Cube-Connected Modules: a Family of Cubic Networks

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Abstract

A family of cubic networks, named cube-connected modules, is proposed in this paper. The cube-connected modules network consists of modules which are interconnected as a hypercube. Any connected graph, e.g., cycle, hypercube graph, and complete graph, can serve as a module. Topological properties are investigated, and the problems of routing, broadcasting, embedding, and finding parallel routing paths are studied. We show that the problem of determining the shortest routing path is NP-hard, and it can be transformed to the asymmetric traveling salesman problem. The broadcasting algorithms on cube-connected modules can be obtained by combining broadcasting algorithms on hypercubes and broadcasting algorithms on modules. We show that if the modules are hamiltonian, then the cube-connected modules are also hamiltonian. Moreover, a sufficient condition is given for the existence of maximum number of parallel paths between any two nodes of cube-connected modules.

1 Introduction

Due to the recent advance in hardware technology and VLSI chip design, it is now feasible to build a large parallel and distributed system involving hundreds or even thousands of processors. Since the interconnection topology of the system plays a significant role in system performance, the design of interconnection networks is becoming one of the most fundamental issues in the study of parallel and distributed systems.

Many interconnection networks have been proposed, e.g., ring, tree, mesh, hypercube, perfect shuffle [23], de Bruijn network [22], Cayley network [1], star network [2], etc. For more references, see [3], [12], [24]. Among them, the hypercube has received much attention because it has a symmetric and recursive topology, and owns many attractive topological properties [21]. Many algorithms [19] have been successfully developed on the hypercube as a consequence of its simple and regular interconnection. There are parallel computers commercially available, e.g., NCUBE series and Intel's iPSC series, whose topologies are based on the hypercube interconnection.

In the past few years, many variants [4], [6]-[9], [15], [18] of the hypercube have been designed, which all are modifications or extensions of the hypercube. We collectively call them *cubic networks*. In this paper, a family of cubic networks, named cube-connected modules, is proposed. We show that the routing problem on cube-connected modules is NP-hard [10], and it can be transformed to the well-known asymmetric traveling salesman problem [13]. Consequently, any algorithm for the latter can be applied to the former. Besides, the problems of broadcasting, embedding, and finding parallel routing paths are also investigated.

In the next section, the structure of cube-connected modules is described.

2 Cube-Connected Modules

An n-dimensional cube-connected modules network, which is denoted by CCM_n, consists of 2^n modules which are interconnected as an n-dimensional hypercube. The modules can be any n-node connected graphs. When the modules are rings, hypercubes, and complete graphs, the resulting networks are denoted by CCR_n, CCQ_n, and CCK_n, respectively. The network CCM_n is

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formally defined as follows.

Definition 2.1. The node set of CCM_n is denoted by $\{(I, J) \mid I \in [0, 2^n-1]$ and $J \in [0, n-1]\}$, where I is module identifier and J is node identifier (within the module I). The adjacency between modules is defined as follows: (I_x, J_x) is adjacent to (I_y, J_y), where $I_x \neq I_y$, if and only if (1) the binary representations of I_x and I_y differ at exactly one bit position, say the bit position with weight 2^k, and (2) $J_x=J_y=k$. The link connecting (I_x, J_x) and (I_y, J_y) is labeled with k.

In Definition 2.1, we have not explicitly defined the adjacency within the modules. Actually, this should conform to the definition of the modules. For example, if the modules are n-node hypercubes, where n is a power of two, the adjacency within the modules should be defined as follows: (I, J_x) is adjacent to (I, J_y) if and only if the binary representations of J_x and J_y differ at exactly one bit position. The structure of CCM_4 is illustrated in Figure 1(a), where $M_0, M_1, ..., M_{15}$ denote the modules. The structure of M_0 is shown in Figure 1(b) where the modures are assumed to be hypercubes.

For easy reference, we call the links between modules the *external links*, and the links within modules the *internal links*. If a path goes through a module, then the two nodes at which the path enters and leaves the module are called the *gateway* nodes.

Define the *Hamming distance* of two values to be the number of different bits between their binary representations. The following is a basic property of CCM_n.

Lemma 2.1. Suppose $S=(I_S, J_S)$ and $T=(I_T, J_T)$ are any two nodes in CCM_n, and k is the Hamming distance between I_S and I_T. Then, the shortest path between S and T contains exactly k external links.

Proof. Omitted.

Lemma 2.1 implies that the shortest path between S and T is the path containing k external links and minimum number of internal links. Also the proof of Lemma 2.1 implies that the shortest path between the farthest two nodes of CCM_n contains n external links. This can be seen as follows. Assume the shortest path between the farthest two nodes of CCM_n contains t external links labeled with $l_1, l_2, ..., l_t$, in this order, where $t<n$. Let us consider another two nodes whose shortest path contains $t+1$ external links labeled with $l_{a_1}, l_{a_2}, ..., l_{a_{t+1}}$, in this order, and the set $\{l_{a_1}, l_{a_2}, ..., l_{a_{t+1}}\}$ minus the set $\{l_1, l_2, ..., l_t\}$ remains one single label, say l_{a_v}, $1 \leq v \leq t+1$. By assumption, the latter path is shorter than the former path. According to the proof of Lemma 2.1, a shorter path will result if the external link with label l_{a_v} is removed from the latter path. Consequently, the path containing t external links labeled with $l_{a_1}, l_{a_2}, ..., l_{a_{v-1}}, l_{a_{v+1}}, ..., l_{a_{t+1}}$ in this order is shorter than that labeled with $l_1, l_2, ..., l_t$ in this order, which is a contradiction. Hence, the following lemma results.

Lemma 2.2. The shortest path between the farthest two nodes of CCM_n contains n external links.

Suppose the diameter of the modules is w. Then, as a consequence of Lemma 2.2, the diameter of CCM_n ranges inclusively from $2n$ to $n(1+w)$. Since $w=1$ for CCK_n, we conclude that the diameter of CCK_n is $2n$. The diameters of CCR_n and CCQ_n are $5n/2$ and $2n$ [15], respectively, because they belong to the classes of cube-connected cycles [18] and hierarchical hypercubes [15], respectively.

There are $n2^n$ nodes and $(e+n/2)2^n$ links contained in CCM_n, where e is the number of links contained in each module. The degree of CCM_n is larger by 1 than the degree of each module. Table 1 summaries the numbers of links, the degrees, the diameters, and the costs for CCR_n,

CCQ_n, CCK_n, and an $(n+m)$-dimensional hypercube, where $n=2^m$ is assumed and the cost is defined as the product of the degree and the diameter.

3 Routing and Broadcasting

The *routing problem* on a network requires a node (the source node) to transmit a message to another node (the destination node). Usually, it is not difficult to determine a path between the source node and the destination node, but, it is somewhat more difficult to find the shortest path. Suppose $S=(I_S, J_S)$ and $T=(I_T, J_T)$ are the source node and the destination node, respectively, in CCM_n. There is a simple approach of routing a message from S to T as follows. First, the message is routed from the module I_S to the module I_T. Then, the message is routed to the node T within the module I_T. Although simple, this approach does not necessarily produce the shortest path between S and T. In the following, we show that the problem of determining the shortest path between two nodes of CCM_n is NP-hard.

Assume the Hamming distance between I_S and I_T is k, and their different bits have weights $2^{a_1}, 2^{a_2}, ..., 2^{a_k}$. By Lemma 2.1, finding the shortest path between S and T is equivalent to determining a permutation of $\{a_1, a_2, ..., a_k\}$, say $a_{i_1}, a_{i_2}, ..., a_{i_k}$, such that the summation of $d(J_S, a_{i_1})+d(a_{i_1}, a_{i_2})+d(a_{i_2}, a_{i_3})+ ... +d(a_{i_{k-1}}, a_{i_k})+d(a_{i_k}, J_T)$ is minimized.

A path in a graph is called a *hamiltonian path* if it traverses every vertex of the graph exactly once. The Hamiltonian Path Problem (HPP, for short) is known as an NP-complete problem [10], which is defined as follows.

Hamiltonian Path Problem (HPP)

Instance: Graph $G=(V, E)$, where V is the vertex set and E is the edge set, and two vertices $v_x, v_y \in V$.

Question: Is there a hamiltonian path between v_x and v_y in G ?

We now show that each instance of the HPP can be transformed in polynomial time into an instance of the routing problem on CCM_n (this transformation is named *Turing reduction* in [10]). Let $G=(\{v_x, v_1, v_2, ..., v_k, v_y\}, E)$ be an arbitrary instance of the HPP. A corresponding instance of the routing problem can be constructed as follows. Suppose $S=(I_S, J_S)$ and $T=(I_T, J_T)$ are the source node and the destination node, respectively, in CCM_n, where $n \geq k$, and the Hamming distance between I_S and I_T is k. Let $a_1, a_2, ..., a_k$ be the labels of the k external links contained in the shortest path between S and T. We set $d(J_S, a_i)=1$ if $(v_x, v_i) \in E$ and $d(J_S, a_i)=2$ if $(v_x, v_i) \notin E$ for all $i \in \{1, 2, ..., k\}$; set $d(a_i, a_j)=1$ if $(v_i, v_j) \in E$ and $d(a_i, a_j)=2$ if $(v_i, v_j) \notin E$ for all $i, j \in \{1, 2, ..., k\}$ and $i \neq j$; set $d(a_i, J_T)=1$ if $(v_i, v_y) \in E$ and $d(a_i, J_T)=2$ if $(v_i, v_y) \notin E$ for all $i \in \{1, 2, ..., k\}$. Then, there exists a hamiltonian path between v_x and v_y in G if and only if there exists a path (the shortest path) of length $2k+1$ between S and T in CCM_n.

According to the discussion above, we reach a conclusion as follows.

Theorem 3.1. The problem of determining the shortest path between two nodes of CCM_n is NP-hard.

Next, we show that solving the shortest-path routing problem on CCM_n can be reduced to solving the Traveling Salesman Problem (TSP, for short). The TSP, which is known as an NP-hard problem [10], is defined as follows.

Traveling Salesman Problem (TSP)

Instance: Set $C=\{c_1, c_2, ..., c_m\}$ of m cities, distance $l(c_i, c_j) \geq 0$ for each pair of cities $c_i, c_j \in C$.

Question: Determine a tour of C, i.e., $<c_{i_1}, c_{i_2}, ..., c_{i_m}>$, such that the summation of $l(c_{i_1}, c_{i_2}) + l(c_{i_2}, c_{i_3}) + ... + l(c_{i_{m-1}}, c_{i_m}) + l(c_{i_m}, c_{i_1})$ is minimized.

The TSP is called *symmetric* if $l(c_i, c_j)=l(c_j, c_i)$ for all $c_i \neq c_j$, and *asymmetric* otherwise. Let us consider an arbitrary instance of the routing problem on CCM_n as follows. $S=(I_S, J_S)$ and $T=(I_T, J_T)$ are the source node and the destination node, respectively, in CCM_n. The Hamming distance between I_S and I_T is k. The k external links contained in the shortest path between S and T are labeled with $a_1, a_2, ..., a_k$. A corresponding instance of the asymmetric TSP can be constructed as follows. Let $C=\{c_x, c_y, c_z, c_1, c_2, ..., c_k\}$ be the set of cities. Set $l(c_i, c_j)=d(a_i, a_j)$ for all $i, j \in \{1, 2, ..., k\}$ and $i \neq j$; $l(c_x, c_i)=d(J_S, a_i)$, $l(c_i, c_y)=d(a_i, J_T)$, and $l(c_i, c_x)=l(c_y, c_i)=l(c_x, c_y)=l(c_y, c_x)=$I for all $i \in \{1, 2, ..., k\}$; $l(c_z, c_x)=(c_y, c_z)=0$ and $l(c_x, c_z)=l(c_z, c_y)=l(c_z, c_i)=l(c_i, c_z)=$I for all $i \in \{1, 2, ..., k\}$. Then, the shortest path between S and T goes through the external links with labels $a_{i_1}, a_{i_2}, ..., a_{i_k}$ in this order if and only if the tour $<a_{i_1}, a_{i_2}, ..., a_{i_k}>$ is the optimal solution to the instance of the asymmetric TSP.

The transformation above also implies that there is a path of length $k+s$ between S and T if and only if there is a tour of length s for the instance of the asymmetric TSP. Since the transformation takes $O(k^2)$ time, the following theorem is an immediate consequence of the discussion above.

Theorem 3.2. Suppose $S=(I_S, J_S)$ and $T=(I_T, J_T)$ are any two nodes in CCM_n, and k is the Hamming distance between I_S and I_T. Then, all algorithms for the asymmetric TSP can be used to determine routing paths between S and T with extra $O(k^2)$ transformation time.

Remarks. The asymmetric TSP is a well-solved problem, and there are many algorithms, exact and approximate, available. Exact algorithms based on dynamic programming and branch-and-bound approaches can be found in [11]. Integer programming approaches appear in [16], [17]. A simple approximation algorithm can be found in [13] (pages 194-203). For most instances, this algorithm produces very near-optimal solutions. Let T denote the value of the approximate solution produced by the algorithm, and T^* the value of the optimal solution. A probabilistic analysis of the algorithm shows that the expected value of $(T- T^*)/T^*$ is bounded above by $O(n^{-1/2})$, where n is the problem size. Thus, T becomes close to T^* as n increases. More approximation algorithms for the asymmetric TSP can be found in [13] (pages 239-243).

A heuristic routing algorithm for CCQ_n can be found in [15]. Suppose $S=(I_S, J_S)$ and $T=(I_T, J_T)$ are source node and destination node, respectively. The routing algorithm can produce a path with length at most $H(I_S, I_T)+n+\log_2 n-1$, where $H(I_S, I_T)$ denotes the Hamming distance between I_S and I_T. Moreover, the routing algorithm can be adapted to CCM_n if the modules contain hamiltonian circuits (explained in the next section). The path produced by the adapted routing algorithm has length at most $H(I_S, I_T)+n+D_M-1$, where D_M denotes the diameter of the modules.

Before ending this section, let us discuss the broadcasting problem on CCM_n. The *broadcasting* problem on a network requires the source node to disseminate a message to all other nodes. There are two models considered, all-port and 1-port. The *all-port model* allows a node to transmit a message along all its incident links in one time unit. Differently, the *1-port model* allows a node to transmit a message along only one of its incident links in one time unit. Broadcasting algorithms for CCM_n can be obtained by combining broadcasting algorithms for an n-dimensional hypercube with broadcasting algorithms for the modules. Considering each module a supernode, we first run a broadcasting algorithm which is executable on an n-dimensional hypercube. Once a supernode gets the message, it executes a broadcasting algorithm which is executable on the modules. Therefore, we have a general result as stated below.

Theorem 3.3. Suppose A_Q is a broadcasting algorithm running in $f_Q(n)$ communication steps on an n-dimensional hypercube, and A_M is a broadcasting algorithm running in $f_M(n)$

communication steps on the modules of CCM_n. Then, there is a broadcasting algorithm A on CCM_n which is the combination of A_Q and A_M, and takes $f_Q(n)+(f_Q(n)+1)f_M(n)$ communication steps. Moreover, A runs under the 1-port model if A_M runs under the 1-port model, or under the all-port model if A_M runs under the all-port model.

In [15], a broadcasting algorithm for CCQ_n was proposed, which takes $2n+\log_2 n-1$ communication steps under the all-port model.

4 Embedding

An *embedding* of a (source) network onto another (target) network is a one-to-one mapping ϕ from the node set of the source network to the node set of the target network. A link in the source network may correspond to a path in the target network. The *dilation* of ϕ is defined as the maximum distance between $\phi(u)$ and $\phi(v)$ for all links (u, v) of the source network.

A *hamiltonian circuit* in a network is a circuit which contains every node of the network exactly once. A network is *hamiltonian* if it contains a hamiltonian circuit. A hamiltonian network can embed a ring with dilation 1. Since the cube-connected cycles network (CCC, for short) is known to be hamiltonian (see Theorem 3.14 in [14]), CCR_n is also hamiltonian. In the following, a sufficient condition for hamiltonian CCM_n is proposed.

Theorem 4.1. CCM_n is hamiltonian if the modules are hamiltonian.

Proof. Omitted.

From the proof of Theorem 4.1, we have the following corollary.

Corollary 4.2. CCM_n contains CCR_n as a subgraph, provided the modules are hamiltonian.

Since hypercubes and complete graphs are hamiltonian, CCQ_n and CCK_n are hamiltonian and contain CCR_n as a subgraph.

Assume $n=r^2$ for some even r. Since a hypercube can embed a torus with dilation 1 [21], the 2^n modules of CCM_n can be arranged as a $2^{n/2} \times 2^{n/2}$ torus (a torus is a mesh with wraparound links in the rows and columns). If the n nodes of each module is further arranged as an $r \times r$ mesh, then the $n2^n$ nodes of CCM_n form an $r2^{n/2} \times r2^{n/2}$ torus. Thus, CCM_n can embed a torus with dilation computed below.

Theorem 4.3. Assume $n=r^2$ for some even r, and let d be the diameter of the modules of CCM_n. Then, CCM_n can embed an $r2^{n/2} \times r2^{n/2}$ torus with dilation not greater than $2d+1$.

5 Parallel Paths

The *node connectivity* [5] of a network is defined as the minimum number of nodes whose removal can disconnect the network. It is well known that there is a relation, i.e., Menger's theorem [5], between the node connectivity and the maximum number of parallel paths. Menger's theorem states that given a network, the maximum number of parallel paths between any two of its nodes is equal to its node connectivity.

Assume the minimum node degree of CCM_n is $k \le n$. Since k is an upper bound on the node connectivity of CCM_n, by Menger's theorem there are at most k parallel paths between any two nodes of CCM_n. In the following, a sufficient condition for the existence of such k parallel paths in CCM_n is proposed

Theorem 5.1. Let $k \leq n$ be the minimum node degree of CCM_n. If there are always k-1 parallel paths from an arbitrary node to any other k-1 nodes, all within the same module, then there are k parallel paths between any two nodes of CCM_n.

Proof. Omitted.

Assume $n=2^m$. The node degrees of CCR_n, CCQ_n, and CCK_n are 3, m+1, and n, respectively. It is clear that the premise of Theorem 5.1 is true for CCR_n and CCK_n. Hence, there are 3 and n parallel paths between any two nodes of CCR_n and CCK_n, respectively. As for CCQ_n, an approach of finding m parallel paths from an arbitrary node to any other m nodes within an m-dimensional hypercube has appeared in [20]. We have a corollary as follows.

Corollary 5.2. Assume $n=2^m$. Three, m+1, and n parallel paths can be constructed between any two nodes of CCR_n, CCQ_n, and CCK_n, respectively.

6 Remarks

So far we give few words about which problems can be solved on CCM_n. Since CCR_n can execute the class of DESCEND/ASCEND algorithms [18], by Corollary 4.2 CCM_n can execute the same class of algorithms as well if the modeules are hamiltonian. Many algorithms, e.g., bitonic merge, bitonic sort, odd-even-merge, odd-even sort, cyclic shift, permutation, shuffle, unshuffle, bit reversal, fast Fourier transform, convolution, matrix transposition, and calculations of symmetric functions, are known to be instances of DESCEND/ASCEND algorithms. Besides, by Theorem 4.1 CCM_n can execute all the algorithms executable on a ring or a linear array if the modules are hamiltonian.

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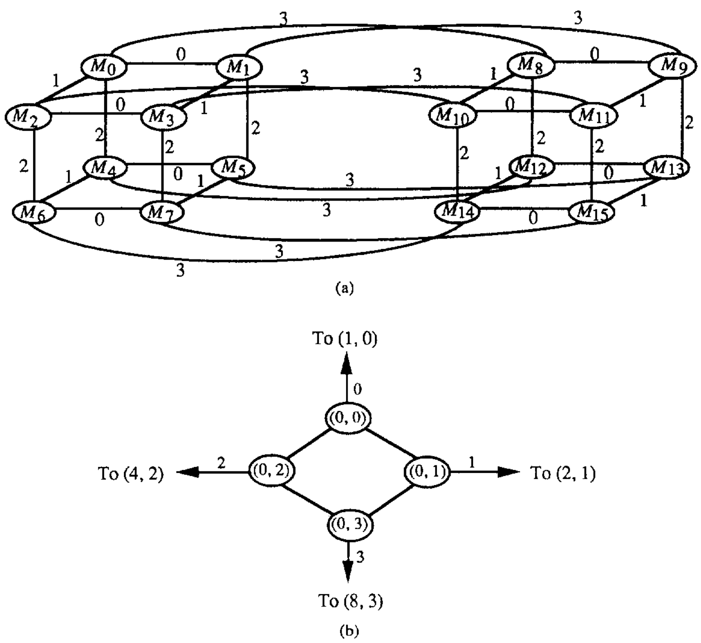

Figure 1. The structure of cube-connected modules networks. (a) The structure of CCM_4. (b) The structure of M_0 when CCM_4 is restricted to CCQ_4.

	CCR_n	CCQ_n	CCK_n	$(n+m)$-dimensional hypercube
Links	$3 \cdot 2^{n+m-1}$	$(m+1)\, 2^{n+m-1}$	$n \cdot 2^{n+m-1}$	$(n+m)\, 2^{n+m-1}$
Degrees	3	$m + 1$	n	$n + m$
Diameters	$\frac{5}{2} n$	$2 \cdot n$	$2 \cdot n$	$n + m$
Costs	$\frac{15}{2} n$	$(m+1) \cdot 2n$	$2 \cdot n^2$	$(n+m)^2$

Table 1. Comparison of CCR_n, CCQ_n, CCK_n, and $(n+m)$-dimensional hypercube.