^c	127.35	50.87	0.252	2.8891
		Present	128.00	51.17	0.251	2.8910
Nicalon/nickel/BMAS	0.01	Displacement	129.11	51.60	0.251	3.0415
		Traction	128.25	51.20	0.253	3.0404
		Present	129.00	51.51	0.252	3.0414
	0.10	Displacement	138.72	55.10	0.259	4.4549
		Traction	137.50	54.54	0.261	4.4496
		Present	138.60	54.99	0.260	4.4341
Nicalon/carbon/BMAS	0.01	Displacement	118.06	47.50	0.243	2.8787
		Traction	117.49	47.27	0.243	2.8767
		Present	118.00	47.43	0.243	2.8817
	0.10	Displacement	80.20	32.94	0.217	2.8410
		Traction	74.96	30.76	0.218	2.8243
		Present	80.18	32.81	0.222	2.8590

^a Thickness parameter = coating thickness/fiber outer radius.^b Pagano and Tandon.³^c Pagano and Tandon.⁴**Table 4. Comparisons of present model with the experimental data of Choy *et al.*¹⁵ on the effective thermoelastic moduli of a composite with uniformly distributed short-coated fibers**

	PPS30cf	(24.3%) ^a	PPS40cf	(33.5%) ^a	PPS40gf	(26.4%) ^a
	Surface	Middle	Surface	Middle	Surface	Middle
λ	4.1	1.9	4.7	2.1	5.3	2.9
C_{11}	36.96 ^b (33.0) ^c	31.04 (31.9)	44.11 (41.7)	36.7 (36.0)	23.02 (24.7)	21.66 (21.2)
C_{22}	10.56 (10.0)	13.86 (12.5)	10.76 (10.8)	14.43 (15.1)	12.11 (12.1)	12.91 (13.0)
C_{33}	9.44 (9.50)	9.44 (10.5)	9.83 (10.3)	9.83 (10.9)	11.88 (11.5)	11.88 (12.0)
C_{44}	1.89 (1.84)	1.93 (2.74)	2.11 (1.99)	2.16 (2.78)	2.23 (2.40)	2.24 (2.65)
C_{55}	2.09 (2.28)	2.06 (3.09)	2.44 (2.55)	2.38 (3.28)	2.34 (2.52)	2.33 (2.95)
C_{66}	4.20 (3.42)	5.56 (5.30)	4.62 (3.86)	6.42 (5.80)	3.02 (3.12)	3.54 (3.45)
C_{23}	5.63 (5.20)	5.45 (5.50)	5.58 (6.10)	5.31 (6.30)	7.42 (7.10)	7.38 (7.10)
C_{13}	5.38 (5.20)	5.22 (5.60)	5.32 (6.20)	5.00 (6.40)	7.07 (7.10)	7.10 (6.90)
C_{12}	7.16 (6.20)	8.29 (5.90)	7.06 (7.20)	8.37 (6.70)	7.64 (7.50)	8.14 (7.20)
E_{11}	31.50 (33.6)	25.3 (28.1)	38.76 (36.2)	30.96 (31.5)	17.42 (19.2)	15.74 (16.1)
E_{22}	6.68 (6.80)	9.59 (9.30)	7.12 (6.90)	10.4 (11.2)	6.93 (7.20)	7.52 (8.20)
E_{33}	6.33 (6.70)	7.05 (7.80)	6.82 (6.70)	7.61 (7.90)	7.02 (7.00)	7.30 (7.60)

^a Fiber volume fraction.^b Our theoretical prediction.^c Experimental data of Choy *et al.*

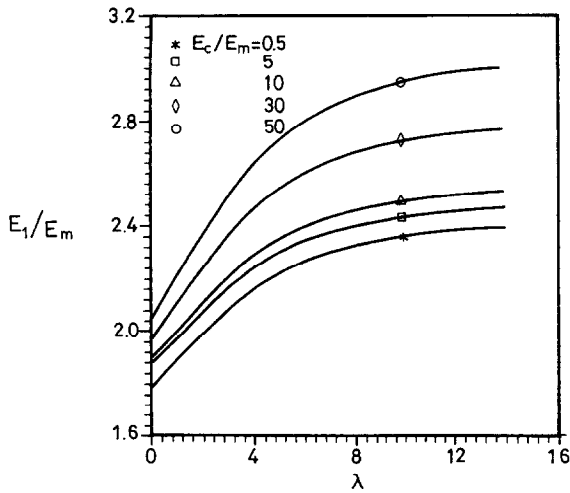


Fig. 1. Equivalent dimensionless Young's modulus E_1/E_m versus fiber orientation parameter λ under various ratios of E_c/E_m .

When the thickness of coating was relatively large, our prediction was closer to the bound one by displacement formulation.

6.3 Example 3

The third example compares our theoretical prediction on the elastic properties with the experimental work of Choy *et al.*¹⁶ In this example, the fibers were uncoated and orientationally distributed in the 1–2 plane (axis 3 was the out-of-plane direction). Thus the composite behaved orthotropically. To mimic this situation, we set angle $\phi = 0$ in relevant formulae. There were three sets of specimen in their experiments. Due to the nature of injection molding, the surface layer and the middle layer of each

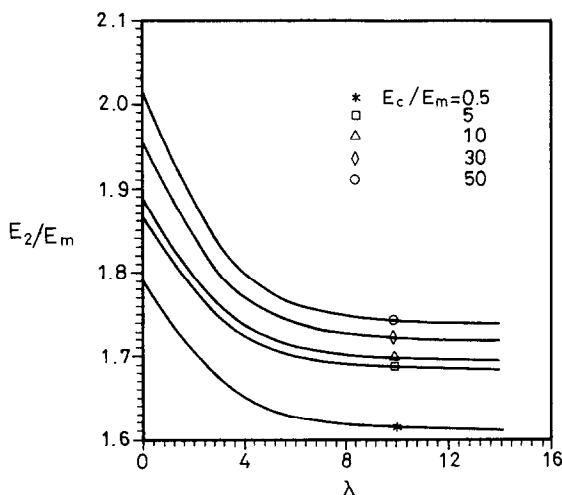


Fig. 2. Equivalent dimensionless Young's modulus E_2/E_m versus fiber orientation parameter λ under various ratios of E_c/E_m .

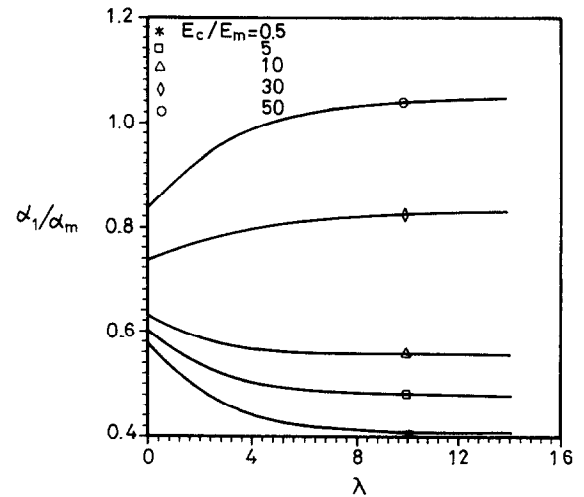


Fig. 3. Equivalent coefficient of thermal expansion α_1/α_m versus fiber orientation parameter λ under various ratios of E_c/E_m .

specimen behaved differently. Thus there were six sets of data. The elastic properties were measured by using an ultrasonic technique. The comparisons are given in Table 4. As claimed by Choy *et al.*, the average of error of their experimental data was about 6%, thus it seemed that our prediction was satisfactory.

6.4 Example 4

In contrast to the above examples, this example is a parametric study for an oriented short-coated fiber composite with orientation distribution described by eqn (62). The numerical results are presented in Figs 1 to 8, where the elastic moduli and the coefficients of thermal expansion are plotted against the shape

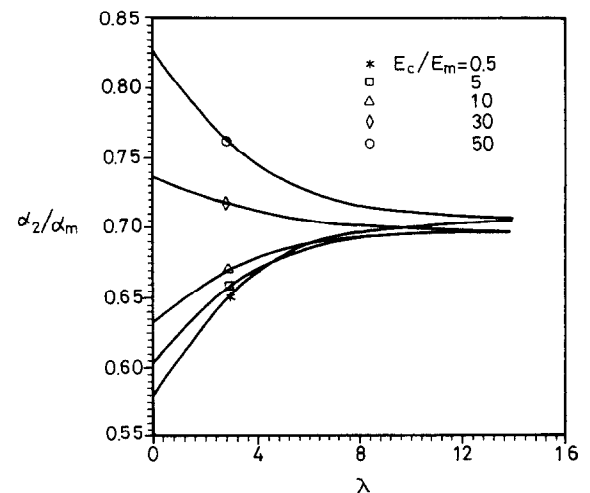


Fig. 4. Equivalent coefficient of thermal expansion α_2/α_m versus fiber orientation parameter λ under various ratios of E_c/E_m .

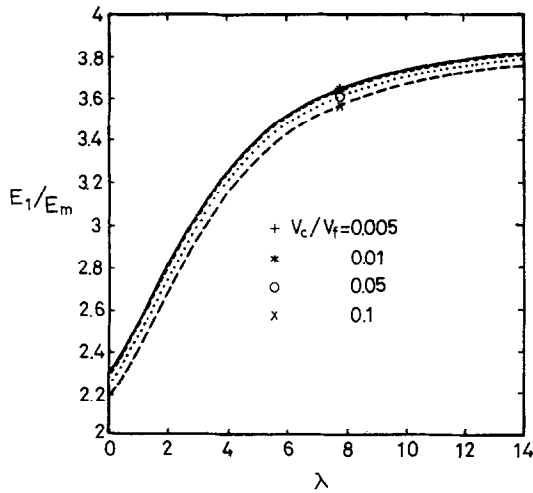


Fig. 5. Equivalent dimensionless Young's modulus E_1/E_m versus fiber orientation parameter λ under various ratios of V_c/V_f .

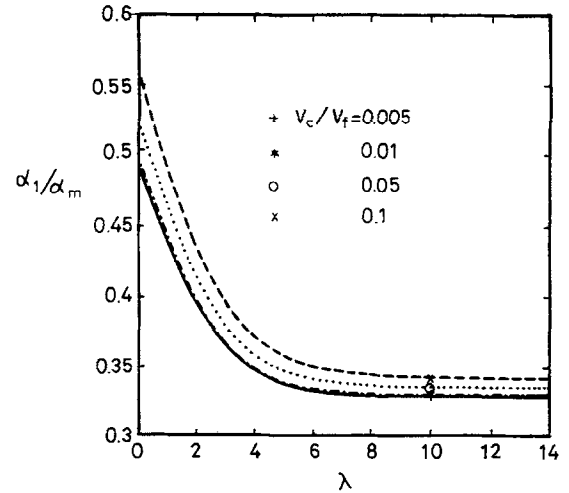


Fig. 7. Equivalent coefficient of thermal expansion α_1/α_m versus fiber orientation parameter λ under various ratios of V_c/V_f .

parameter λ . The unchanged material properties for each figure are: $E_f/E_m = 5$, $\nu_c = \nu_f = 0.03$, $\psi_m = 0.25$, $\alpha_f/\alpha_m = 0.2$, $\alpha_c/\alpha_m = 3.3$, $l/d = 10$, $V_f = 0.4$.

In addition, in Figs 1–4 the ratio $V_c/V_f = 0.04$ but E_c/E_m were allowed to vary from 0.5 to 50; in contrast, in Figs 5–8, the ratio $E_c/E_m = 0.5$ but the ratio V_c/V_f varied from 0.005 to 0.1. Saturation of the moduli and coefficients of thermal expansion as λ increases (i.e. fibers were more aligned) was expected.

6.5 Example 5

This example is also a parametric study for an aligned short coated fiber composite with length distribution described by eqn (59). The numerical results are presented in Figs 9–12, where the dimensionless

equivalent elastic modulus E_1/E_m and the dimensionless coefficient of thermal expansion α_1/α_m are plotted against the fiber volume fraction V_f . The unchanged material properties for each figure were: $E_c/E_m = 0.5$, $\nu_c = \nu_f = 0.25$, $\nu_m = 0.3$, $\alpha_f/\alpha_m = 0.2$, $\alpha_c/\alpha_m = 2$, $c = 1.25$, $b = 21.4755$.

In addition, in Figs 9 and 10 the ratio $E_f/E_m = 20$ but V_c/V_f were assigned to be 0.01 and 0.05; in contrast, in Figs 11 and 12, the ratio $V_c/V_f = 0.05$ but the ratios E_f/E_m were given as 5 and 20 respectively. In each of these figures, two sets of data were obtained: one based on the length distribution, eqn (59), the other based on the mean of this distribution. As expected, the differences of the values of the equivalent Young's modulus E_1/E_m and the coefficient of thermal expansion α_1/α_m calculated based on the distribution and those calculated based on the mean

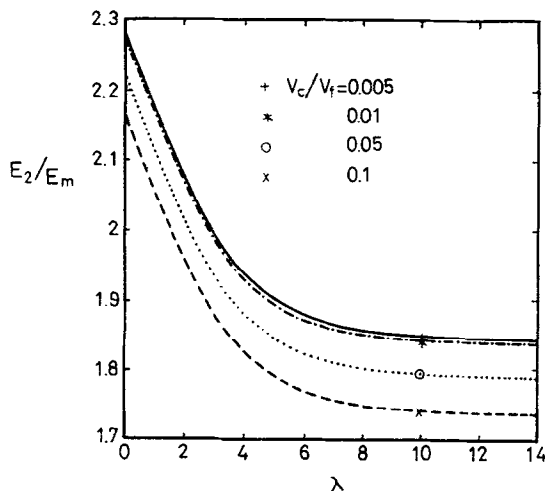


Fig. 6. Equivalent dimensionless Young's modulus E_2/E_m versus fiber orientation parameter λ under various ratios of V_c/V_f .

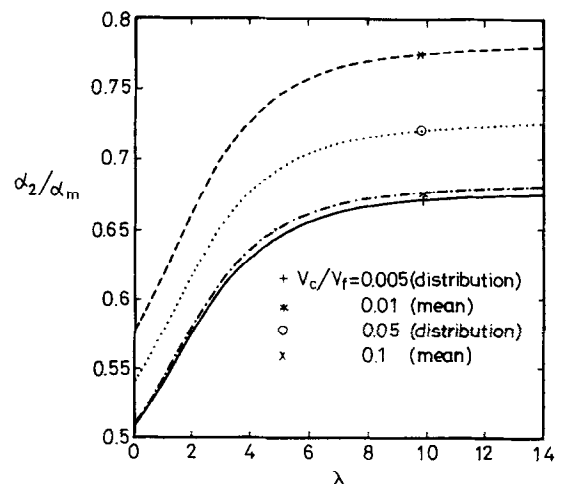


Fig. 8. Equivalent coefficient of thermal expansion α_2/α_m versus fiber orientation parameter λ under various ratios of V_c/V_f .

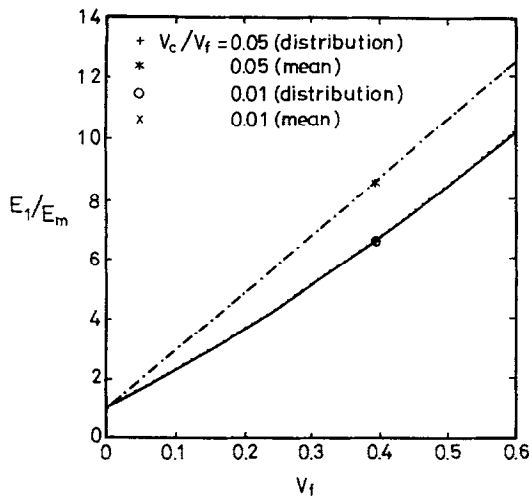


Fig. 9. Equivalent dimensionless Young's modulus E_1/E_m versus fiber volume fraction V_f calculated based on distributional fiber lengths and mean fiber length under various ratios of V_c/V_f .

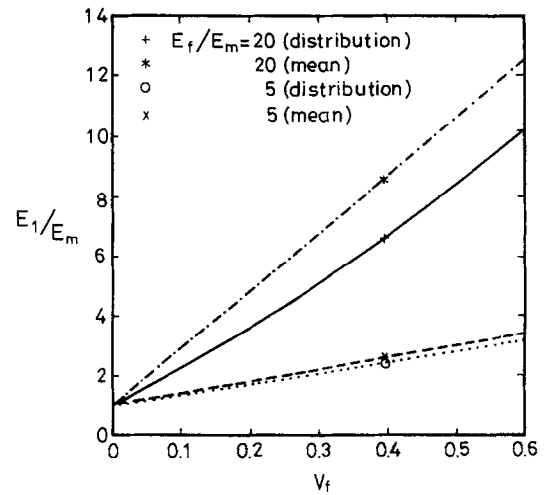


Fig. 11. Equivalent dimensionless Young's modulus E_1/E_m versus fiber volume fraction V_f calculated based on distributional fiber lengths and mean fiber length under various ratios of E_c/E_m .

were prominent; but for other moduli, the differences were less significant and hence not shown here.

7 CONCLUDING REMARKS

In this work, the closed form formulae for effective thermoelastic properties of a composite reinforced by the orientation distributed and/or length distributed coated fibers were derived under the assumption of thin coating and constant variation of stresses and strains through the thickness of the coating. Comparisons with existing theoretical and experimental data were made and the results were satisfactory. Additional numerical examples for some parametric studies were also presented for illustration.

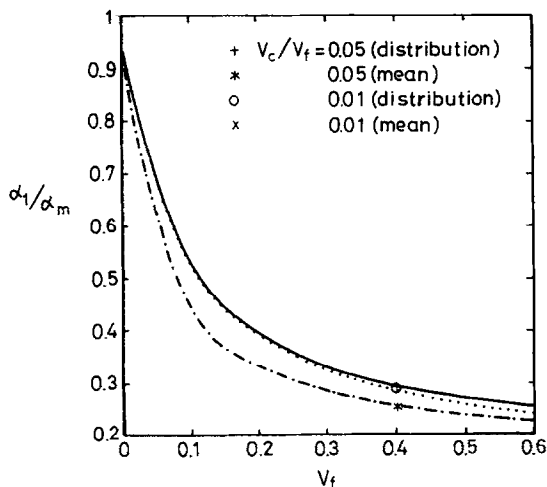


Fig. 10. Equivalent coefficient of thermal expansion α_1/α_m versus fiber volume fraction V_f calculated based on distributional fiber lengths and mean fiber length under various ratios of V_c/V_f .

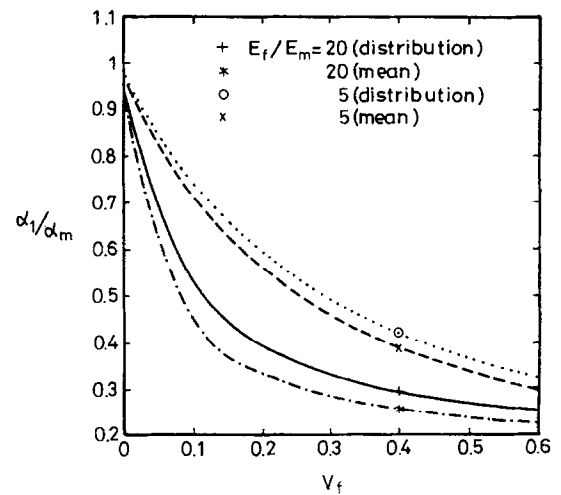


Fig. 12. Equivalent coefficient of thermal expansion α_1/α_m versus fiber volume fraction V_f calculated based on distributional fiber lengths and mean fiber length under various ratios of E_c/E_m .

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