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Formulations of Support Vector Machines: A Note from an Optimization Point of View

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In this article, we discuss issues about formulations of support vector machines (SVM) from an optimization point of view. First, SVMs map training data into a higher- (maybe infinite-) dimensional space. Currently primal and dual formulations of SVM are derived in the finite dimensional space and readily extend to the infinite-dimensional space. We rigorously discuss the primal-dual relation in the infinite-dimensional spaces. Second, SVM formulations contain penalty terms, which are different from unconstrained penalty functions in optimization. Traditionally unconstrained penalty functions approximate a constrained problem as the penalty parameter increases. We are interested in similar properties for SVM formulations. For two of the most popular SVM formulations, we show that one enjoys properties of exact penalty functions, but the other is only like traditional penalty functions, which converge when the penalty parameter goes to infinity.

1 Introduction

The support vector machine (SVM) is a new and promising classification technique (for surveys of SVM, see Vapnik, 1995, 1998). Given training vectors $x_i \in R^n$, $i = 1, \ldots, l$ in two classes and a vector $y \in R^n$ such that $y_i \in \{1, -1\}$, the SVM requires the solution of the following quadratic programming problem (Boser, Guyon, & Vapnik, 1992):

$$P_0: \qquad \min \frac{1}{2}\langle w, w\rangle$$
$$y_i(\langle w, \phi(x_i)\rangle + b) \geq 1, \quad i = 1, \ldots, l,$$

where $\langle \cdot, \cdot \rangle$ is the inner product of two finite or infinite vectors.

Training vectors x_i are mapped into a higher-dimensional space by the function ϕ. In this new space, we seek a separating hyperplane $\langle w, \phi(x)\rangle + b = 0$ with coefficients w and b such that all training vectors are separated. Note that this higher-dimensional space can be finite or infinite. In this article, to stress the difference between finite and infinite inner products, we represent them as $x^T y$ and $\langle x, y\rangle$, respectively.

If in a higher-dimensional space, data are still not separable, Cortes and Vapnik (1995) propose adding a penalty term $C\sum_{i=1}^{l}\xi_i$ in the objective function, where C is a large, positive number:

$$P_C: \qquad \min \frac{1}{2}\langle w, w\rangle + C\sum_{i=1}^{l}\xi_i$$
$$y_i(\langle w, \phi(x_i)\rangle + b) \geq 1 - \xi_i,$$
$$\xi_i \geq 0, \quad i = 1, \ldots, l.$$

The technique of introducing variables ξ_i can be traced back to, for example, Bennet and Mangasarian (1992) for linear programming formulations.

As P_C becomes a problem with many (or infinite) variables, currently the standard method (see, for example, Cortes & Vapnik, 1995, and Vapnik, 1995, 1998) for solving problem P_C is through the following dual, which is a finite problem with l variables:

$$D_C: \qquad \min \frac{1}{2}\alpha^T Q\alpha - e^T\alpha$$
$$y^T\alpha = 0, \quad 0 \leq \alpha_i \leq C, \quad i = 1, \ldots, l,$$

where $Q_{ij} \equiv y_i y_j\langle\phi(x_i), \phi(x_j)\rangle$ and e is the vector of all ones. If α is a solution of (D_C), $w \equiv \sum_{i=1}^{l} y_i\alpha_i\phi(x_i)$ is a solution of problem P_C. Usually $K(x_i, x_j) \equiv \langle\phi(x_i), \phi(x_j)\rangle$ is called the kernel. Although $\phi(x)$ is in a very high-dimensional space, if a special ϕ is used, $K(x_i, x_j)$ is finite and can be easily calculated. Algorithms for solving D_C can be seen in, for example, Schölkopf, Burges, and Smola (1998). Similarly, the dual of problem P_0 is as follows:

$$D_0: \qquad \min \frac{1}{2}\alpha^T Q\alpha - e^T\alpha$$
$$y^T\alpha = 0, \quad \alpha_i \geq 0, \quad i = 1, \ldots, l.$$

However, in the above references, the derivation of D_C was through the standard convex analysis in finite-dimensional spaces. Note that $\phi(x)$ and w can be vectors in an infinite-dimensional Hilbert space. Since properties of convex analysis in finite spaces may not always hold in infinite spaces, we must be careful with any generalization. In this article, all results are derived for infinite-dimensional formulations.

The first issue is on the relation between P_C and D_C: $C \geq 0$. For example, we may ask whether it might happen that P_0 has an optimal solution but D_0 is not solvable. Note that it is possible that a convex programming problem has a lower bound on all objective values but has no feasible points achieving this bound. In section 2, we show that P_0 and D_0 are either both solvable

or P_0 is infeasible and D_0 is unbounded. In addition, P_C and D_C, $C > 0$ are both solvable.

Cortes and Vapnik (1995) and, more recently, Friess, Cristianini, and Campbell (1998) offer a different SVM formulation:

$$\bar{P}_C: \quad \begin{array}{l} \min \frac{1}{2}\langle w, w\rangle + \frac{C}{2}\sum_{i=1}^{l}\xi_i^2 \\ y_i(\langle w, \phi(x_i)\rangle + b) \geq 1 - \xi_i, \\ i = 1, \ldots, l. \end{array} \qquad \bar{D}_C \quad \begin{array}{l} \min \frac{1}{2}\alpha^T\left(Q + \frac{I}{C}\right)\alpha - e^T\alpha \\ y^T\alpha = 0, \alpha_i \geq 0, \\ i = 1, \ldots, l. \end{array}$$

Here I is the identity matrix. Using similar techniques, we show that both $\bar{P}_C$ and $\bar{D}_C$, $C > 0$ are always solvable.

The second topic of this article is the penalty parameter. In P_C (or $\bar{P}_C$), the penalty term $C\sum_{i=1}^{l}\xi_i$ (or $\frac{C}{2}\sum_{i=1}^{l}\xi_i^2$) is added to reduce the occurrence of incorrectly classified data. Note that such an approach is different from traditional penalty functions, which modify constrained problems to unconstrained ones. As the penalty parameter goes to infinity, the solutions of this sequence of unconstrained problems approximate a solution of the original problem. Here we are interested in similar properties for SVM formulations. In section 3, we show that for P_C, after C is greater than a certain number, all solutions of P_C are a solution of P_0. In other words, P_C is like an exact penalty function approach. On the other hand, this result may not hold for $\bar{P}_C$ but as $C \to \infty$, solutions of $\bar{P}_C$ still converge to a solution of P_0.

We hope that we demonstrate a framework on optimization properties of SVM formulations. Hence, when new formulations are proposed, similar tools and techniques can be applied without many modifications. Finally we make conclusions and discussions in section 4.

2 Primal-Dual Relation of SVM Formulations

For the discussion in the rest of this article, we need the following theorem, which shows the necessary and sufficient optimality conditions of a convex optimization problem. This is an extension of the KKT condition in finite-dimensional spaces. Note that the standard necessary condition usually requires some constraint qualifications (e.g., Slater condition). However, for problems with linear constraints, such an assumption is not necessary.

Theorem 1. *Consider an optimization problem*

$$\min f(x) \; s.t. \; \langle v_i, x\rangle - b_i \geq 0, \quad i = 1, \ldots, l,$$

where $x \in X$, a Hilbert space, $v_i \in X$, $i = 1, \ldots, l$ and $b_i \in R$, $i = 1, \ldots, l$. If f is a real convex and differentiable function, then a feasible element x is an optimal solution if and only if there exist real numbers $\lambda_1, \ldots, \lambda_l$ such that

$$\nabla f(x) - \lambda_1 v_1 - \cdots - \lambda_l v_l = 0$$
$$\lambda_i \geq 0, \lambda_i(\langle v_i, x\rangle - b_i) = 0, \forall i = 1, \ldots, l.$$

Proof. The necessary condition when constraints are linear can be seen in, for example, Girsanov (1972, theorem 11.3). The sufficient condition does not require any constraint qualification. An example of proof can be seen in corollary 1.1 of Barbu and Precupanu (1986, chap. 3).

Note that the definition of $\nabla f(x)$ here is in the sense of Frechet differentiability. Theorem 1 can be extended to linear equality constraints as a linear equality can be written as two linear inequalities.

Then the following theorem shows the relation between P_0 and D_0:

Theorem 2. *For problems P_0 and D_0, only one of the following two situations can happen:*

P_0 solvable and D_0 solvable,

P_0 infeasible and D_0 unbounded.

On the other hand, P_C and D_C, $C > 0$, are both solvable.

Proof. First we prove that if D_0 is solvable, P_0 is solvable. Suppose that an optimal solution of D_0 is α, by theorem 1 (necessary condition), there exists a b such that

$$\alpha_i(Q\alpha - e - by)_i = 0, i = 1, \ldots, l, \text{ and } Q\alpha - e - by \geq 0.$$

By defining $w \equiv \sum_{i=1}^{l} \alpha_i y_i \phi(x_i)$, we have (w, b) which satisfies the optimality condition of P_0. From theorem 1 (sufficient condition), (w, b) is an optimal solution of P_0. Hence P_0 is solvable.

Next we show that if P_0 is feasible, D_0 is solvable. Consider the following problem:

$$\max_{\alpha \geq 0} \left\{ \inf_{w,b} \left[\frac{1}{2}\langle w, w\rangle - \sum_{i=1}^{l} \alpha_i (y_i(\langle w, \phi(x_i)\rangle + b) - 1) \right] \right\}. \tag{2.1}$$

When $y^T\alpha = 0$, $\inf_{w,b}[\frac{1}{2}\langle w, w\rangle - \sum_{i=1}^{l} \alpha_i(y_i(\langle w, \phi(x_i)\rangle + b) - 1)]$ has a global minimum at $w = \sum_{i=1}^{l} \alpha_i y_i \phi(x_i)$ (see, for example, Debnath & Mikusinski, 1999, example 9.3.3). Thus equation 2.1 is equivalent to D_0. It can be clearly seen that if P_0 is feasible with any feasible solution w^*, $\frac{1}{2}\langle w^*, w^*\rangle$ is an upper bound of equation 2.1. That is, $-\frac{1}{2}\langle w^*, w^*\rangle$ is a lower bound of D_0. From corollary 27.3.1 of Rockafellar (1970), any feasible bounded finite-dimensional space quadratic convex function over a polyhedral attains at least an optimal solution. Since D_0 is always feasible, we know that it is solvable.

Furthermore, because a feasible P_0 implies a solvable D_0 and a solvable D_0 implies a solvable P_0, we know that if P_0 is feasible, its minimum

is attained. Therefore, we conclude that there are only two possible situations:

P_0 solvable and D_0 solvable.

P_0 infeasible and D_0 unsolvable.

We then use the same property that if D_0 is bounded, it is solvable. Therefore, D_0 unsolvable means D_0 is unbounded, so the proof is complete.

Since P_C and D_C, $C > 0$, are both feasible, by the same arguments above, they are both solvable.

Using a similar method, we can prove that $\bar{P}_C$ and $\bar{D}_C$ are also both solvable.

3 Penalty Parameters

In P_C (or $\bar{P}_C$), the penalty term $C\sum_{i=1}^{l}\xi_i$ (or $\frac{C}{2}\sum_{i=1}^{l}\xi_i^2$) is used to reduce the occurrence of wrongly classified data. This is different from the penalty approach in mathematical programming. Originally this method approximates a constrained problem by unconstrained minimization. When the penalty parameter is large enough, minima of unconstrained problems are close to a solution. Hence it is important to know that if P_0 is separable, how close a solution of P_C is to a solution of P_0 as C increases. This can be a criterion to judge the appropriateness of using the penalty term and will be the main topic of this section. First we prove the following technical lemma:

Lemma 1. *For any $C \geq 0$, the vector w of optimal solutions of P_C is unique.*

Proof. Assume that (w, b, ξ) and $(\bar{w}, \bar{b}, \bar{\xi})$ with $w \neq \bar{w}$ are both optimal solutions of P_C. For any $0 \leq \alpha \leq 1$, $(\hat{w}, \hat{b}, \hat{\xi}) = \alpha(w, b, \xi) + (1-\alpha)(\bar{w}, \bar{b}, \bar{\xi})$ is feasible to P_C. Therefore,

$$\frac{1}{2}\langle \hat{w}, \hat{w}\rangle + C\sum_{i=1}^{l}\hat{\xi}_i \geq \frac{1}{2}\langle w, w\rangle + C\sum_{i=1}^{l}\xi_i = \frac{1}{2}\langle \bar{w}, \bar{w}\rangle + C\sum_{i=1}^{l}\bar{\xi}_i. \tag{3.1}$$

Note that equation 3.1 can be treated as two inequalities. By multiplying the first part by α and the second part by $(1-\alpha)$,

$$\frac{1}{2}\alpha^2\langle w, w\rangle + \alpha(1-\alpha)\langle w, \bar{w}\rangle + \frac{1}{2}(1-\alpha)^2\langle \bar{w}, \bar{w}\rangle \geq \frac{1}{2}\alpha\langle w, w\rangle + \frac{1}{2}(1-\alpha)\langle \bar{w}, \bar{w}\rangle.$$

Thus,

$$\alpha(1-\alpha)\langle w - \bar{w}, w - \bar{w}\rangle \leq 0, \ \forall 0 \leq \alpha \leq 1.$$

Since $w - \bar{w}$ is in a Hilbert space, $w - \bar{w} = 0$. This causes the contradiction, so w must be unique.

For other variables such as b and ξ of P_C and α of D_C, there are always examples where multiple solutions occur. Next we present a relation between solutions of P_C, $C > 0$ and those of P_0.

Theorem 3. *If P_0 is solvable, there is a C^* such that for all $C \geq C^*$, any solution (w, b) of P_0 is a solution of P_C.*

Proof. If (w, b) is a solution of P, by theorem 1, there are α and b such that

$$w = \sum_{i=1}^{l} y_i \alpha_i \phi(x_i),\ y^T \alpha = 0, \quad \alpha \geq 0$$
$$\alpha_i (y_i \langle \phi(x_i), w \rangle + b - 1) = 0, \quad i = 1, \ldots, l.$$

Let $C^* > \max(\alpha_i)$. For all $C > C^*$, we can set $\xi = 0$ and obtain

$$w = \sum_{i=1}^{l} y_i \alpha_i \phi(x_i),\ y^T \alpha = 0$$
$$\alpha_i (y_i \langle \phi(x_i), w \rangle + b - 1 + \xi_i) = 0$$
$$0 \leq \alpha_i \leq C, \quad \xi_i (C - \alpha_i) = 0, \quad i = 1, \ldots, l.$$

Therefore, from theorem 1, (w, b, ξ) is also a solution of P_C.

Theorem 3 proves a property similar to exact penalty functions. That is, after the penalty parameter is large enough, any solution of the original problem is also a solution of penalty functions. However, according to Di Pillo (1994), they are only weak exact penalty functions, though most early literature on penalty functions studied this result. Since the penalty approach is an attempt to solve P_0, it is more important to study the converse property, which ensures that solutions of P_C are solutions of P_0. It is shown in the following theorem and corollary:

Theorem 4. *If P_0 is solvable, there is a C^* such that for all $C \geq C^*$, any solution (w, b, ξ) of P_C satisfies that $\xi = 0$ and (w, b) is a solution of P_0.*

Proof. Consider any C^* such that theorem 3 holds and assume (w, b) is a solution of P_0. From theorem 3, $(w, b, 0)$ is also an optimal solution of P_C. Now if we solve P_C and obtain $(\bar{w}, \bar{b}, \bar{\xi})$, from lemma 1, $\bar{w} = w$. Therefore,

$$\frac{1}{2}\langle \bar{w}, \bar{w} \rangle + C \sum_{i=1}^{l} \bar{\xi}_i \leq \frac{1}{2}\langle w, w \rangle.$$

Thus, $\bar{\xi} = 0$ and $(\bar{w}, \bar{b})$ is an optimal solution of P_0.

Corollary 1. *If P_0 is solvable and*

$$C^* \equiv \min \left\{ \max_{i=1,\ldots,l} \alpha_i \mid \alpha \text{ is a solution of } D_0 \right\},$$

each of $P_C, C < C^*$ has a different solution w from the solution of P_0. On the contrary, $P_C, C \geq C^*$ has the same solution w as that of P_0.

Proof. When $C \geq C^*$, the result is given by theorem 4. For the case of $C < C^*$, assume there is a P_C with a solution (w_C, b_C, ξ_C) such that w_C is also part of a solution of P_0. If (w, b) is a solution of P_0, (w, b) is feasible to P_C, so

$$\frac{1}{2}\langle w_C, w_C\rangle + C\sum_{i=1}^{l}(\xi_C)_i \leq \frac{1}{2}\langle w, w\rangle.$$

This implies $\xi_C = 0$. Hence any solution of D_C is also a solution of D_0. Since $C < C^*$, this contradicts the definition of C^*.

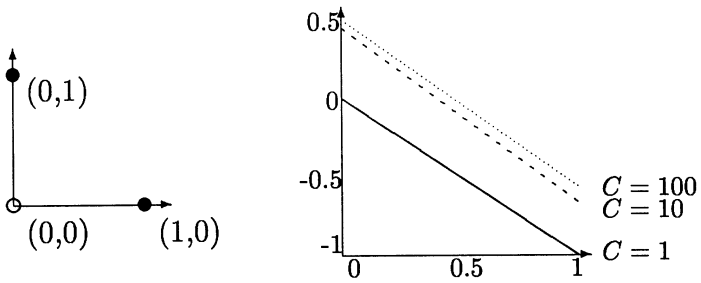
As we show only that vectors w are different, when $C < C^*$, it is still possible to obtain the same separating hyperplane. Note that $\langle w, \phi(x)\rangle + b = 0$ is the hyperplane so any $\alpha(w, b), \alpha \neq 0$, are appropriate coefficients. However, when $C < C^*$, the separating plane generated by P_C can also be very different from that of P_0. For example, consider a problem with three points, 0, 1, and 100, where 0 is in the first class, 1 and 100 are in the other class, and $K(x_i, x_j) = x_i^T x_j$. When $C \geq 2$, $\alpha = [2, 2, 0]$ and the hyperplane is $2x - 1 = 0$. When $1 \leq C \leq 2$, $\alpha = [C, C, 0]$ and $w = C$. As b can be any number that satisfies $-1 \leq b \leq 1 - C$, $b = -C/2$ can be a choice. Therefore, $2x - 1 = 0$ can still be a solution. However, when C becomes smaller, the hyperplane starts moving toward the training point 100. In addition, the distance between two planes $\langle w, x\rangle + b = \pm 1$ becomes very large. This shows that if small Cs are used, some careful analysis should be conducted.

Next we switch the target to $\bar{P}_C$ and $\bar{D}_C$. Interestingly they are not like exact penalty functions. In the following example, we show that no matter how large C is, solutions of $\bar{P}_C$ are not solutions of P_0. In Figure 1a, there are three training vectors in two different classes. If $Q_{ij} = y_i y_j x_i^T x_j$, the optimal separating hyperplane for P_0 is $[2\ 2]x - 1 = 0$. The optimality condition of $\bar{P}_C$ is

$$\begin{bmatrix} \frac{1}{C} & 0 & 0 \\ 0 & 1+\frac{1}{C} & 0 \\ 0 & 0 & 1+\frac{1}{C} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}.$$

At the solution, all three vectors are support vectors so $\lambda = 0$. Solving the above equation with $y^T\alpha = 0$ obtains

$$b = \frac{C-1}{-C-3}, \alpha_1 = (1+b)C, \quad \alpha_2 = \alpha_3 = \frac{(1-b)C}{C+1} = \frac{2C}{C+3}.$$



(a) Training data (b) Separating hyperplanes

Figure 1: A simple example.

The separating hyperplane is

$$\left[\frac{C(1-b)}{C+1}\ \frac{C(1-b)}{C+1}\right]x + b = 0,$$

that is,

$$[2C\ 2C]x + (1-C) = 0. \tag{3.2}$$

No matter what C is, equation 3.2 is different from $[2\ 2]x - 1 = 0$. However, it can be seen in Figure 1b that as $C \to \infty$, equation 3.2 approaches $[2\ 2]x - 1 = 0$. In the following theorems we show that $\bar{P}_C$ has properties of only traditional penalty functions. That is, as $C \to \infty$, solutions of $\bar{P}_C$ approach solutions of P_0.

Lemma 2. *Let $(w_{C_j}, b_{C_j}, \xi_{C_j})$ be a solution of $\bar{P}_{C_j}$, α_{C_j} be the solution of $\bar{D}_{C_j}$, $j = 1, 2, \ldots$, and assume $\lim_{j\to\infty} C_j = \infty$. If P_0 is solvable and $\lim_{j\to\infty} \alpha_{C_j}$ exists, then $\lim_{j\to\infty} w_{C_j}$ exists and is the solution of P_0.*

Proof. From the optimality condition (see theorem 1) of $\bar{D}_C$, $\alpha_{C_j} = C\xi_{C_j}$. Since $\lim_{j\to\infty} \alpha_{C_j}$ exists, $\lim_{j\to\infty} \xi_{C_j} = 0$. As $w_{C_j} = \sum_{i=1}^{l} y_i(\alpha_{C_j})_i \phi(x_i)$ and $\lim_{j\to\infty} \alpha_{C_j}$ exists, $\lim_{j\to\infty} w_{C_j}$ exists and we can denote it as w. Note that

$$\max_{i:\ y_i=1}(1 - (\xi_{C_j})_i - \langle w_{C_j}, \phi(x_i)\rangle) \le b_{C_j} \le \min_{i:\ y_i=-1}(-1 + (\xi_{C_j})_i + \langle w_{C_j}, \phi(x_i)\rangle).$$

As $C_j \to \infty$, $\lim_{j\to\infty} \xi_{C_j} = 0$ so

$$\max_{i:\ y_i=1}(1 - \langle w, \phi(x_i)\rangle) \le \min_{i:\ y_i=-1}(-1 + \langle w, \phi(x_i)\rangle).$$

Hence w is feasible to P_0. Furthermore, as P_0 is solvable, we can assume $\bar{w}$ is a solution. Since $\bar{w}$ is feasible to all $\bar{P}_C$, $C > 0$ and w_{C_j} is the optimal solution of $\bar{P}_{C_j}$, we have

$$\frac{1}{2}\langle w_{C_j}, w_{C_j}\rangle \leq \frac{1}{2}\langle \bar{w}, \bar{w}\rangle.$$

As $j \to \infty$, $\frac{1}{2}\langle w, w\rangle \leq \frac{1}{2}\langle \bar{w}, \bar{w}\rangle$. Hence $\lim_{j\to\infty} w_{C_j} = w$ is an optimal solution of P_0.

Theorem 5. *Let (w_C, b_C, ξ_C) be a solution of $\bar{P}_C$. If P_0 is solvable, then $\lim_{C\to\infty} w_C$ exists and is the solution of P_0.*

Proof. First we prove that $\{\alpha_C \mid C \geq \epsilon\}$ is bounded, where α_C is a solution of D_C and ϵ is any positive number. If $\bar{\alpha}$ is a solution of D_0, $\bar{\alpha}$ is feasible to $\bar{D}_C$ and α_C is feasible to D_0, so

$$\begin{aligned}\frac{1}{2}\bar{\alpha}^T Q\bar{\alpha} - e^T\bar{\alpha} &\leq \frac{1}{2}\alpha_C^T Q\alpha_C - e^T\alpha_C \\ &\leq \frac{1}{2}\alpha_C^T\left(Q + \frac{I}{C}\right)\alpha_C - e^T\alpha_C \leq \frac{1}{2}\bar{\alpha}^T\left(Q + \frac{I}{C}\right)\bar{\alpha} - e^T\bar{\alpha}.\end{aligned}$$

The above formula provides lower and upper bounds on $\frac{1}{2}\alpha_C^T(Q + \frac{I}{C})\alpha_C - e^T\alpha_C$ when $C \geq \epsilon > 0$. From the optimality condition (see theorem 1) of $\bar{D}_C$,

$$-\frac{1}{2}\alpha_C^T\left(Q + \frac{I}{C}\right)\alpha_C + e^T\alpha_C = \frac{1}{2}\alpha_C^T\left(Q + \frac{I}{C}\right)\alpha_C.$$

Hence $\frac{1}{2}\alpha_C^T\left(Q + \frac{I}{C}\right)\alpha_C$ is bounded and so as $e^T\alpha_C$. Since $\alpha_C \geq 0$, this implies that $\{\alpha_C\}$ is bounded after C is large enough.

Since $\{\alpha_C \mid C \geq \epsilon\}$ is bounded, there exists a convergent sequence $\{\alpha_{C_j}\}$. From lemma 2, $\lim_{j\to\infty} w_{C_j}$ exists, where w_{C_j} is part of a solution of $\bar{P}_{C_j}$. If $\lim_{C\to\infty} w_C \neq w$, there is a sequence $\{w_{C_k}\}$ and $\delta > 0$ such that $\|w_{C_k} - w\| \geq \delta$, for all $k = 1, 2, \ldots$. However, $\{\alpha_{C_k}\}$ is bounded so we can find a subsequence of $\{\alpha_{C_k}\}$ which satisfies conditions of lemma 2. Hence this subsequence converges to a solution of P_0, that is, w. This contradicts the assumption that $\|w_{C_k} - w\| \geq \delta, \forall k$. Therefore, $\lim_{C\to\infty} w_C = w$

4 Conclusions

Through the use of optimization techniques, we have a better understanding about formulations of SVMs. This work can be a good example of analyzing future SVM formulations by optimization techniques.

SVM formulations are very simple convex quadratic problems in Hilbert spaces. Thus we can use the simple Wolfe dual (see equation 2.1) to derive

the primal-dual relation instead of using the standard conjugate duality. For different SVM formulations, the results of theorem 2 could be quite different. For example, for the new ν-SVM proposed by Schölkopf, Smola, Williamson, and Bartlett (2000), the primal problem can be unbounded while the dual problem can be infeasible (see Crisp & Burges, in press, and Chang & Lin, 2000).

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References

Barbu, V., & Precupanu, T. (1986). *Convexity and optimization in banach spaces*. Dordrecht: D. Reidel.

Bennett, K. P., & Mangasarian, O. L. (1992). Robust linear programming discrimination of two linear inseparable sets. *Optimization Methods and Software 1*, 23–34.

Boser, B., Guyon, I., & Vapnik, V. (1992). A training algorithm for optimal margin classifiers. In *Proceedings of the Fifth Annual Workshop on Computational Learning Theory*.

Chang, C.-C., & Lin, C.-J. (2000). *Some analysis on ν-support vector classification* (Tech. Rep.) Taipei, Taiwan: Department of Computer Science and Information Engineering, National Taiwan University.

Cortes, C., & Vapnik, V. (1995). Support-vector network. *Machine Learning, 20*, 273–297.

Crisp, D. J., & Burges, C. J. C. (In press). A geometric interpretation of ν-svn classifiers. In *Advances in neural information processing systems*. Cambridge, MA: MIT Press.

Debnath, L., & Mikusinski, P. (1999). *Introduction to Hilbert spaces with applications* (2nd ed.). San Diego: Academic Press.

Di Pillo, G. (1994). Exact penalty methods. In I. Ciocco (Ed.), *Algorithms for continuous optimization*. Dordrecht: Kluwer.

Friess, T.-T., Cristianini, N., & Campbell, C. (1998). The kernel Adatron algorithm: A fast and simple learning procedure for support vector machines. In *Proceedings of the 15th Intl. Conf. Machine Learning*. San Mateo, CA: Morgan Kaufmann.

Girsanov, I. V. (1972). *Lectures on mathematical theory of extremum problems*. Berlin: Springer-Verlag.

Rockafellar, R. T. (1970). *Convex analysis*. Princeton, NJ: Princeton University Press.

Schölkopf, B., Burges, C. J. C., & Smola, A. J. (Eds.) (1998). *Advances in kernel methods—Support vector learning*. Cambridge, MA: MIT Press.

Schölkopf, B., Smola, A., Williamson, R. C., & Bartlett, P. L. (2000). New support vector algorithms. *Neural Computation, 12*, 1083–1121.

Vapnik, V. (1995). *The nature of statistical learning theory*. New York: Springer-Verlag.

Vapnik, V. (1998). *Statistical learning theory*. New York: Wiley.

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