行政院國家科學委員會補助專題研究計畫成果報告

※※※※※※※※※※※※※※※※※※※※※※※※※
※　　　　　　　　　　　　　　　　　　　　　　　※
※　　　　複合彈性楔形體之應力奇異性　　　　　　※
※　　　　及相關路徑無關積分之研究　　　　　　　※
※　　　　　　　　　　　　　　　　　　　　　　　※
※※※※※※※※※※※※※※※※※※※※※※※※※

計畫類別：■個別型計畫　　□整合型計畫
計畫編號：NSC　89－2212－E－002－096－
執行期間：2000 年 8 月 1 日至　2001 年 7 月 31 日

計畫主持人：吳光鐘
共同主持人：

本成果報告包括以下應繳交之附件：
□赴國外出差或研習心得報告一份
□赴大陸地區出差或研習心得報告一份
□出席國際學術會議心得報告及發表之論文各一份
□國際合作研究計畫國外研究報告書一份

執行單位：國立台灣大學應用力學研究所

中　華　民　國　　　90 年 10 月 23 日

行政院國家科學委員會專題研究計畫成果報告

複合彈性楔形體之應力奇異性及相關路徑無關積分之研究

On the Stress Singularities and Associated Path-Independent Integrals for Composite Elastic Wedge

計畫編號：NSC 89-2212-E-002-096

執行期限： 200 年 8 月 1 日至 2001 年 7 月 31 日

主持人：吳光鐘　　台灣大學應用力學研究所

一、中文摘要

本計畫分析複合彈性楔形體之尖端奇異應力場。該複合楔形體是由 N 個不同均質異向彈性楔形體完全接合而成。楔形體表面考慮一系列的邊界條件。本計畫建立決定應力奇異性的條件。本計畫進一步導得連結尖端奇異應力場與遠域場的與路徑無關的積分。

關鍵詞：奇異應力場、複合楔形體、路徑無關積分。

Abstract

The asymptotic fields in an elastic anisotropic composite wedge are considered for a wide range of boundary conditions. It is shown that the eigenfunctions for the near-field and far-field are dual as they are generated by the same set of eigenvalues in general. If the boundary conditions on the wedge faces are the same, an additional eigenfunction may appear in the far-field. Moreover the dual eigenfunctions are used to derive path-independent integrals that relate the near-field to the far-field.

Keywords: Singular Stress Field, Composite Wedge, Path-independent Integral Integral Equation.

2、INTRODUCTION

The singular stress fields near the tip of an elastic wedge play an important role in the failure analysis of a notched body. The problem of finding stress singularities was first considered by Williams [1] for homogeneous isotropic media. A systematic approach to finding the singularities in a composite wedge was presented by Dempsey and Sinclair [2], who also provided extensive references on the subject. Similar problems for anisotropic materials was started by Sih et. al. [3] and was followed by Bogy [4] and Bogy and Kuo [5], among others. Following the analysis of Dempsey and Sinclair [2] and using Stroh's formalism [6], Ting and Chou [7] presented possible forms of stress distribution near the vertex of a wedge or a composite wedge of general anisotropic materials. In the approach of Ting and Chou for an N-material wedge, the determinant of a $6N \times 6N$ matrix must be considered. For traction-free homogeneous anisotropic wedges, Wu and Chang [8] used Stroh's formalism to derive a simple characteristic equation, given as "zero determinant condition" of a 3×3 matrix. The lower dimension of the linear system facilitates not only analytic manipulations but also numerical computations. Wu and Chang [8] also gave a simple condition for the existence of the logarithmic stress singularities. The approach of Wu and Chang was generalized by Mantic, et. al. [9] to more general boundary conditions. Belov and Kirchner [10] employed a six-dimensional formalism developed by Kirchner [11] to analyze power-law stress singularities for an traction-free angularly inhomogeneous wedge. The stress singularities were also determined by the determinant of a 3×3 matrix.

The singularity analysis only determines the local fields up to some proportionality coefficients, which are referred to as the stress intensity factors. For homogeneous media with notches, Wu and Chang [8] developed path-independent integrals for stress intensity factors associated with power-law and logarithmic singularities using Betti's reciprocity theorem. The approach is similar to the weight function of Bueckner [12]. The weight function theory was originally developed for cracked bodies and was subsequently extended to a piecewise homogeneous notched structure under anti-plane strain deformation [13]. Belov and Kirchner [10] further generalized the weight function to angularly inhomogeneous solids with notches and cracks for power-law singularities.

Another type of problems related to an elastic wedge is that of a force or a couple acting at the tip. The force problem was considered by Barnett and Lothe [14] for an anisotropic wedge and by Ting [15] for a composite wedge. The couple problem was studied by Carothers [16], Sternberg and Koiter [17] for isotropic materials and by Ting [15] and Hwu and Ting [18] for anisotropic materials. The solution of the couple problem exhibits anomalous behavior at some critical angles. The issue has been addressed by Sternberg and Koiter [17], Hwu and Ting [18], and Belov and Kirchner [19]. Sternberg and Koiter [17] showed that the solution of a couple was essentially the far-field of an infinite wedge with self-equilibrium tractions applied on finite segments of the lateral faces.

The solutions to the aforementioned two types of problems are both related to the eigenfunctions that satisfy the prescribed boundary conditions on the wedge faces. In the near-field problem, the eigenfunctions with bounded displacements at the tip are sought; whereas in the far-field problem those with stresses vanishing as r^{-s}, r being the radial distance and $s\geq1$ at infinity are required. The objective of the present paper is to show that these two types of eigenfunctions are dual in the sense that the existence of one implies the other. If the boundary conditions on the wedge faces are the same, an additional eigenfunction may appear in the far-field. A full range of boundary conditions are considered. The determinant conditions for eigenfunctions with power-law dependence on r are established and further conditions are derived for eigenfunctions that contain $\log r$. The dual sets of eigenfunctions are used to obtain path-independent integrals that relate the near-field to the far-field

3、DUAL SETS OF EIGENFUNCTIONS

In an angularly inhomogeneous medium in which the elastic constants vary with the polar angle θ, the stresses σ_{ij} are related to the displacements u_k by

$$\sigma_{ij} = C_{ijks}(\theta)\frac{\partial u_k}{\partial x_s} \tag{1}$$

where $C_{ijks}(\theta)$ are the elastic constants and repeated indices imply summation from 1 to 3. In the absence of body forces, the stresses can be expressed as

$$\sigma_{i1} = -\frac{\partial \phi_i}{\partial x_2}, \quad \sigma_{i2} = \frac{\partial \phi_i}{\partial x_1}$$

(1) can be expressed in terms **u** and ϕ as [11]

$$\frac{\partial \mathbf{w}}{r\partial \theta} = \mathbf{N}(\theta)\frac{\partial \mathbf{w}}{\partial r} \tag{2}$$

where $\mathbf{w} = \left[\mathbf{u}^{\mathrm{T}}, \phi^{\mathrm{T}}\right]$ is a six-dimensional vector and $\mathbf{N}(\theta)$ is a 6×6 matrix consisting of three 3×3 sub-matrices $\mathbf{N}_1(\theta)$, $\mathbf{N}_2(\theta)$ and $\mathbf{N}_3(\theta)$ as

$$\mathbf{N}(\theta) = \begin{bmatrix} \mathbf{N}_1(\theta) & \mathbf{N}_2(\theta) \\ \mathbf{N}_3(\theta) & \mathbf{N}_1^{\mathrm{T}}(\theta) \end{bmatrix}$$

Consider the wedge that occupies the region $\alpha \le \theta \le \beta$ as depicted in Fig.1. Let the solution of **w** be expressed as

$$\mathbf{w}(r,\theta) = r^{\lambda}\tilde{\mathbf{w}}(\theta) \tag{3}$$

where $\lambda = \mu + i\gamma$ with μ and γ being real constants. Substitution of Eq. (3) into Eq. (2) yields

$$\frac{d\tilde{\mathbf{w}}(\theta)}{d\theta} = \lambda \mathbf{N}(\theta)\tilde{\mathbf{w}}(\theta) \tag{4}$$

Eq. (4) is a system of first order linear differential equations. The general solution of can be expressed as [10]

$$\tilde{\mathbf{w}}(\theta) = \mathbf{W}(\theta,\alpha;\lambda)\tilde{\mathbf{w}}(\alpha) \tag{5}$$

where $\mathbf{W}(\theta,\alpha;\lambda)$ satisfies

$$\frac{d\mathbf{W}(\theta,\alpha;\lambda)}{d\theta} = \lambda \mathbf{N}(\theta)\mathbf{W}(\theta,\alpha;\lambda) \tag{6}$$

From Eq. (5), at $\theta = \beta$

$$\tilde{\mathbf{w}}(\beta) = \mathbf{W}(\beta,\alpha;\lambda)\tilde{\mathbf{w}}(\alpha) \tag{7}$$

The expression for $\mathbf{W}(\beta,\alpha;\lambda)$ for the composite wedge is

$$\mathbf{W}(\beta,\alpha;\lambda) = \mathbf{W}^{(N)}(\theta,\theta_{N-1};\lambda)\cdots\mathbf{W}^{(j)}(\theta_1,\alpha;\lambda) \tag{8}$$

Let the boundary conditions on the wedge face at $\theta = \alpha$ and $\theta = \beta$ be specified as [20]

$$\mathbf{B}_1^{\mathrm{T}}\frac{\partial \mathbf{w}(r,\alpha)}{\partial r} = \mathbf{0} \tag{9}$$

$$\mathbf{B}_2^{\mathrm{T}}\frac{\partial \mathbf{w}(r,\beta)}{\partial r} = \mathbf{0} \tag{10}$$

where $\mathbf{B}_j^{\mathrm{T}} = \left(\mathbf{X}_j^{\mathrm{T}}, \mathbf{Y}_j^{\mathrm{T}}\right)$, $\mathbf{X}_j^{\mathrm{T}}$ and $\mathbf{Y}_j^{\mathrm{T}}$, $j = 1,2$, are 3×3 matrices given by

$$\mathbf{X}_j = \frac{1}{2}\left(\mathbf{\Omega}_{1j} + \mathbf{\Omega}_{2j}\right), \mathbf{Y}_j = \frac{1}{2}\left(\mathbf{\Omega}_{1j} - \mathbf{\Omega}_{2j}\right)$$

and $\mathbf{\Omega}_{ij}$, $i = 1,2$ are arbitrary orthogonal matrices. The matrices $\mathbf{B}_i$, $i = 1,2$ have the following properties

$$\mathbf{B}_i^{\mathrm{T}}\mathbf{B}_i = \mathbf{B}_i\mathbf{B}_i^{\mathrm{T}} = \mathbf{I} \tag{11}$$

$$\mathbf{B}_i^{\mathrm{T}}\mathbf{J}\mathbf{B}_i = \mathbf{B}_i\mathbf{J}\mathbf{B}_i^{\mathrm{T}} = \mathbf{0} \tag{12}$$

Eqs. (9) and (10) cover a very wide range of boundary conditions. For example, the traction-free conditions are characterized by $\mathbf{X}_i = \mathbf{0}$ and $\mathbf{Y}_i = \mathbf{I}$ while the clamped conditions by $\mathbf{X}_i = \mathbf{I}$ and $\mathbf{Y}_i = \mathbf{0}$. The boundary conditions (9) and (10) are satisfied if

$$\lambda \mathbf{B}_1^{\mathrm{T}}\tilde{\mathbf{w}}(\alpha) = \mathbf{0} \tag{13}$$

$$\lambda \mathbf{B}_2^{\mathrm{T}}\tilde{\mathbf{w}}(\beta) = \mathbf{0} \tag{14}$$

From Eqs. (12) and (13), for $\lambda \ne 0$, $\tilde{\mathbf{w}}(\alpha)$ can be expressed as

$$\tilde{\mathbf{w}}(\alpha) = \mathbf{J}\mathbf{B}_1\mathbf{q} \tag{15}$$

where **q** is a three-dimensional vector. Substituting Eq. (15) into Eq. (7) and effecting Eq. (14) leads to

$$\mathbf{C}(\lambda)\mathbf{q}=\mathbf{0} \tag{16}$$

where the 3×3 matrix is given by

$$\mathbf{C}(\lambda)=\lambda\mathbf{B}_2^{\mathrm{T}}\mathbf{W}(\beta,\alpha;\lambda)\mathbf{J}\mathbf{B}_1 \tag{17}$$

Wu and Chang [8] derived Eq. (16) for traction-free homogeneous anisotropic wedges. Belov and Kirchner [10] have obtained Eq. (16) for traction-free angularly inhomogeneous wedges. To have non-trivial solution of **q**, λ must be a root of the following determinant equation

$$\left|\mathbf{C}(\lambda)\right|=0 \tag{18}$$

Eq. (18) yields an infinite set of roots Λ:

$$\Lambda=\{\lambda_k=\mu_k+i\gamma_k, k=1,2,\cdots\infty\}$$

each of which corresponds to an eigenfunction in the form of Eq. (3). It is assumed that $\mu_k>0$ so that the corresponding displacement is finite at $r=0$. The set of eigenfunctions is physically admissible in a finite region containing the wedge tip. For $\mu_k<1$ the corresponding eigenfunctions give rise to singular stresses at the tip.

Consider an alternative form given as [8]

$$\mathbf{w}^*(r,\theta)=r^{-\lambda}\tilde{\mathbf{w}}^*(\theta) \tag{19}$$

where

$$\tilde{\mathbf{w}}^*(\theta)=\mathbf{W}(\theta,\beta;-\lambda)\tilde{\mathbf{w}}^*(\beta) \tag{20}$$

For an *N*-material composite wedge with perfectly bonded interfaces at $\theta=\theta_i$, $i=1\sim N-1$, the expression for $\tilde{\mathbf{w}}^*(\theta)$ in the region $\theta_{j-1}<\theta<\theta_j$, $j=1\sim N$, $\theta_0=\alpha$, $\theta_N=\beta$ is given by

$$\tilde{\mathbf{w}}^*(\theta)=\mathbf{W}^{(j)}(\theta,\theta_j;-\lambda)\tilde{\mathbf{w}}^*(\theta_j)$$

From Eq. (20), at $\theta=\alpha$,

$$\tilde{\mathbf{w}}^*(\alpha)=\mathbf{W}^{(j)}(\alpha,\beta;-\lambda)\tilde{\mathbf{w}}^*(\beta) \tag{21}$$

Let $\tilde{\mathbf{w}}^*(\alpha)$ and $\tilde{\mathbf{w}}^*(\beta)$ satisfy the same boundary conditions as described by Eq. (13) and Eq. (14) respectively, for $\tilde{\mathbf{w}}(\alpha)$ and $\tilde{\mathbf{w}}(\beta)$. From Eq. (14) for $\lambda\neq0$ can be expressed as

$$\tilde{\mathbf{w}}^*(\beta)=\mathbf{J}\mathbf{B}_2\mathbf{q}^* \tag{22}$$

where **q** is a three-dimensional vector. Substituting Eq. (15) into Eq. (7) and effecting Eq. (14) leads to

$$\mathbf{C}^{\mathrm{T}}(\lambda)\mathbf{q}^*=\mathbf{0} \tag{23}$$

where the identity

$$\mathbf{W}(\alpha,\beta;\lambda)=\mathbf{J}\mathbf{W}(\beta,\alpha;-\lambda)^{\mathrm{T}}\mathbf{J} \tag{24}$$

is used [10]. The existence of a non-trivial solution of $\mathbf{q}^*$ is ensured if $\lambda\in\Lambda$. Thus the set Λ generates not only a set of eigenfunctions of (3) but also a dual set of eigenfunctions of (19). Note that the dual eigenfunctions are unbounded at the tip but vanish as $r\to\infty$. The set of dual eigenfunctions is thus physically admissible in an infinite region not including the tip.

4. PATH-INDEPENDENT INTEGRALS

Consider Betti's reciprocal work theorem expressed in the following form [8]

$$\int_C \mathbf{w}^{*\mathrm{T}}\mathbf{J}\frac{\partial\mathbf{w}}{\partial s}ds=0,$$

where $\mathbf{w}^{*\mathrm{T}}$ and **w** are two admissible elasticity solutions and *C* is a closed contour parametrized by the arc length *s*. The region bounded by *C* contains no body forces or dislocations. Let $C=\Gamma\bigcup\Gamma^*\bigcup\Gamma_\alpha\bigcup\Gamma_\beta$, where Γ and Γ^* are two non-intersecting contours starting from $\theta=\alpha$ and terminating at $\theta=\beta$, Γ_α and Γ_β, respectively, are the parts of *C* coinciding with $\theta=\alpha$ and $\theta=\beta$. If

$$\mathbf{w}^{*\mathrm{T}}\mathbf{J}\frac{\partial\mathbf{w}}{\partial r}=0, \tag{25}$$

on the wedge faces, then

$$Q\left(\mathbf{w}^*, \mathbf{w}; \Gamma\right) = Q\left(\mathbf{w}^*, \mathbf{w}; \Gamma^*\right) \qquad (26)$$

where the two-state integral $Q\left(\mathbf{w}^*, \mathbf{w}; \Gamma\right)$ is defined as

$$Q\left(\mathbf{w}^*, \mathbf{w}; \Gamma\right) = \int_\Gamma \mathbf{w}^{*\mathrm{T}} \mathbf{J} \frac{\partial \mathbf{w}}{\partial s} ds. \qquad (27)$$

Thus $Q\left(\mathbf{w}^*, \mathbf{w}; \Gamma\right)$ is independent of the choice of Γ The integral $Q\left(\mathbf{w}, \mathbf{w}^*; \Gamma\right)$ is also path-independent and is related to $Q\left(\mathbf{w}^*, \mathbf{w}; \Gamma\right)$ by

$$Q\left(\mathbf{w}, \mathbf{w}^*; \Gamma\right) = -Q\left(\mathbf{w}^*, \mathbf{w}; \Gamma\right), \qquad (28)$$

provided that

$$\mathbf{w}^{*\mathrm{T}} \mathbf{J} \mathbf{w} = 0 \qquad (29)$$

on the wedge faces. Conditions (25) and (29) hold if $\mathbf{w}^*$ and $\mathbf{w}$ are any two eigenfunctions satisfying boundary conditions (13) and (14).

Let $\mathbf{w}_t^{(j)}$ and $\mathbf{w}_s^{*(i)}$ be the eigenfunctions, respectively, generated by λ_t and $\lambda_s \in \Lambda$. With Γ chosen as a circular arc of r = R, Eq. (27) yields

$$Q\left(\mathbf{w}_s^{*(i)}, \mathbf{w}_t^{(j)}; \Gamma\right) = R^{(\lambda_t - \lambda_s)} \tilde{Q}^{(ij)}, \qquad (30)$$

where

$$\tilde{Q}^{(ij)} = \int_\alpha^\beta \tilde{\mathbf{w}}_s^{*(i)}(\theta)^{\mathrm{T}} \frac{d}{d\theta}\left(\tilde{\mathbf{w}}_t^{(j)}(\theta)\right) d\theta \qquad (31)$$

if $\lambda_s \neq \lambda_t$, since (30) is independent of R, we must have

$$\tilde{Q}^{(ij)}\left(\tilde{\mathbf{w}}_s^*, \tilde{\mathbf{w}}_t\right) = 0, \qquad (32)$$

When $\lambda_s = \lambda_t$ the integrals $\tilde{Q}_{k\ell}^{(ij)}\left(\mathbf{w}_s^*, \mathbf{w}_t\right)$ can be integrated explicitly as

$$\tilde{Q}^{(ij)}\left(\tilde{\mathbf{w}}_s^*, \tilde{\mathbf{w}}_t\right) = \Delta_{ij}{}'\left(\lambda_s\right) \qquad (33)$$

where $\Delta_{ij}(\lambda) = \mathbf{q}_i^{*\mathrm{T}} \mathbf{C}(\lambda) \mathbf{q}_j$. (33) is a path-independent integral that can be used to determine the near-tip singular stress field.

5. REFRENCES

1. Williams, M. L., "Stress singularities resulting from various conditions in an angular corners of plates in extension," ASME *Journal of Applied Mechanics*, Vol. 19, pp. 526-528 (1952).
2. Dempsey, J. P. and Sinclair, G. B., "On the stress singularities in the plane elasticity of the composite wedge," *Journal of Elasticity*, Vol. 9, pp. 373-391 (1979).
3. Sih, G. C., Paris, P. C. and Irwin, G. R., "On cracks in rectilinearly anisotropic bodies," *International Journal of Fracture Mechanics*, Vol. 1, pp. 189-302 (1965).
4. Bogy, D. B., "The plane solution for anisotropic elastic wedges under normal and shear loading," ASME *Journal of Applied Mechanics*, Vol. 39, pp. 1103-1109 (1972).
5. Kuo, M. C. and Bogy, D. B., "Plane solutions for the displacement and traction-displacement problems for anisotropic elastic wedges," ASME *Journal of Applied Mechanics*, Vol. 41, pp. 197-203 (1974).
6. Stroh, A. N., "Steady State Problems in Anisotropic Elasticity," Journal of *Mathematical Physics*, Vol. \41, pp. 77-103 (1962).
7. Ting, T. C. T. and Chou, S., "Edge singularities in anisotropic composites," *International Journal of Solids and Structures*, Vol. 17, pp. 1057-1068 (1981).
8. Wu, K.-C. and Chang, F.-T., "Near-tip fields in a notched body with dislocations and body forces," ASME *Journal of*

Applied Mechanics, Vol. 60, pp. 936-941 (1993).

9. Mantic, V. Paris, F. and Canas, J., Stress singularities in 2D orthotropic corners," *International Journal of Fracture*, Vol. 83, pp. 67-90 (1997).
10. Belov, A. Y. and Kirchner, H. O. K., "Universal weight functions for elastically angularly inhomogeneous media with notches and cracks," *Philosophical Magazine*, Vol. A73, pp. 1621-1646 (1996).
11. Kirchner, H. O. K., "Elastically anisotropic angularly inhomogeneous media I. a new formalism," *Philosophical Magazine*, Vol. B60, pp. 423-432 (1989).
12. Bueckner, H. F., "A novel principle for the computation of stress intensity factors," *Zeitschrift Fur Angewandte Mathematik und Mechanik*, Vol. 50, pp. 529-546 (1970).
13. Sham, T. L. and Bueckner, H. F., "The weight function theory for piecewise homogeneous isotropic notches in antiplane strain," ASME *Journal of Applied Mechanics*, Vol. 55, pp. 596-603 (1988).
14. Barnett, D. M. and Lothe, J., "Line force loadings on anisotropic half-spaces and wedges," *Physica Norvegica*, Vol. 8, p,p. 13-22 (1975).
15. Ting, T. C. T., "Line forces and dislocations in anisotropic elastic composite wedges and spaces," *Physica Status Solidi* B, Vol. 146, pp. 81-90 (1988).
16. Carothers, S. D., "Plane strain in a wedge," *Proceedings of the Royal Society of Edingburgh*, Vol. 23, pp. 292-306 (1912).
17. Sternberg, E. and Koiter, W., "The Wedge under A Concentrated Couple: A Paradox in The Two-Dimensional Theory of Elasticity," ASME *Journal of Applied Mechanics*, Vol. 25, pp. 575-581 (1958).
18. Hwu, C. and Ting, T. C. T., "Solutions for the anisotropic wedges at critical angles," *Journal of Elasticity*, Vol. 24, pp. 1-20 (1990).
19. Belov, A. Y. and Kirchner, H. O. K., "Critical angles in bending of rotationally inhomogeneous elastic wedges," ASME *Journal of Applied Mechanics*, Vol.62, pp. 429-440 (1995).
20. Ting, T. C. T., *Anisotropic Elasticity-Theory and Application*, Oxford University Press, New York (1996).

Kirchner, H. O. K., "Weight functions for notches: constructive and variational definition," ASME *Journal of Applied Mechanics*, Vol. 64, pp. 270-274 (1997).