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# Noncanonical Poisson brackets for elastic and micromorphic solids

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## Abstract

This paper investigates the Lagrangian-to-Eulerian transformation approach to the construction of noncanonical Poisson brackets for the conservative part of elastic solids and micromorphic elastic solids. The Dirac delta function links Lagrangian canonical variables and Eulerian state variables, producing noncanonical Poisson brackets from the corresponding canonical brackets. Specifying the Hamiltonian functionals generates the evolution equations for these state variables from the Poisson brackets. Different elastic strain tensors, such as the Green deformation tensor, the Cauchy deformation tensor, and the higher-order deformation tensor, are appropriate state variables in Poisson bracket formalism since they are quantities composed of the deformation gradient. This paper also considers deformable directors to comprise the three elastic strain density measures for micromorphic solids. Furthermore, the technique of variable transformation is also discussed when a state variable is not conserved along with the motion of the body.


*Keywords:* Noncanonical Poisson bracket; Continuum mechanics; Microcontinuum; Lagrangian-to-Eulerian transformation

## 1. Introduction

The Poisson bracket formulation of Hamilton's mechanics was originally developed for discrete particle systems. The application of Poisson bracket formalism to continuous systems began with Arnold (1966), Arnold (1978), Morrison (1980), and Marsden and Weinstein (1982). Later, Kaufman (1984), Morrison (1984), and Grmela (1984) almost simultaneously made a more general extension to nonconservative continuum systems by introducing a dissipative bracket into the time evolution equation for a system functional $F$ as

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, E\} + [F, S], \tag{1}$$

where $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ represent the Poisson bracket and the dissipative bracket, respectively. $E$ and $S$ in (1) are the total energy and the total entropy of a system. In this framework, the Poisson bracket characterizes the

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conservative dynamics of the system. On the other hand, the dissipative bracket represents the dissipative dynamics of the system (Beris and Edwards, 1994; Edwards, 1998; Öttinger, 1999; Beris, 2001).

Through Poisson bracket formalism, the equations of motion for a system can be directly derived from the general equation (1). The only difficulty of this derivation is constructing the two brackets. Even for a conservative continuous system, in which the dissipative bracket can be discarded, the explicit form of its Poisson bracket is noncanonical since a continuous system is usually described by field variables (Eulerian variables) rather than canonical variables (Lagrangian variables) such as the positions and momenta of particles. The difference between the adoptions of the two types of variables lies in the different descriptions for a system, i.e., the Lagrangian description and the Eulerian description. In several studies such as the immersed boundary method (Peskin, 2002), Lagrangian hydrodynamics (Grmela, 2002, 2003) and mesoscopic dynamics (Grmela, 2004), both Eulerian and Lagrangian variables are simultaneously involved. The two descriptions can be linked by the Dirac delta function so that the noncanonical Eulerian Poisson bracket can be constructed by its corresponding canonical Lagrangian form. Using the Dirac delta function, Abarbanel et al. (1988) propose the Lagrangian-to-Eulerian (LE) transformation of state variables to derive the noncanonical Poisson bracket for inviscid flows. Using the same transformation, Edwards and Beris (1991) develop the Poisson bracket for nonlinear elasticity. Another approach to deriving the noncanonical Poisson bracket is to identify the underlying Lie algebraic structure of the state space expressed by the state variables of a system (Marsden and Ratiu, 1994).

Despite the mathematical complexities of LE transformation, which might draw researcher's attention away from development of this method, this approach clearly shows that the noncanonical Poisson bracket can be directly constructed from its canonical counterpart in the Lagrangian description. Successful examples of LE transformation are elastic fluids (Edwards and Beris, 1991), and anisotropic fluids (Edwards and Beris, 1998). This paper extends LE transformation to the study of noncanonical Poisson brackets for elastic solids and micromorphic elastic solids. After a preliminary review of the Poisson bracket for a discrete system in Section 2, Section 3 studies the LE transformation of a set of suitable state variables in the case of an elastic solid. We use this same method in Section 4 to obtain noncanonical Poisson bracket for a micromorphic solid. No previous study has used this method for this task. Section 5 presents Poisson brackets for an elastic system with three different types of state variables: the Cauchy deformation tensor, the gradient of deformation tensor, and the nonconservative state variable. Finally, the paper closes with a summary and concluding remarks.

## 2. Brief review of the Poisson bracket for a discrete system

The equations of motion for an N-particle discrete system are expressed by the Hamilton's canonical equations:

$$\dot{\bar{\mathbf{x}}}^\alpha = \frac{\mathrm{d}\bar{\mathbf{x}}^\alpha}{\mathrm{d}t} = \frac{\partial H}{\partial \bar{\mathbf{p}}^\alpha}, \qquad \dot{\bar{\mathbf{p}}}^\alpha = \frac{\mathrm{d}\bar{\mathbf{p}}^\alpha}{\mathrm{d}t} = -\frac{\partial H}{\partial \bar{\mathbf{x}}^\alpha}, \tag{2}$$

where $H(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \ldots, \bar{\mathbf{x}}^N, \bar{\mathbf{p}}^1, \bar{\mathbf{p}}^2, \ldots, \bar{\mathbf{p}}^N)$ is the Hamiltonian of the system, and $\bar{\mathbf{x}}^\alpha(t)$ and $\bar{\mathbf{p}}^\alpha(t)$ are the coordinate and the momentum of particle $\alpha$. The Hamilton's equations can be derived under the framework of Poisson bracket formalism:

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_{\alpha=1}^{N}\left(\frac{\partial F}{\partial \bar{\mathbf{x}}^\alpha}\cdot\frac{\mathrm{d}\bar{\mathbf{x}}^\alpha}{\mathrm{d}t} + \frac{\partial F}{\partial \bar{\mathbf{p}}^\alpha}\cdot\frac{\mathrm{d}\bar{\mathbf{p}}^\alpha}{\mathrm{d}t}\right) = \{F, H\}_L, \tag{3}$$

with the introduction of the Poisson bracket of the system as

$$\{F, G\}_L = \sum_{\alpha=1}^{N}\left(\frac{\partial F}{\partial \bar{\mathbf{x}}^\alpha}\cdot\frac{\partial G}{\partial \bar{\mathbf{p}}^\alpha} - \frac{\partial G}{\partial \bar{\mathbf{x}}^\alpha}\cdot\frac{\partial F}{\partial \bar{\mathbf{p}}^\alpha}\right). \tag{4}$$

Here $F$ and $G$ are arbitrary functions with arguments $(\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \ldots, \bar{\mathbf{x}}^N, \bar{\mathbf{p}}^1, \bar{\mathbf{p}}^2, \ldots, \bar{\mathbf{p}}^N)$. The subscript "$L$" in the Poisson bracket indicates the Lagrangian description, by which the motion of a definite particle can be depicted. Note that the Poisson bracket is bilinear and antisymmetric in $F$ and $G$.

The Lagrangian-to-Eulerian transformation enables extension of Poisson bracket formalism in a discrete system, described by the Lagrangian description, to that in a continuous system, usually characterized by the Eulerian description. Consider a material point in a continuous system. Let vector $\mathbf{X}$ be its position vector at time $t = 0$ and function $\bar{\mathbf{x}}(\mathbf{X}, t)$ be its position function at time $t$. The function $\bar{\mathbf{x}}(\mathbf{X}, t)$ with the initial condition $\bar{\mathbf{x}}(\mathbf{X}, 0) = \mathbf{X}$ specifies the motion of the continuum. The continuum occupies the region $\Omega$ with the boundary $\partial\Omega$ at time $t = 0$, and due to the motion of the continuum, this region changes to $\Omega'$ with the boundary $\partial\Omega'$ at time $t$. The dynamical variables of a continuum in the Lagrangian description are the position $\bar{\mathbf{x}}(\mathbf{X}, t)$ and the momentum per unit volume $\bar{\mathbf{u}}(\mathbf{X}, t) = \rho_0(\mathbf{X})(\partial\bar{\mathbf{x}}(\mathbf{X}, t)/\partial t)$, where $\rho_0(\mathbf{X})$ is the mass density at time $t = 0$.

For an analogous manner in a discrete system, the Poisson bracket for a continuous system without dissipation can be expressed as

$$\{F, G\}_L = \int_\Omega \left(\frac{\delta F}{\delta\bar{\mathbf{x}}} \cdot \frac{\delta G}{\delta\bar{\mathbf{u}}} - \frac{\delta F}{\delta\bar{\mathbf{u}}} \cdot \frac{\delta G}{\delta\bar{\mathbf{x}}}\right) \mathrm{d}^3 X, \tag{5}$$

for arbitrary functionals $F = F[\bar{\mathbf{x}}, \bar{\mathbf{u}}]$ and $G = G[\bar{\mathbf{x}}, \bar{\mathbf{u}}]$. In Eq. (5), the notation of the Volterra functional derivative

$$\frac{\delta F}{\delta a} := \frac{\partial f}{\partial a} - \nabla \cdot \frac{\partial f}{\partial(\nabla a)}, \tag{6}$$

has been used (Beris, 2001), and the derivatives are defined through the following variation on a functional $F\left(= \int_\Omega f(a, \nabla a)\ \mathrm{d}^3 X\right)$:

$$\delta F = \int_\Omega \left(\left(\frac{\partial f}{\partial a} - \nabla \cdot \frac{\partial f}{\partial(\nabla a)}\right)\delta a\right) \mathrm{d}^3 X = \int_\Omega \left(\frac{\delta F}{\delta a}\delta a\right) \mathrm{d}^3 X. \tag{7}$$

in which $f$ is a scalar density function. Here and henceforth, the boundary terms in the variational operation are not considered because it is assumed that the system is free from the boundary. A boundary usually causes disturbances in a system, forcing it into thermodynamic nonequilibrium states. This issue is beyond the scope of the present study and is an important topic for future research.

## 3. Poisson bracket for an elastic solid

Lagrangian description traces a moving particle, but Eulerian description focuses on a spatial point $\mathbf{x}$, through which different particles flow at different times. The two descriptions can be linked to each other by introducing the relation of motion $\mathbf{x} = \bar{\mathbf{x}}(\mathbf{X}, t)$ and the 3-dimensional delta function $\delta^3$. Hence, the transformation relations for the mass density $\rho(\mathbf{x}, t)$, the momentum density $\mathbf{u}(\mathbf{x}, t)$, and the entropy density $s(\mathbf{x}, t)$ of a continuous system are

$$\rho(\mathbf{x}, t) = \int_{\Omega'} \rho(\bar{\mathbf{x}}(\mathbf{X}, t), t)\delta^3[\bar{\mathbf{x}}(\mathbf{X}, t) - \mathbf{x}]\mathrm{d}^3\bar{x} = \int_\Omega \rho_0(\mathbf{X})\delta^3[\bar{\mathbf{x}}(\mathbf{X}, t) - \mathbf{x}]\mathrm{d}^3 X, \tag{8}$$

$$\mathbf{u}(\mathbf{x}, t) = \rho(\mathbf{x}(\mathbf{X}, t), t)\dot{\bar{\mathbf{x}}}(\mathbf{X}, t) = \int_\Omega \bar{\mathbf{u}}(\mathbf{X}, t)\delta^3[\bar{\mathbf{x}}(\mathbf{X}, t) - \mathbf{x}]\mathrm{d}^3 X, \tag{9}$$

$$s(\mathbf{x}, t) = \int_\Omega s_0(\mathbf{X})\delta^3[\bar{\mathbf{x}}(\mathbf{X}, t) - \mathbf{x}]\mathrm{d}^3 X, \tag{10}$$

using the relation of conservation of mass $\mathrm{d}^3\bar{x} = J\mathrm{d}^3 X$ or $\rho(\bar{\mathbf{x}}, t) = J^{-1}\rho_0(\mathbf{X})$, with the Jacobian of the motion $J(= \det(\partial\bar{x}_i/\partial X_J))$. Note that the isentropic process is assumed here so that the relation $s(\bar{\mathbf{x}}, \mathbf{t})\,\mathrm{d}^3\bar{x} = s_0(\mathbf{X})\,\mathrm{d}^3 X$ is held.

In order to delineate the behavior of an elastic solid, an extra state variable to characterize the elastic deformation of the body should be included. A second order symmetric tensor $\hat{\mathbf{C}}(\mathbf{x}, t)$, which is the density of the Green deformation tensor $\mathbf{C}(\mathbf{x}, t)$, is adopted and its index form is expressed as $\hat{C}_{KL} = \rho C_{KL} = \rho\bar{x}_{i,K}\bar{x}_{i,L}$. Similarly, the transformation relation for $\hat{\mathbf{C}}$ is

$$\hat{\mathbf{C}}(\mathbf{x},t) = \int_{\Omega} \rho_0(\mathbf{X})\mathbf{C}(\bar{\mathbf{x}}(\mathbf{X},t),t)\delta^3[\bar{\mathbf{x}}(\mathbf{X},t) - \mathbf{x}]\mathrm{d}^3X. \tag{11}$$

Now all the state variables for an elastic solid, i.e., $(\rho, \mathbf{u}, s, \hat{\mathbf{C}})$, have been determined and note that all of them are quantities per unit volume. The transformation relations for these variables (8)–(11) are essential to the following derivations. Their fundamental significance lies in the correlation of these field variables and the Lagrangian variables.

In order to produce the noncanonical Poisson bracket in the Eulerian description, use the chain rule of differentiation and arrive at

$$\frac{\delta F}{\delta \bar{z}_n} = \int_{\Omega'} \left( \frac{\delta F}{\delta \rho(\mathbf{x},t)} \frac{\delta \rho(\mathbf{x},t)}{\delta \bar{z}_n(\mathbf{X},t)} + \frac{\delta F}{\delta u_j(\mathbf{x},t)} \frac{\delta u_j(\mathbf{x},t)}{\delta \bar{z}_n(\mathbf{X},t)} + \frac{\delta F}{\delta s} \frac{\delta s}{\delta \bar{z}_n} + \frac{\delta F}{\delta \hat{C}_{KL}} \frac{\delta \hat{C}_{KL}}{\delta \bar{z}_n} \right) \mathrm{d}^3x, \tag{12}$$

for the functional $F = F[\rho, \mathbf{u}, s, \hat{\mathbf{C}}] = \int_{\Omega'} f(\rho, \mathbf{u}, s, \hat{\mathbf{C}})\mathrm{d}^3x$, where $\bar{z}_n$ stands for $\bar{x}_n$ or $\bar{u}_n$. Notably, when the functional $F$ is expressed in terms of the integration of its field density $f$ over the space $\Omega'$, the Volterra functional derivative $\delta F/\delta\rho$ is referred to as $\partial f/\partial\rho - \nabla\cdot(\partial f/\partial(\nabla\rho))$. Now insert (12) into (5) and form

$$\begin{aligned}
\{F,G\}_E = & \int_{\Omega'}\int_{\Omega'} \left( \frac{\delta F}{\delta \rho(\mathbf{x},t)} \frac{\delta G}{\delta u_j(\mathbf{z},t)} - \frac{\delta G}{\delta \rho(\mathbf{x},t)} \frac{\delta F}{\delta u_j(\mathbf{z},t)} \right) \{\rho(\mathbf{x},t), u_j(\mathbf{z},t)\}_L \mathrm{d}^3z\mathrm{d}^3x \\
& + \int_{\Omega'}\int_{\Omega'} \left( \frac{\delta F}{\delta s(\mathbf{x},t)} \frac{\delta G}{\delta u_j(\mathbf{z},t)} - \frac{\delta G}{\delta s(\mathbf{x},t)} \frac{\delta F}{\delta u_j(\mathbf{z},t)} \right) \{s(\mathbf{x},t), u_j(\mathbf{z},t)\}_L \mathrm{d}^3z\mathrm{d}^3x \\
& + \int_{\Omega'}\int_{\Omega'} \frac{\delta F}{\delta u_k(\mathbf{x},t)} \frac{\delta G}{\delta u_j(\mathbf{z},t)} \{u_k(\mathbf{x},t), u_j(\mathbf{z},t)\}_L \mathrm{d}^3z\mathrm{d}^3x, \\
& + \int_{\Omega'}\int_{\Omega'} \left( \frac{\delta F}{\delta \hat{C}_{IJ}(\mathbf{x},t)} \frac{\delta G}{\delta u_k(\mathbf{z},t)} - \frac{\delta G}{\delta \hat{C}_{IJ}(\mathbf{x},t)} \frac{\delta F}{\delta u_k(\mathbf{z},t)} \right) \{\hat{C}_{IJ}(\mathbf{x},t),\ u_k(\mathbf{z},t)\}_L \mathrm{d}^3z\mathrm{d}^3x \\
& + \int_{\Omega'}\int_{\Omega'} \frac{\delta F}{\delta \hat{C}_{IJ}(\mathbf{x},t)} \frac{\delta G}{\delta \hat{C}_{KL}(\mathbf{z},t)} \{\hat{C}_{IJ}(\mathbf{x},t), \hat{C}_{KL}(\mathbf{z},t)\}_L \mathrm{d}^3z\mathrm{d}^3x,
\end{aligned} \tag{13}$$

where the subscript "$E$" in the Poisson Bracket indicates the Eulerian description and $\mathbf{z}$ is the spatial coordinate, playing the same role as $\mathbf{x}$. To find the five Poisson brackets $\{\bullet,\bullet\}_L$ in (13), the functional derivatives of $\rho(\mathbf{x},t)$, $\mathbf{u}(\mathbf{x},t)$, $s(\mathbf{x},t)$ and $\hat{\mathbf{C}}(\mathbf{x},t)$ with respect to $\bar{\mathbf{x}}(\mathbf{X},t)$ and $\bar{\mathbf{u}}(\mathbf{X},t)$, which can be derived from the transformation relations (8)–(11), are required:

$$\begin{cases}
\frac{\delta\rho(\mathbf{x},t)}{\delta\bar{x}_n(\mathbf{X},t)} = \rho_0(\mathbf{X})\frac{\partial\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]}{\partial\bar{x}_n(\mathbf{X},t)}, \quad \frac{\delta\rho(\mathbf{x},t)}{\delta\bar{u}_n(\mathbf{X},t)} = 0, \\
\frac{\delta s(\mathbf{x},t)}{\delta\bar{x}_n(\mathbf{X},t)} = s_0(\mathbf{X})\frac{\partial\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]}{\partial\bar{x}_n(\mathbf{X},t)}, \quad \frac{\delta s(\mathbf{x},t)}{\delta\bar{u}_n(\mathbf{X},t)} = 0, \\
\frac{\delta u_j(\mathbf{x},t)}{\delta\bar{x}_n(\mathbf{X},t)} = \bar{u}_j(\mathbf{X},t)\frac{\partial\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]}{\partial\bar{x}_n(\mathbf{X},t)}, \quad \frac{\delta u_j(\mathbf{x},t)}{\delta\bar{u}_n(\mathbf{X},t)} = \delta_{jn}\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}], \\
\frac{\delta\hat{C}_{KL}(\mathbf{x},t)}{\delta\bar{x}_n(\mathbf{X},t)} = \rho_0(\mathbf{X})C_{KL}\frac{\partial\delta^3[\bar{\mathbf{x}}-\mathbf{x}]}{\partial\bar{x}_n(\mathbf{X},t)} - \frac{\partial}{\partial X_M}\left(\rho_0(\mathbf{X})\delta^3[\bar{\mathbf{x}}-\mathbf{x}]\frac{\partial C_{KL}}{\partial\bar{x}_{n,M}}\right), \\
\frac{\delta\hat{C}_{KL}(\mathbf{x},t)}{\delta\bar{u}_n(\mathbf{X},t)} = 0.
\end{cases} \tag{14}$$

Thus

$$\begin{cases}
\{\rho(\mathbf{x},t), u_j(\mathbf{z},t)\}_L = \rho(\mathbf{z},t)\frac{\partial\delta^3[\mathbf{z}-\mathbf{x}]}{\partial z_j}, \\
\{u_k(\mathbf{x},t), u_j(\mathbf{z},t)\}_L = u_k(\mathbf{z},t)\frac{\partial\delta^3[\mathbf{z}-\mathbf{x}]}{\partial z_j} - u_j(\mathbf{x},t)\frac{\partial\delta^3[\mathbf{x}-\mathbf{z}]}{\partial x_k}, \\
\{s(\mathbf{x},t), u_j(\mathbf{z},t)\}_L = s(\mathbf{z},t)\frac{\partial\delta^3[\mathbf{z}-\mathbf{x}]}{\partial z_j}, \qquad \{\hat{C}_{IJ}(\mathbf{x},t), \hat{C}_{KL}(\mathbf{z},t)\}_L = 0, \\
\{\hat{C}_{KL}(\mathbf{x},t), u_k(\mathbf{z},t)\}_L \\
\quad = \hat{C}_{KL}(\mathbf{z},t)\frac{\partial\delta^3[\mathbf{z}-\mathbf{x}]}{\partial z_k} - \frac{\partial}{\partial z_l}\left(\rho(\mathbf{z},t)\delta^3[\mathbf{z}-\mathbf{x}]\left(\frac{\partial z_k}{\partial X_L}\frac{\partial z_l}{\partial X_K} + \frac{\partial z_k}{\partial X_K}\frac{\partial z_l}{\partial X_L}\right)\right).
\end{cases} \tag{15}$$

Substituting (15) into (13) yields $\{F,G\}_E = \{F,G\}_E^{\rho} + \{F,G\}_E^{s} + \{F,G\}_E^{u} + \{F,G\}_E^{\hat{C}}$, where

$$\begin{aligned}\{F,G\}_E^{\rho} &= \int_{\Omega'}\int_{\Omega'}\left(\frac{\delta F}{\delta\rho(\mathbf{x},t)}\frac{\delta G}{\delta u_j(\mathbf{z},t)} - \frac{\delta G}{\delta\rho(\mathbf{x},t)}\frac{\delta F}{\delta u_j(\mathbf{z},t)}\right)\rho(\mathbf{z},t)\frac{\partial\delta^3[\mathbf{z}-\mathbf{x}]}{\partial z_j}\mathrm{d}^3z\mathrm{d}^3x \\ &= \int_{\Omega'}\left(-\frac{\delta F}{\delta\rho}\left(\rho\frac{\delta G}{\delta u_j}\right)_{,j} + \frac{\delta G}{\delta\rho}\left(\rho\frac{\delta F}{\delta u_j}\right)_{,j}\right)\mathrm{d}^3x, \end{aligned} \tag{16}$$

$$\{F,G\}_E^{s} = \int_{\Omega'}\left(-\frac{\delta F}{\delta s}\left(s\frac{\delta G}{\delta u_j}\right)_{,j} + \frac{\delta G}{\delta s}\left(s\frac{\delta F}{\delta u_j}\right)_{,j}\right)\mathrm{d}^3x, \tag{17}$$

$$\{F,G\}_E^{u} = \int_{\Omega'}\left(-\frac{\delta F}{\delta u_k}\left(u_k\frac{\delta G}{\delta u_j}\right)_{,j} + \frac{\delta G}{\delta u_k}\left(u_k\frac{\delta F}{\delta u_j}\right)_{,j}\right)\mathrm{d}^3x, \tag{18}$$

$$\begin{aligned}\{F,G\}_E^{\hat{C}} = \int_{\Omega'}\Bigg(&-\frac{\delta F}{\delta\hat{C}_{KL}}\left(\hat{C}_{KL}\frac{\delta G}{\delta u_k}\right)_{,k} + \rho\frac{\delta F}{\delta\hat{C}_{KL}}\left(\frac{\delta G}{\delta u_k}\right)_{,l}(x_{k,L}x_{l,K}+x_{k,K}x_{l,L}) + \frac{\delta G}{\delta\hat{C}_{KL}}\left(\hat{C}_{KL}\frac{\delta F}{\delta u_k}\right)_{,k} \\ &-\rho\frac{\delta G}{\delta\hat{C}_{KL}}\left(\frac{\delta F}{\delta u_k}\right)_{,l}(x_{k,L}x_{l,K}+x_{k,K}x_{l,L})\Bigg)\mathrm{d}^3x, \end{aligned} \tag{19}$$

where the subscript "$_{,j}$" stands for $\partial/\partial x_j$. Rearranging Eqs.(16)–(19) produces the final expression of the generalized Poisson bracket in Eulerian description

$$\begin{aligned}\{F,G\}_E = \int_{\Omega'}\Bigg(&-\frac{\delta F}{\delta\rho}\left(\frac{\delta G}{\delta u_j}\rho\right)_{,j} + \frac{\delta G}{\delta\rho}\left(\frac{\delta F}{\delta u_j}\rho\right)_{,j} - \frac{\delta F}{\delta u_k}\left(\frac{\delta G}{\delta u_j}u_k\right)_{,j} + \frac{\delta G}{\delta u_k}\left(\frac{\delta F}{\delta u_j}u_k\right)_{,j} \\ &-\frac{\delta F}{\delta s}\left(\frac{\delta G}{\delta u_j}s\right)_{,j} + \frac{\delta G}{\delta s}\left(\frac{\delta F}{\delta u_j}s\right)_{,j} - \frac{\delta F}{\delta\hat{C}_{KL}}\left(\frac{\delta G}{\delta u_j}\ \hat{C}_{KL}\right)_{,j} \\ &+\frac{\delta G}{\delta\hat{C}_{KL}}\left(\frac{\delta F}{\delta u_j}\ \hat{C}_{KL}\right)_{,j} + \rho\frac{\delta F}{\delta\hat{C}_{KL}}\left(\frac{\delta G}{\delta u_k}\right)_{,l}(x_{k,L}x_{l,K}+x_{k,K}x_{l,L}) \\ &-\rho\frac{\delta G}{\delta\hat{C}_{KL}}\left(\frac{\delta F}{\delta u_k}\right)_{,l}(x_{k,L}x_{l,K}+x_{k,K}x_{l,L})\Bigg)\mathrm{d}^3x. \end{aligned} \tag{20}$$

Obviously, this noncanonical Poisson bracket has the properties of bilinearity and antisymmetricity. Choosing the Hamiltonian functional $H[\rho,\mathbf{u},s,\hat{\mathbf{C}}]$ in the form of the composition of the kinetic energy $|\mathbf{u}^2|/2\rho$ and the internal energy $\varepsilon(\rho,s,\hat{\mathbf{C}})$ as

$$H[\rho,\mathbf{u},s,\hat{\mathbf{C}}] = \int_{\Omega'}\left(\frac{|\mathbf{u}(\mathbf{x},t)|^2}{2\rho(\mathbf{x},t)} + \varepsilon\Big(\rho(\mathbf{x},t),s(\mathbf{x},t),\hat{\mathbf{C}}(\mathbf{x},t)\Big)\right)\mathrm{d}^3x, \tag{21}$$

then the independence of $\delta F/\delta\rho$, $\delta F/\delta\mathbf{u}$, $\delta F/\delta s$, and $\delta F/\delta\hat{\mathbf{C}}$ in the evolution equation of the system

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F,H\}_E$$

leads to the evolution equations for the dynamical state variables of an elastic solid:

$$\frac{\partial\rho}{\partial t} = -\left(\frac{\delta H}{\delta u_j}\rho\right)_{,j} = -(v_j\rho)_{,j}, \tag{22}$$

$$\frac{\partial u_i}{\partial t} = -(\rho v_k v_i)_{,k} + \tau_{ki,k}, \tag{23}$$

$$\frac{\partial s}{\partial t} = -(v_j s)_{,j}, \tag{24}$$

$$\frac{\partial\hat{C}_{KL}}{\partial t} = -(v_k\hat{C}_{KL})_{,k} + \rho v_{k,l}(x_{k,L}x_{l,K}+x_{k,K}x_{l,L}), \tag{25}$$

in which Cauchy stress tensor $\tau_{ki}$ are defined as

$$\tau_{ki} := -p\delta_{ki} + \rho \frac{\partial \varepsilon}{\partial \hat{C}_{KL}}(x_{i,L}x_{k,K} + x_{i,K}x_{k,L}), \tag{26}$$

with the pressure

$$p = -\varepsilon + \rho \frac{\partial \varepsilon}{\partial \rho} + s\frac{\partial \varepsilon}{\partial s} + \hat{C}_{KL}\frac{\partial \varepsilon}{\partial \hat{C}_{KL}}. \tag{27}$$

Eqs. (22)–(24) are standard forms for the equations of mass, linear momentum, and entropy, respectively. Eq. (25) is the evolution equation for the state variable $\hat{C}_{KL}$ and it can be checked by taking the material time derivative of the deformation gradient $\partial x_i/\partial X_K$.

Note that standard continuum mechanics usually adopts the internal energy per unit mass, denoted by $\psi$, rather than the internal energy per unit volume $\varepsilon$. The energy $\psi$ for an elastic solid is usually a function of the set $(\eta, \mathbf{C})$, where $\eta = s/\rho$ is the entropy per unit mass. If we set $\varepsilon(\rho, s, \hat{\mathbf{C}}) = \rho\psi(\eta, \mathbf{C})$, then it is easy to find

$$\frac{\partial \varepsilon}{\partial s} = \frac{\partial \psi}{\partial \eta}, \qquad \frac{\partial \varepsilon}{\partial \hat{\mathbf{C}}} = \frac{\partial \psi}{\partial \mathbf{C}}, \qquad \text{and} \qquad \rho\frac{\partial \varepsilon}{\partial \rho} = \varepsilon - s\frac{\partial \varepsilon}{\partial s} - \hat{C}_{KL}\frac{\partial \psi}{\partial \hat{C}_{KL}}. \tag{28}$$

Eq. (28)[3] implies that the pressure $p$ in Eq. (27) is equal to zero and the Cauchy stress will purely come from the deformation tensor $\hat{\mathbf{C}}$.

With the above evolution equations for the state variables, the internal energy equation can be determined by taking the time derivative of the internal energy density, $\varepsilon(\mathbf{x}, t) = \tilde{\varepsilon}(\rho(\mathbf{x}, t), s(\mathbf{x}, t), \hat{\mathbf{C}}(\mathbf{x}, t))$, at a fixed spatial position,

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial \tilde{\varepsilon}}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial \tilde{\varepsilon}}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial \tilde{\varepsilon}}{\partial \hat{C}_{KL}}\frac{\partial \hat{C}_{KL}}{\partial t}. \tag{29}$$

Inserting the evolution Eqs. (22), (24), and (25) into Eq. (29) generates

$$\frac{\partial \varepsilon}{\partial t} = -(\varepsilon v_k)_{,k} + \tau_{kl}v_{l,k}. \tag{30}$$

## 4. Poisson bracket for a micromorphic solid

In the microcontinuum field theory (Eringen, 1999; Chen et al., 2004; Lee et al., 2004), the kinematics of a volume element can be divided into two parts. The first part is the motion of a macroelement, described by the macromotion $\mathbf{X} \to \mathbf{x} = \tilde{\mathbf{x}}(\mathbf{X}, t)$, where $\mathbf{x}$ and $\mathbf{X}$ are the position vectors of the center of mass for the macroelement in the current and reference configurations. A macroelement is comprised of many microelements. The relative position vector of a microelement to the center of mass of the macroelement is represented by the vector $\boldsymbol{\xi}$ in the current configuration, or its counterpart $\boldsymbol{\Xi}$ in the reference configuration. The second part of kinematics is associated with the motion of the microelements, characterized by the micromotion $\boldsymbol{\Xi} \to \boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}(\mathbf{X}, \boldsymbol{\Xi}, t)$. The two motions are mathematically expressed by the deformation gradient $\mathbf{F}$ and the deformable directors $\boldsymbol{\chi}$, which are

$$\mathrm{d}\mathbf{x} = \mathbf{F}\cdot \mathrm{d}\mathbf{X}, \qquad \boldsymbol{\xi} = \boldsymbol{\chi}\cdot\boldsymbol{\Xi}. \tag{31}$$

Note that both the deformation gradient and the deformable directors are two-point tensors, and their inverses are denoted by $\nabla_{\mathbf{x}}\mathbf{X}$ and $\boldsymbol{\mathcal{X}}$, i.e., $F^{-1}_{Kk} = X_{K,k}$ and $\chi^{-1}_{Kk} = \mathcal{X}_{Kk}$. The two motions allow the definition of the following three independent strain tensors as

$$\hat{\Upsilon}_{KL} = \rho x_{k,K}\mathcal{X}_{Lk}, \qquad \hat{\mathcal{C}}_{KL} = \rho\chi_{kK}\chi_{kL}, \qquad \hat{\Gamma}_{KLM} = \rho\mathcal{X}_{Kk}\chi_{kL,M}, \tag{32}$$

where $\hat{\boldsymbol{\Upsilon}}$, $\hat{\boldsymbol{\mathcal{C}}}$, and $\hat{\boldsymbol{\Gamma}}$ are the deformation density tensor, the microdeformation density tensor, and the wryness density tensor, respectively. These variables are also the density counterparts for the deformation tensor $\boldsymbol{\Upsilon}$, the microdeformation tensor $\boldsymbol{\mathcal{C}}$, and the wryness tensor $\boldsymbol{\Gamma}$.

A complete description of the motion of a micromorphic solid in the Hamilton's approach should include these strain measures as the state variables in addition to the mass density $\rho$, the momentum density $\mathbf{u}$, the entropy density $s$, the microinertia density $\hat{\mathbf{i}}$, and the micromomentum density $\hat{\mathbf{m}}$. The latter two densities are given as

$$\hat{i}_{kl} = \rho i_{kl} = \frac{1}{\Delta v}\int_{\Delta B_x} \rho'(\mathbf{x},\boldsymbol{\xi},t)\xi_k\xi_l \mathrm{d}v'_x, \tag{33}$$

$$\hat{m}_{kl} = \rho m_{kl} = \frac{1}{\Delta v}\int_{\Delta B_x} \rho'(\mathbf{x},\boldsymbol{\xi},t)\xi_k\dot{\xi}_l \mathrm{d}v'_x, \tag{34}$$

where $\mathbf{i}$ and $\mathbf{m}$ are the microinertia tensor and the micromomentum tensor. The microgyration tensor $\boldsymbol{v}$ is related to the material time rate of the vector $\boldsymbol{\xi}$ through the relation

$$\dot{\xi}_k = v_{kl}\xi_l. \tag{35}$$

In Eqs. (33) and (34), $\Delta B_x$ is the macroelement at position $\mathbf{x}$ with volume $\Delta v$. The macroelement is composed of many microelements, distinguished by vector $\boldsymbol{\xi}$, with volume elements $\mathrm{d}v'_x$ and mass density $\rho'$.

The set of state variables for a micromorphic solid is now represented by $(\rho, \mathbf{u}, s, \hat{\mathbf{i}}, \hat{\mathbf{m}}, \hat{\boldsymbol{\Upsilon}}, \hat{\boldsymbol{\mathcal{C}}}, \hat{\boldsymbol{\Gamma}})$, and the Lagrangian-to-Eulerian transformation relations for this set can be written as

$$\begin{cases} \rho(\mathbf{x},t) = \int_\Omega \rho_0(\mathbf{X})\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \\ \mathbf{u}(\mathbf{x},t) = \int_\Omega \bar{\mathbf{u}}(\mathbf{X},t)\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \\ s(\mathbf{x},t) = \int_\Omega s_0(\mathbf{X})\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \\ \hat{\mathbf{i}}(\mathbf{x},t) = \int_\Omega(\int_{\Delta B_X} f'_0(\mathbf{X},\boldsymbol{\Xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}\mathrm{d}V'_X)\rho_0(\mathbf{X},t)\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \\ \hat{\mathbf{m}}(\mathbf{x},t) = \int_\Omega(\int_{\Delta B_X} f'_0(\mathbf{X},\boldsymbol{\Xi})\boldsymbol{\xi}\otimes\bar{\mathbf{p}}^{\xi}\mathrm{d}V'_X)\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \end{cases} \tag{36}$$

and

$$\begin{cases} \hat{\boldsymbol{\Upsilon}}(\mathbf{x},t) = \int_\Omega \rho_0(\mathbf{X})\boldsymbol{\Upsilon}(\bar{\mathbf{x}},t)\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \\ \hat{\boldsymbol{\mathcal{C}}}(\mathbf{x},t) = \int_\Omega \rho_0(\mathbf{X})\boldsymbol{\mathcal{C}}(\bar{\mathbf{x}},t)\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \\ \hat{\boldsymbol{\Gamma}}(\mathbf{x},t) = \int_\Omega \rho_0(\mathbf{X})\boldsymbol{\Gamma}(\bar{\mathbf{x}},t)\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\mathrm{d}^3X, \end{cases} \tag{37}$$

where $\otimes$ denotes the tensor product. In the sets of Eqs. (36) and (37), $\Delta B_X$, $\Delta V$, $\mathrm{d}V'_X$, and $\rho'_0$ are the counterparts of $\Delta B_x$, $\mathrm{d}v'_x$, $\Delta v$, and $\rho'$ in the reference configuration, respectively. Moreover, $f'_0$ and $f'$ are defined as $\rho'_0/\rho_0\Delta V$ and $\rho'/\rho\Delta v$, and they are satisfied by the relation of the mass conservation for a microelement:

$$f'\mathrm{d}v'_x = \left(\frac{\rho'}{\rho\Delta v}\right)\mathrm{d}v'_x = \left(\frac{\rho'_0}{\rho_0\Delta V}\right)\mathrm{d}V'_X = f'_0\mathrm{d}V'_X. \tag{38}$$

Eq. (3) shows the generalized equation of motion for an arbitrary functional $F$ in the Poisson bracket formulation. Consider the expression of the Poisson bracket for a multi-particle system. In the center-of-mass coordinate system, the Poisson bracket of this system can be shown to be

$$\begin{aligned} \{F,G\}_L = {} & \left(\frac{\partial F}{\partial \mathbf{r}_c}\cdot\frac{\partial G}{\partial \mathbf{p}_c} - \frac{\partial F}{\partial \mathbf{p}_c}\cdot\frac{\partial G}{\partial \mathbf{r}_c}\right) + \sum_{\alpha=1}^{N}\left(\frac{\partial F}{\partial \mathbf{s}^{(\alpha)}}\cdot\frac{\partial G}{\partial \mathbf{p}^{(\alpha)}} - \frac{\partial F}{\partial \mathbf{p}^{(\alpha)}}\cdot\frac{\partial G}{\partial \mathbf{s}^{(\alpha)}}\right) \\ & + \left(\left(\sum_{\alpha=1}^{N}\frac{m^{(\alpha)}}{M}\frac{\partial F}{\partial \mathbf{p}^{(\alpha)}}\right)\cdot\frac{\partial G}{\partial \mathbf{r}_c} - \frac{\partial F}{\partial \mathbf{r}_c}\cdot\left(\sum_{\alpha=1}^{N}\frac{m^{(\alpha)}}{M}\frac{\partial G}{\partial \mathbf{p}^{(\alpha)}}\right)\right), \end{aligned} \tag{39}$$

where $M$ is the total mass of the system, $\mathbf{r}_c$ and $\mathbf{p}_c$ are the position and the momentum of the center of mass, and $\mathbf{s}^{(\alpha)}$ and $\mathbf{p}^{(\alpha)}$ represent the position and momentum of particle $\alpha$ relative to the center of mass. Analogous to the Poisson bracket in (39), the Poisson bracket for a microcontinuum is proposed to be

$$\begin{aligned}\{F,G\}_L = &\int_\Omega \left(\frac{\delta F}{\delta \bar{\mathbf{x}}}\cdot\frac{\delta G}{\delta \bar{\mathbf{u}}} - \frac{\delta F}{\delta \bar{\mathbf{u}}}\cdot\frac{\delta G}{\delta \bar{\mathbf{x}}}\right)\mathrm{d}^3X \\ &+ \int_\Omega\int_{\Delta B_X} \frac{1}{f_0'(\mathbf{X},\boldsymbol{\Xi})(\Delta V_X')^2}\left(\frac{\delta F}{\delta \boldsymbol{\xi}}\cdot\frac{\delta G}{\delta \bar{\mathbf{p}}^\xi} - \frac{\delta F}{\delta \bar{\mathbf{p}}^\xi}\cdot\frac{\delta G}{\delta \boldsymbol{\xi}}\right)\mathrm{d}V_X'\mathrm{d}^3X \\ &+ \int_\Omega\int_{\Delta B_X} \frac{1}{\Delta V_X'}\left(\frac{\delta F}{\delta \bar{\mathbf{p}}^\xi}\cdot\frac{\delta G}{\delta \bar{\mathbf{x}}} - \frac{\delta F}{\delta \bar{\mathbf{x}}}\cdot\frac{\delta G}{\delta \bar{\mathbf{p}}^\xi}\right)\mathrm{d}V_X'\mathrm{d}^3X. \end{aligned} \tag{40}$$

Here $\bar{\mathbf{p}}^\xi(=\rho_0\dot{\boldsymbol{\xi}})$ is the micromomentum density. This bracket contains the information of the Volterra functional derivatives, which can be generated from

$$\frac{\delta F}{\delta \bar{z}_n} = \int_{\Omega'}\left(\frac{\delta F}{\delta \rho}\frac{\delta \rho}{\delta \bar{z}_n} + \frac{\delta F}{\delta u_j}\frac{\delta u_j}{\delta \bar{z}_n} + \frac{\delta F}{\delta s}\frac{\delta s}{\delta \bar{z}_n} + \frac{\delta F}{\delta \hat{i}_{kl}}\frac{\delta \hat{i}_{kl}}{\delta \bar{z}_n} + \frac{\delta F}{\delta \hat{m}_{kl}}\frac{\delta \hat{m}_{kl}}{\delta \bar{z}_n} + \frac{\delta F}{\delta \hat{\Upsilon}_{KL}}\frac{\delta \hat{\Upsilon}_{KL}}{\delta \bar{z}_n} + \frac{\delta F}{\delta \hat{\mathcal{C}}_{KL}}\frac{\delta \hat{\mathcal{C}}_{KL}}{\delta \bar{z}_n} + \frac{\delta F}{\delta \hat{\Gamma}_{KLM}}\frac{\delta \hat{\Gamma}_{KLM}}{\delta \bar{z}_n}\right)\mathrm{d}^3x, \tag{41}$$

where $\bar{z}_n$ stands for one of the components in the set $(\bar{x}_n, \bar{u}_n, \xi_n, \bar{p}_n^\xi)$. In Eq. (41), the eight derivatives $\delta\rho/\delta\bar{z}_n$, $\delta u_j/\delta\bar{z}_n$, $\delta s/\delta\bar{z}_n$, $\delta\hat{i}_{kl}/\delta\bar{z}_n$, $\delta\hat{m}_{kl}/\delta\bar{z}_n$, $\delta\hat{\Upsilon}_{KL}/\delta\bar{z}_n$, $\delta\hat{\mathcal{C}}_{KL}/\delta\bar{z}_n$, and $\delta\hat{\Gamma}_{KLM}/\delta\bar{z}_n$ are determined from the Lagrangian-to-Eulerian transformation relations for the eight state variables in Eqs. (36) and (37):

$$(\mathrm{I})\left\{\frac{\delta\rho}{\delta\bar{x}_n} = \rho_0(\mathbf{X})\frac{\partial\delta^3(\bar{\mathbf{x}}-\mathbf{x})}{\partial\bar{x}_n},\quad \frac{\delta\rho}{\delta\bar{u}_n}=0,\quad \frac{\delta\rho}{\delta\xi_n}=0,\quad \frac{\delta\rho}{\delta\bar{p}_n^\xi}=0,\right. \tag{42}$$

$$(\mathrm{II})\begin{cases}\frac{\delta u_k(\mathbf{x},t)}{\delta\bar{x}_n(\mathbf{X},t)} = \bar{u}_k(\mathbf{X},t)\frac{\partial\delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]}{\partial\bar{x}_n(\mathbf{X},t)}, & \frac{\delta u_k}{\delta\xi_n}=0,\\ \frac{\delta u_k(\mathbf{x},t)}{\delta\bar{u}_n(\mathbf{X},t)} = \delta^3[\bar{\mathbf{x}}(\mathbf{X},t)-\mathbf{x}]\delta_{kn}, & \frac{\delta u_k}{\delta\bar{p}_n^\xi}=0,\end{cases} \tag{43}$$

$$(\mathrm{III})\left\{\frac{\delta s}{\delta\bar{x}_n} = s_0(\mathbf{X})\frac{\partial\delta^3(\bar{\mathbf{x}}-\mathbf{x})}{\partial\bar{x}_n},\quad \frac{\delta s}{\delta\bar{u}_n}=0,\quad \frac{\delta s}{\delta\xi_n}=0,\quad \frac{\delta s}{\delta\bar{p}_n^\xi}=0,\right. \tag{44}$$

$$(\mathrm{IV})\begin{cases}\frac{\delta\hat{i}_{kl}}{\delta\bar{x}_n} = \rho_0(\mathbf{X})\frac{\partial\delta^3(\bar{\mathbf{x}}-\mathbf{x})}{\partial\bar{x}_n}i_{kl}(\bar{\mathbf{x}},t), & \frac{\delta\hat{i}_{kl}}{\delta\bar{u}_n}=\frac{\delta\hat{i}_{kl}}{\delta\bar{p}_n^\xi}=0,\\ \frac{\delta\hat{i}_{kl}}{\delta\xi_n} = \rho_0(\mathbf{X})\delta^3(\bar{\mathbf{x}}-\mathbf{x})(\delta_{kn}\xi_l+\delta_{ln}\xi_k)f_0'\Delta V_X', & \end{cases} \tag{45}$$

$$(\mathrm{V})\begin{cases}\frac{\delta\hat{m}_{kl}}{\delta\bar{x}_n} = \rho_0(\mathbf{X})\frac{\partial\delta^3(\bar{\mathbf{x}}-\mathbf{x})}{\partial\bar{x}_n}m_{kl}(\bar{\mathbf{x}},t), & \frac{\delta\hat{m}_{kl}}{\delta\bar{u}_n}=0,\\ \frac{\delta\hat{m}_{kl}}{\delta\xi_n} = \delta^3(\bar{\mathbf{x}}-\mathbf{x})\delta_{kn}\bar{p}_l^\xi f_0'\Delta V_X', & \frac{\delta\hat{m}_{kl}}{\delta\bar{p}_n^\xi}=\delta^3(\bar{\mathbf{x}}-\mathbf{x})\xi_k\delta_{ln}f_0'\Delta V_X',\end{cases} \tag{46}$$

$$(\mathrm{VI})\begin{cases}\frac{\delta\hat{\Upsilon}_{KL}}{\delta\bar{x}_n} = \rho_0(\mathbf{X})\frac{\partial\delta^3(\bar{\mathbf{x}}-\mathbf{x})}{\partial\bar{x}_n}\Upsilon_{KL}(\bar{\mathbf{x}},t) - \frac{\partial}{\partial X_J}\left(\rho_0\delta^3(\bar{\mathbf{x}}-\mathbf{x})\frac{\partial\Upsilon_{KL}}{\partial\bar{x}_{n,J}}\right), & \\ \frac{\delta\hat{\Upsilon}_{KL}}{\delta\xi_n}\xi_i = -\rho_0\delta^3(\bar{\mathbf{x}}-\mathbf{x})\bar{x}_{i,K}\mathcal{X}_{Ln}, & \frac{\delta\hat{\Upsilon}_{KL}}{\delta\bar{u}_n}=\frac{\delta\hat{\Upsilon}_{KL}}{\delta\bar{p}_n^\xi}=0,\end{cases} \tag{47}$$

$$(\mathrm{VII})\begin{cases}\frac{\delta\hat{\mathcal{C}}_{KL}}{\delta\bar{x}_n} = \rho_0(\mathbf{X})\frac{\partial\delta^3(\bar{\mathbf{x}}-\mathbf{x})}{\partial\bar{x}_n}\mathcal{C}_{KL}(\bar{\mathbf{x}},t), & \frac{\delta\hat{\mathcal{C}}_{KL}}{\delta\bar{u}_n}=\frac{\delta\hat{\mathcal{C}}_{KL}}{\delta\bar{p}_n^\xi}=0,\\ \frac{\delta\hat{\mathcal{C}}_{KL}}{\delta\xi_n}\Xi_M = \rho_0\delta^3(\bar{\mathbf{x}}-\mathbf{x})(\chi_{kK}\delta_{kn}\delta_{LM}+\chi_{kL}\delta_{kn}\delta_{KM}), & \end{cases} \tag{48}$$

$$(\mathrm{VIII})\begin{cases}\frac{\delta\hat{\Gamma}_{KLM}}{\delta\bar{x}_n} = \rho_0(\mathbf{X})\frac{\partial\delta^3(\bar{\mathbf{x}}-\mathbf{x})}{\partial\bar{x}_n}\Gamma_{KLM}(\bar{\mathbf{x}},t), & \frac{\delta\hat{\Gamma}_{KLM}}{\delta\bar{u}_n}=\frac{\delta\hat{\Gamma}_{KLM}}{\delta\bar{p}_n^\xi}=0,\\ \frac{\delta\hat{\Gamma}_{KLM}}{\delta\xi_n}\Xi_I = -\rho_0\delta^3(\bar{\mathbf{x}}-\mathbf{x})\chi_{kL,M}\mathcal{X}_{Kn}\mathcal{X}_{Ik} - \frac{\partial}{\partial X_M}(\rho_0\delta^3(\bar{\mathbf{x}}-\mathbf{x})\mathcal{X}_{Kn}\delta_{IL}). & \end{cases} \tag{49}$$

Inserting the relations (42) to (49) into Eq. (41), and then substituting Eq. (41) into the Poisson bracket (40) easily produces the noncanonical Poisson bracket for a micromorphic elastic solid

$$
\begin{aligned}
\{F,H\}_E = \int_{\Omega'} \Bigg( & -\frac{\delta F}{\delta \rho}\left(\rho \frac{\delta H}{\delta u_j}\right)_{,j} - \frac{\delta F}{\delta u_j}\left(u_j \frac{\delta H}{\delta u_k}\right)_{,k} - \frac{\delta F}{\delta s}\left(s \frac{\delta H}{\delta u_j}\right)_{,j} - \frac{\delta F}{\delta \hat{i}_{kl}}\left(\hat{i}_{kl} \frac{\delta H}{\delta u_j}\right)_{,j} \\
& - \frac{\delta F}{\delta \hat{m}_{kl}}\left(\hat{m}_{kl} \frac{\delta H}{\delta u_j}\right)_{,j} + \frac{\delta F}{\delta \hat{i}_{kl}} \frac{\delta H}{\delta \hat{m}_{jk}} \hat{i}_{lj} + \frac{\delta F}{\delta \hat{i}_{kl}} \frac{\delta H}{\delta \hat{m}_{jl}} \hat{i}_{kj} + \frac{\delta F}{\delta \hat{m}_{kl}} \frac{\delta H}{\delta \hat{m}_{jk}} \hat{m}_{jl} \\
& - \frac{\delta F}{\delta \hat{\Upsilon}_{KL}}\left(\hat{\Upsilon}_{KL} \frac{\delta H}{\delta u_j}\right)_{,j} + \frac{\delta F}{\delta \hat{\Upsilon}_{KL}}\left(\frac{\delta H}{\delta u_j}\right)_{,k} \rho x_{k,K} \mathcal{X}_{Lj} - \frac{\delta F}{\delta \hat{\Upsilon}_{KL}} \frac{\delta H}{\delta \hat{m}_{jk}} \rho x_{j,K} \mathcal{X}_{Lk} \\
& + \frac{\delta F}{\delta \hat{\mathcal{C}}_{KL}} \frac{\delta H}{\delta \hat{m}_{ik}} \rho (\chi_{kL}\chi_{iK} + \chi_{kK}\chi_{iL}) + \frac{\delta F}{\delta \hat{\Gamma}_{KLM}}\left(\frac{\delta H}{\delta \hat{m}_{kl}}\right)_{,m} \rho \mathcal{X}_{Kl}\chi_{kL} x_{m,M} \\
& - \frac{\delta F}{\delta \hat{\mathcal{C}}_{KL}}\left(\hat{\mathcal{C}}_{KL} \frac{\delta H}{\delta u_j}\right)_{,j} - \frac{\delta F}{\delta \hat{\Gamma}_{KLM}}\left(\hat{\Gamma}_{KLM} \frac{\delta H}{\delta u_j}\right)_{,j} - (F \Longleftrightarrow H) \Bigg) \mathrm{d}^3x, \qquad (50)
\end{aligned}
$$

where $(F \Longleftrightarrow H)$ represents the above corresponding terms with the interchange of $F$ and $H$. Since

$$
\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}t} &= \{F,H\}_E \\
&= \int_{\Omega'} \left( \frac{\delta F}{\delta \rho}\frac{\partial \rho}{\partial t} + \frac{\delta F}{\delta u_j}\frac{\partial u_j}{\partial t} + \frac{\delta F}{\delta s}\frac{\partial s}{\partial t} + \frac{\delta F}{\delta \hat{i}_{kl}}\frac{\partial \hat{i}_{kl}}{\partial t} + \frac{\delta F}{\delta \hat{m}_{kl}}\frac{\partial \hat{m}_{kl}}{\partial t} + \frac{\delta F}{\delta \hat{\Upsilon}_{KL}}\frac{\hat{\Upsilon}_{KL}}{\partial t} + \frac{\delta F}{\delta \hat{\mathcal{C}}_{KL}}\frac{\partial \hat{\mathcal{C}}_{KL}}{\partial t} + \frac{\delta F}{\delta \hat{\Gamma}_{KLM}}\frac{\partial \hat{\Gamma}_{KLM}}{\partial t} \right) \mathrm{d}^3x, \qquad (51)
\end{aligned}
$$

the independence of the eight state variables yields the following eight evolution equations:

$$
\frac{\partial \rho}{\partial t} = -\left(\rho \frac{\delta H}{\delta u_j}\right)_{,j}, \qquad (52)
$$

$$
\begin{aligned}
\frac{\partial u_j}{\partial t} = & \left(-u_j \frac{\delta H}{\delta u_k} + \rho x_{k,K} \mathcal{X}_{Lj} \frac{\delta H}{\delta \hat{\Upsilon}_{KL}}\right)_{,k} - \rho\left(\frac{\delta H}{\delta \rho}\right)_{,j} - u_k\left(\frac{\delta H}{\delta u_k}\right)_{,j} - s\left(\frac{\delta H}{\delta s}\right)_{,j} - \hat{i}_{kl}\left(\frac{\delta H}{\delta \hat{i}_{kl}}\right)_{,j} - \hat{m}_{kl}\left(\frac{\delta H}{\delta \hat{m}_{kl}}\right)_{,j} \\
& - \hat{\Upsilon}_{KL}\left(\frac{\delta H}{\delta \hat{\Upsilon}_{KL}}\right)_{,j} - \hat{\mathcal{C}}_{KL}\left(\frac{\delta H}{\delta \hat{\mathcal{C}}_{KL}}\right)_{,j} - \hat{\Gamma}_{KLM}\left(\frac{\delta H}{\delta \hat{\Gamma}_{KLM}}\right)_{,j}, \qquad (53)
\end{aligned}
$$

$$
\frac{\partial s}{\partial t} = -\left(s \frac{\delta\, H}{\delta u_j}\right)_{,j}, \qquad (54)
$$

$$
\frac{\partial \hat{i}_{kl}}{\partial t} = -\left(\hat{i}_{kl} \frac{\delta H}{\delta u_j}\right)_{,j} + (\hat{i}_{li}\delta_{kj} + \hat{i}_{ki}\delta_{lj}) \frac{\delta H}{\delta \hat{m}_{ij}}, \qquad (55)
$$

$$
\begin{aligned}
\frac{\partial \hat{m}_{kl}}{\partial t} = & \left(-\hat{m}_{kl} \frac{\delta H}{\delta u_j} + \rho \mathcal{X}_{Kl}\chi_{kL} x_{j,M} \frac{\delta H}{\delta \hat{\Gamma}_{KLM}}\right)_{,j} + \left(\hat{m}_{jl} \frac{\delta H}{\delta \hat{m}_{jk}}\right) - \left(\hat{m}_{kj} \frac{\delta H}{\delta \hat{m}_{lj}}\right) - \left(\hat{i}_{kj} \frac{\delta H}{\delta \hat{i}_{lj}} + \hat{i}_{kj} \frac{\delta H}{\delta \hat{i}_{jl}}\right) \\
& + \frac{\delta H}{\delta \hat{\Upsilon}_{KL}} \rho x_{k,K} \mathcal{X}_{Ll} - \frac{\delta H}{\delta \hat{\mathcal{C}}_{KL}} \rho (\chi_{lL}\chi_{kK} + \chi_{lK}\chi_{kL}), \qquad (56)
\end{aligned}
$$

$$
\frac{\partial \hat{\Upsilon}_{KL}}{\partial t} = -\left(\hat{\Upsilon}_{KL} \frac{\delta H}{\delta u_j}\right)_{,j} + \left(\frac{\delta H}{\delta u_j}\right)_{,k} \rho x_{k,K} \mathcal{X}_{Lj} - \frac{\delta H}{\delta \hat{m}_{jk}} \rho x_{j,K} \mathcal{X}_{Lk}, \qquad (57)
$$

$$
\frac{\partial \hat{\mathcal{C}}_{KL}}{\partial t} = -\left(\hat{\mathcal{C}}_{KL} \frac{\delta H}{\delta u_j}\right)_{,j} + \frac{\delta H}{\delta \hat{m}_{ik}} \rho (\chi_{kL}\chi_{iK} + \chi_{kK}\chi_{iL}), \qquad (58)
$$

$$
\frac{\partial \hat{\Gamma}_{KLM}}{\partial t} = -\left(\hat{\Gamma}_{KLM} \frac{\delta H}{\delta u_j}\right)_{,j} + \left(\frac{\delta H}{\delta \hat{m}_{kl}}\right)_{,m} \rho \mathcal{X}_{Kl}\chi_{kL} x_{m,M}. \qquad (59)
$$

The explicit forms of all of the eight evolution equations are determined by the Hamiltonian $H$. For a micromorphic solid, it is appropriate to assume that $H$ is composed of kinetic energy and internal energy. The expression of kinetic energy is clearly established, and it includes the kinetic energy of the center of mass and the kinetic energy relative to the center of mass. Internal energy is usually denoted by $\int_{\Omega'} \varepsilon \mathrm{d}^3x$, where $\varepsilon$ is the internal energy density and can be assumed to be a function of mass density $\rho$, entropy density $s$, and the three strain densities: $\hat{\Upsilon}_{KL}$, $\hat{\mathcal{C}}_{KL}$, and $\hat{\mathcal{C}}_{KL}$. Hence, from

$$H = \int_{\Omega'} h\mathrm{d}^3x = \int_{\Omega'} \left(\frac{1}{2}\rho v_k v_k + \frac{1}{2}\rho i_{kl} v_{mk} v_{ml} + \varepsilon(\rho, s, \hat{\Upsilon}_{KL}, \hat{\mathcal{C}}_{KL}, \hat{\Gamma}_{KLM})\right)\mathrm{d}^3x$$
$$= \int_{\Omega'} \left(\frac{u_k u_k}{2\rho} + \frac{1}{2}\hat{i}^{-1}_{pq}\hat{m}_{pk}\hat{m}_{qk} + \varepsilon(\rho, s, \hat{\Upsilon}_{KL}, \hat{\mathcal{C}}_{KL}, \hat{\Gamma}_{KLM})\right)\mathrm{d}^3x, \tag{60}$$

it follows that

$$\begin{cases} \frac{\delta H}{\delta \rho} = \frac{-u_k u_k}{2\rho^2} + \frac{\partial \varepsilon}{\partial \rho}, & \frac{\delta H}{\delta u_k} = v_k, \quad \frac{\delta H}{\delta s} = \frac{\partial \varepsilon}{\partial s} \quad \frac{\delta H}{\delta \hat{i}_{kl}} = \frac{-1}{2}\hat{m}_{pr}\hat{m}_{qr}\hat{i}^{-1}_{pk}\hat{i}^{-1}_{ql}, \\ \frac{\delta H}{\delta \hat{m}_{kl}} = v_{lk}, & \frac{\delta H}{\delta \hat{\Upsilon}_{KL}} = \frac{\partial \varepsilon}{\partial \hat{\Upsilon}_{KL}}, \quad \frac{\delta H}{\delta \hat{\mathcal{C}}_{KL}} = \frac{\partial \varepsilon}{\partial \hat{\mathcal{C}}_{KL}}, \quad \frac{\delta H}{\delta \hat{\Gamma}_{KLM}} = \frac{\partial \varepsilon}{\partial \hat{\Gamma}_{KLM}}. \end{cases} \tag{61}$$

Substituting the set (61) into Eqs. (52) to (56) results in

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho v_{k,k}, \tag{62}$$
$$\rho\frac{\mathrm{d}v_k}{\mathrm{d}t} = \tau_{lk,l}, \tag{63}$$
$$\frac{\mathrm{d}s}{\mathrm{d}t} = -s v_{k,k}, \tag{64}$$
$$\frac{\mathrm{d}i_{kl}}{\mathrm{d}t} = i_{kj}v_{lj} + i_{lj}v_{kj}, \tag{65}$$
$$\rho\sigma_{lk} = \gamma_{jlk,j} + \tau_{kl} - s_{kl}, \tag{66}$$

where stress tensor $\tau_{kl}$, microstress tensor $s_{kl}$, couple stress tensor $\gamma_{jlk}$, and spin inertia per unit mass $\sigma_{kl}$ are defined and written as

$$\tau_{kl} := -p\delta_{kl} + \rho\frac{\partial\varepsilon}{\partial\hat{\Upsilon}_{KL}}x_{k,K}\mathcal{X}_{Ll}, \tag{67}$$
$$s_{kl} := 2\rho\frac{\partial\varepsilon}{\partial\hat{\mathcal{C}}_{KL}}\chi_{lL}\chi_{kK}, \tag{68}$$
$$\gamma_{jlk} := \rho\frac{\partial\varepsilon}{\partial\hat{\Gamma}_{KLM}}x_{j,M}\mathcal{X}_{Kl}\chi_{kL}, \tag{69}$$
$$\rho\sigma_{lk} := \dot{v}_{lm}\hat{i}_{mk} + v_{lm}v_{mn}\hat{i}_{nk} = \frac{\partial\hat{m}_{kl}}{\partial t} + (v_j\hat{m}_{kl})_j - v_{km}\hat{i}_{mn}v_{ln}, \tag{70}$$

with the pressure

$$p := -\varepsilon + \rho\frac{\partial\varepsilon}{\partial\rho} + s\frac{\partial\varepsilon}{\partial s} + \hat{\Upsilon}_{KL}\frac{\partial\varepsilon}{\partial\hat{\Upsilon}_{KL}} + \hat{\mathcal{C}}_{KL}\frac{\partial\varepsilon}{\partial\hat{\mathcal{C}}_{KL}} + \hat{\Gamma}_{KLM}\frac{\partial\varepsilon}{\partial\hat{\Gamma}_{KLM}}. \tag{71}$$

In the same manner as the previous section, let the energy per unit mass $\psi$ be a function of the set $(\eta, \Upsilon_{KL}, \mathcal{C}_{KL}, \Gamma_{KLM})$ and assume that $\rho\psi(\eta, \Upsilon_{KL}, \mathcal{C}_{KL}, \Gamma_{KLM}) = \varepsilon(\rho, s, \hat{\Upsilon}_{KL}, \hat{\mathcal{C}}_{KL}, \hat{\Gamma}_{KLM})$, then it is easy to find

$$\begin{cases} \frac{\partial\varepsilon}{\partial s} = \frac{\partial\psi}{\partial\eta}, \quad \frac{\partial\varepsilon}{\partial\hat{\Upsilon}_{KL}} = \frac{\partial\psi}{\partial\Upsilon_{KL}}, \quad \frac{\partial\varepsilon}{\partial\hat{\mathcal{C}}_{KL}} = \frac{\partial\psi}{\partial\mathcal{C}_{KL}}, \quad \frac{\partial\varepsilon}{\partial\hat{\Gamma}_{KLM}} = \frac{\partial\psi}{\partial\Gamma_{KLM}}, \\ \left.\frac{\partial\varepsilon}{\partial\rho}\right|_{\hat{\Upsilon}_{KL},\hat{\mathcal{C}}_{KL},\hat{\Gamma}_{KLM}} = \psi - \eta\frac{\partial\psi}{\partial\eta} - \Upsilon_{KL}\frac{\partial\psi}{\partial\Upsilon_{KL}} - \mathcal{C}_{KL}\frac{\partial\psi}{\partial\mathcal{C}_{KL}} - \Gamma_{KLM}\frac{\partial\psi}{\partial\Gamma_{KLM}} \\ \qquad = \frac{1}{\rho}\left(\varepsilon - s\frac{\partial\varepsilon}{\partial s} - \hat{\Upsilon}_{KL}\frac{\partial\varepsilon}{\partial\hat{\Upsilon}_{KL}} - \hat{\mathcal{C}}_{KL}\frac{\partial\varepsilon}{\partial\hat{\mathcal{C}}_{KL}} - \hat{\Gamma}_{KLM}\frac{\partial\varepsilon}{\partial\hat{\Gamma}_{KLM}}\right). \end{cases} \tag{72}$$

The last relation in (72) shows from Eq. (71) that pressure $p$ should be equal to zero. Hence, the four evolution equations (62) to (66) will be reduced to the sourceless balance equations of mass, linear momentum, micro-inertia, and momentum moment in the field theory of micromorphic elastic solids (Eringen, 1999).

Moreover, because of the relations in (72), the last three evolution equations (57) to (59) can be written as

$$\frac{\partial \hat{\Upsilon}_{KL}}{\partial t} = -(\hat{\Upsilon}_{KL} v_j)_{,j} + \rho v_{j,k} x_{k,K} \mathcal{X}_{Lj} - \rho v_{kj} x_{j,K} \mathcal{X}_{Lk}, \tag{73}$$

$$\frac{\partial \hat{\mathcal{C}}_{KL}}{\partial t} = -(\hat{\mathcal{C}}_{KL} v_j)_{,j} + \rho v_{ki} (\chi_{kL} \chi_{iK} + \chi_{kK} \chi_{iL}), \tag{74}$$

$$\frac{\partial \hat{\Gamma}_{KLM}}{\partial t} = -(\hat{\Gamma}_{KLM} v_j)_{,j} + \rho v_{lk,m} \mathcal{X}_{Kl} \chi_{kL} x_{m,M}, \tag{75}$$

which are exactly the same as those obtained by taking the material time derivatives of the three strain densities $\hat{\Upsilon}_{KL}$, $\hat{\mathcal{C}}_{KL}$, and $\hat{\Gamma}_{KLM}$ in (32).

The field equation for the internal energy density can also be determined by taking its partial time derivative, $\varepsilon(\mathbf{x},t) = \tilde{\varepsilon}(\rho(\mathbf{x},t), \hat{\Upsilon}_{KL}(\mathbf{x},t), \hat{\mathcal{C}}_{KL}(\mathbf{x},t), \hat{\Gamma}_{KLM}(\mathbf{x},t))$,

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial \tilde{\varepsilon}}{\partial \rho}\frac{\partial \rho}{\partial t} + \frac{\partial \tilde{\varepsilon}}{\partial \hat{\Upsilon}_{KL}}\frac{\partial \hat{\Upsilon}_{KL}}{\partial t} + \frac{\partial \tilde{\varepsilon}}{\partial \hat{\mathcal{C}}_{KL}}\frac{\partial \hat{\mathcal{C}}_{KL}}{\partial t} + \frac{\partial \tilde{\varepsilon}}{\partial \hat{\Gamma}_{KLM}}\frac{\partial \hat{\Gamma}_{KLM}}{\partial t}. \tag{76}$$

Inserting the evolution Eqs. (52), (57), (58), and (59) into Eq. (76) gives

$$\frac{\partial \varepsilon}{\partial t} = -(\varepsilon v_k)_{,k} + \tau_{kj}(v_{j,k} - v_{jk}) + s_{kj} v_{jk} + \gamma_{mlk} v_{lk,m}. \tag{77}$$

## 5. State variable selection for LE transformation

The previous two sections show that for nondissipative systems, the evolution of the state variables can be determined purely by the LE transformation while the suitable state variables of the system have been selected. The number of state variables amounts to the number of evolution equations for a system. However, the choice of the state variables is not unique and could depend on the perception of the system. This section discusses three types of state variables: the Cauchy deformation tensor $c_{ij}$, the gradient of deformation tensor $F_{iJK}$, and the internal energy density $\varepsilon$.

### 5.1. Other elastic strain measures

The Green deformation tensor $C_{IJ}$ is not the only variable to describe the elastic deformation of an elastic solid discussed in Section 3. The Cauchy deformation tensor $c_{ij} = (\partial X_K/\partial x_i)(\partial X_K/\partial x_j)$ could replace the Green deformation tensor as a suitable strain measure. If the density $\hat{c}_{ij} (= \rho c_{ij})$ is the state variable for an elastic solid, then the Poisson bracket associated with $\hat{c}_{ij}$ can be derived by performing the same manipulation used in previous sections:

$$\begin{aligned}\{F,G\}_E^{\hat{c}} = \int_{\Omega'} \Bigg[ &-\frac{\delta F}{\delta \hat{c}_{ij}}\left(\hat{c}_{ij}\frac{\delta G}{\delta u_k}\right)_{,k} + \frac{\delta G}{\delta \hat{c}_{ij}}\left(\hat{c}_{ij}\frac{\delta F}{\delta u_k}\right)_{,k} - \hat{c}_{jk}\frac{\delta F}{\delta \hat{c}_{ij}}\left(\frac{\delta G}{\delta u_k}\right)_{,i} + \hat{c}_{jk}\frac{\delta G}{\delta \hat{c}_{ij}}\left(\frac{\delta F}{\delta u_k}\right)_{,i} - \hat{c}_{ik}\frac{\delta F}{\delta \hat{c}_{ij}}\left(\frac{\delta G}{\delta u_k}\right)_{,j} \\ &+ \hat{c}_{ik}\frac{\delta G}{\delta \hat{c}_{ij}}\left(\frac{\delta F}{\delta u_k}\right)_{,j}\Bigg] \mathrm{d}^3x. \end{aligned} \tag{78}$$

The full Poisson bracket should be the sum of $\{F,G\}_E^{\rho}$ in (16), $\{F,G\}_E^{s}$ in (17), $\{F,G\}_E^{u}$ in (18), and $\{F,G\}_E^{\hat{c}}$ in (78). If the Hamiltonian functional $H[\rho,\mathbf{u},s,\hat{\mathbf{c}}]$ is expressed as

$$H[\rho,\mathbf{u},s,\hat{\mathbf{c}}] = \int_{\Omega'} \left( \frac{|\mathbf{u}(\mathbf{x},t)|^2}{2\rho(\mathbf{x},t)} + \varepsilon(\rho(\mathbf{x},t), s(\mathbf{x},t), \hat{\mathbf{c}}(\mathbf{x},t)) \right) \mathrm{d}^3x, \tag{79}$$

with internal energy $\varepsilon(\rho,s,\hat{\mathbf{c}})$, then accounting for the equation $\mathrm{d}F/\mathrm{d}t = \{F,H\}_E$ reveals the evolution equations, which resemble those in Section 3 except that the equation for $\hat{C}_{IJ}$ is replaced by

$$\frac{\partial \hat{c}_{ij}}{\partial t} = -(v_k \hat{c}_{ij})_{,k} - (v_{k,i}\hat{c}_{kj} + v_{k,j}\hat{c}_{ki}), \tag{80}$$

and Cauchy stress tensor $\tau_{ki}$ are defined by

$$\tau_{ki} = \left(\varepsilon - \rho\frac{\partial \varepsilon}{\partial \rho} - s\frac{\partial \varepsilon}{\partial s} - \hat{c}_{ij}\frac{\partial \varepsilon}{\partial \hat{c}_{ij}}\right)\delta_{ki} - 2\left(\hat{c}_{jk}\frac{\partial \varepsilon}{\partial \hat{c}_{ij}}\right). \tag{81}$$

In addition to the aforementioned two strain measures, $C_{KL}$ and $c_{ij}$, other strain versions can describe the deformation of a material body: the Piola strain tensor $C_{KL}^{-1} = (\partial X_K/\partial x_i)(\partial X_L/\partial x_i)$, the Finger strain tensor $c_{ij}^{-1} = (\partial x_i/\partial X_K)(\partial x_j/\partial X_K)$, the Lagrangian strain tensor $E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL})$, the Eulerian strain tensor $e_{ij} = \frac{1}{2}(\delta_{ij} - c_{ij})$, etc. These strains are all clearly expressed in terms of the deformation gradient $F_{iK}$ and, hence, the constructions of the LE transformation relations for these strains are straightforward.

### 5.2. Higher strain gradient tensor

Poisson bracket formalism can be generalized to the case of a second-gradient theory (Mindlin, 1964; Forest and Sievert, 2006) where the higher gradient of strain is incorporated into the set of state variables. Assume that the whole set of variables is $(\rho, \mathbf{u}, s, \hat{F}_{iJ}, \hat{F}_{iJK})$, with $\hat{F}_{iJ}$ and $\hat{F}_{iJK}$ being the densities of the deformation gradient and the second-gradient of the deformation gradient. The LE transformation relations for the two variables are given as

$$\hat{F}_{iJ}(\mathbf{x}, t) = \rho\frac{\partial x_i}{\partial X_J} = \int_{\Omega} \rho_0(\mathbf{X}) F_{iJ}(\bar{\mathbf{x}}(\mathbf{X}, t), t)\delta^3[\bar{\mathbf{x}}(\mathbf{X}, t) - \mathbf{x}]\mathrm{d}^3X, \tag{82}$$

$$\hat{F}_{iJK}(\mathbf{x}, t) = \rho\frac{\partial^2 x_i}{\partial X_J \partial X_K} = \int_{\Omega} \rho_0(\mathbf{X}) F_{iJK}(\bar{\mathbf{x}}(\mathbf{X}, t), t)\delta^3[\bar{\mathbf{x}}(\mathbf{X}, t) - \mathbf{x}]\mathrm{d}^3X, \tag{83}$$

which lead to

$$\begin{cases} \frac{\delta \hat{F}_{iJ}(\mathbf{x},t)}{\delta \bar{x}_n(\mathbf{X},t)} = \rho_0(\mathbf{X})F_{iJ}\frac{\partial \delta^3[\bar{\mathbf{x}}-\mathbf{x}]}{\partial \bar{x}_n(\mathbf{X},t)} - \frac{\partial}{\partial X_J}(\delta_{in}\rho_0(\mathbf{X})\delta^3[\bar{\mathbf{x}} - \mathbf{x}]), \\ \frac{\delta \hat{F}_{iJK}(\mathbf{x},t)}{\delta \bar{x}_n(\mathbf{X},t)} = \rho_0(\mathbf{X})F_{iJK}\frac{\partial \delta^3[\bar{\mathbf{x}}-\mathbf{x}]}{\partial \bar{x}_n(\mathbf{X},t)} + \frac{\partial}{\partial X_J}\frac{\partial}{\partial X_K}(\delta_{in}\rho_0(\mathbf{X})\delta^3[\bar{\mathbf{x}} - \mathbf{x}]). \end{cases} \tag{84}$$

After applying the same procedure used in the previous sections, the Poisson bracket for this case should be expressed as

$$\{F, G\}_E = \{F, G\}_E^{\rho} + \{F, G\}_E^{s} + \{F, G\}_E^{u} + \{F, G\}_E^{F}, \tag{85}$$

where the first, second, and third brackets on the right hand side have been defined in Eqs. (16)–(18), and the last one is

$$\begin{aligned} \{F, G\}_E^F = \int_{\Omega'} \Bigg[ & -\frac{\delta F}{\delta \hat{F}_{iJ}}\left(\hat{F}_{iJ}\frac{\delta G}{\delta u_k}\right)_{,k} + \frac{\delta G}{\delta \hat{F}_{iJ}}\left(\hat{F}_{iJ}\frac{\delta F}{\delta u_k}\right)_{,k} + \hat{F}_{kJ}\frac{\delta F}{\delta \hat{F}_{iJ}}\left(\frac{\delta G}{\delta u_i}\right)_{,k} \\ & - \hat{F}_{kJ}\frac{\delta G}{\delta \hat{F}_{iJ}}\left(\frac{\delta F}{\delta u_i}\right)_{,k} - \frac{\delta F}{\delta \hat{F}_{iJK}}\left(\hat{F}_{iJK}\frac{\delta G}{\delta u_k}\right)_{,k} + \frac{\delta G}{\delta \hat{F}_{iJK}}\left(\hat{F}_{iJK}\frac{\delta F}{\delta u_k}\right)_{,k} \\ & + \frac{\delta F}{\delta \hat{F}_{iJK}}\hat{F}_{kK}\left(\frac{\delta G}{\delta u_i}x_{j,J}\right)_{,jk} - \frac{\delta G}{\delta \hat{F}_{iJK}}\hat{F}_{kK}\left(\frac{\delta F}{\delta u_i}x_{j,J}\right)_{,jk} \Bigg]\mathrm{d}^3x. \end{aligned} \tag{86}$$

On account of the equation $\mathrm{d}F/\mathrm{d}t = \{F, H\}_E$ and the functional expression for the Hamiltonian functional $H$,

$$H[\rho, u_i, s, \hat{F}_{iJ}, \hat{F}_{iJK}] = \int_{\Omega'} \left(\frac{u_i u_i}{2\rho} + \varepsilon(\rho, s, \hat{F}_{iJ}, \hat{F}_{iJK})\right)\mathrm{d}^3x, \tag{87}$$

with the internal energy $\varepsilon(\rho, s, \hat{F}_{iJ}, \hat{F}_{iJK})$, the independence of the five quantities, $\delta F/\delta\rho$, $\delta F/\delta u_i$, $\delta F/\delta s$, $\delta F/\delta \hat{F}_{iJ}$, and $\delta F/\delta \hat{F}_{iJK}$, generates the evolution equations for the state variables $(\rho, u_i, s, \hat{F}_{iJ}, \hat{F}_{iJK})$. The first three equations are identical to Eqs. (22)–(24), whereas, Cauchy stress tensor $\tau_{ki}$ should be changed to

$$\tau_{ki} = -\left(\rho\frac{\partial\varepsilon}{\partial\rho} + s\frac{\partial\varepsilon}{\partial s} + \hat{F}_{iJ}\frac{\partial\varepsilon}{\partial\hat{F}_{iJ}} + \hat{F}_{iJK}\frac{\partial\varepsilon}{\partial\hat{F}_{iJK}} - \varepsilon\right)\delta_{ki} + \frac{\partial\varepsilon}{\partial\hat{F}_{kJ}}\hat{F}_{iJ} - \left(\frac{\partial\varepsilon}{\partial\hat{F}_{kJK}}\hat{F}_{iJ}\right)_{,K}. \tag{88}$$

The second part of the stress tensor is related to the first Piola-Kirchhoff tensor and the third part can be considered the Piola-Kirchhoff hyperstress. Moreover, the evolution equations for the variables $(\hat{F}_{iJ}, \hat{F}_{iJK})$ are written as

$$\frac{\partial\hat{F}_{iJ}}{\partial t} = -(v_k\hat{F}_{iJ})_{,k} + \hat{F}_{kJ}v_{i,k}, \tag{89}$$

$$\frac{\partial\hat{F}_{iJK}}{\partial t} = -(v_k\hat{F}_{iJK})_{,k} + \hat{F}_{kK}(v_i x_{j,J})_{,jk}, \tag{90}$$

which are exactly the forms obtained by taking the material time derivatives of $\hat{F}_{iJ}$ and $\hat{F}_{iJK}$.

### 5.3. Nonconservative state variable

Not all state variables share the LE transformation relations as thosediscussed above. Section 3 uses an elastic solid as an example. The internal energy density $\varepsilon$ is not conserved along with the motion of a material element, even in an isentropic process. However, the Poisson bracket for the system, in which the nonconservative quantity is introduced as a state variable, can be constructed by the direct mappings (Edwards and Beris, 1998)

$$\left.\frac{\delta F'}{\delta\mathbf{u}}\right|_{\rho,s,\hat{\mathbf{C}}} = \left.\frac{\delta F}{\delta\mathbf{u}}\right|_{\rho,\varepsilon,\hat{\mathbf{C}}}, \qquad \left.\frac{\delta F'}{\delta s}\right|_{\rho,\mathbf{u},\hat{\mathbf{C}}} = \left.\frac{\delta F}{\delta\varepsilon}\right|_{\rho,\mathbf{u},\hat{\mathbf{C}}}\left.\frac{\delta\varepsilon}{\delta s}\right|_{\rho,\hat{\mathbf{C}}} = T\left.\frac{\delta F}{\delta\varepsilon}\right|_{\rho,\mathbf{u},\hat{\mathbf{C}}},$$
$$\left.\frac{\delta F'}{\delta\rho}\right|_{s,\mathbf{u},\hat{\mathbf{C}}} = \left.\frac{\delta F}{\delta\rho}\right|_{\varepsilon,\mathbf{u},\hat{\mathbf{C}}} + \left.\frac{\delta F}{\delta\varepsilon}\right|_{\rho,\mathbf{u},\hat{\mathbf{C}}}\left.\frac{\delta\varepsilon}{\delta\rho}\right|_{s,\hat{\mathbf{C}}}, \qquad \left.\frac{\delta F'}{\delta\hat{\mathbf{C}}}\right|_{\rho,\mathbf{u},s} = \left.\frac{\delta F}{\delta\hat{\mathbf{C}}}\right|_{\rho,\mathbf{u},\varepsilon} + \left.\frac{\delta F}{\delta\varepsilon}\right|_{\rho,\mathbf{u},\hat{\mathbf{C}}}\left.\frac{\delta\varepsilon}{\delta\hat{\mathbf{C}}}\right|_{\rho,s},$$

which transform the Poisson bracket with variable $(\rho, \mathbf{u}, s, \hat{\mathbf{C}})$ and functional $F' = F'[\rho, \mathbf{u}, s, \hat{\mathbf{C}}]$ into a Poisson bracket with variable $(\rho, \mathbf{u}, \varepsilon, \hat{\mathbf{C}})$ and functional $F = F[\rho, \mathbf{u}, \varepsilon, \hat{\mathbf{C}}]$.

The LE transformation can also be used to construct a Poisson bracket for an elastic solid with the state variables $(\rho, \mathbf{u}, \varepsilon, \hat{\mathbf{C}})$. For the arbitrary functionals $F[\rho, \mathbf{u}, \varepsilon, \hat{\mathbf{C}}]$ and $G[\rho, \mathbf{u}, \varepsilon, \hat{\mathbf{C}}]$,

$$\frac{\delta F}{\delta\bar{z}_n} = \int_{\Omega'}\left(\frac{\delta F}{\delta\rho}\frac{\delta\rho}{\delta\bar{z}_n} + \frac{\delta F}{\delta u_j}\frac{\delta u_j}{\delta\bar{z}_n} + \frac{\delta F}{\delta\varepsilon}\frac{\delta\varepsilon}{\delta\bar{z}_n} + \frac{\delta F}{\delta\hat{C}_{KL}}\frac{\delta\hat{C}_{KL}}{\delta\bar{z}_n}\right)\mathrm{d}^3x, \tag{91}$$

in which $\bar{z}_n$ stands for $\bar{x}_n$ or $\bar{u}_n$. Inserting (91) into (5) yields the Poisson bracket as

$$\{F,G\}_E = \{F,G\}_E^{\rho} + \{F,G\}_E^{u} + \{F,G\}_E^{C} + \{F,G\}_E^{\varepsilon}. \tag{92}$$

The first three brackets are described in Section 3 and the final bracket is

$$\int_{\Omega'}\int_{\Omega'}\left(\frac{\delta F}{\delta\ \varepsilon(\mathbf{x},t)}\frac{\delta G}{\delta u_j(\mathbf{z},t)} - \frac{\delta G}{\delta\varepsilon(\mathbf{x},t)}\frac{\delta F}{\delta u_j(\mathbf{z},t)}\right)\{\varepsilon(\mathbf{x},t), u_j(\mathbf{z},t)\}_L \mathrm{d}^3z\mathrm{d}^3x. \tag{93}$$

The energy of a material element is not conserved with the movement of the element. Hence, the LE transformation relation for the energy density does not exist, impeding calculation for $\{\varepsilon(\mathbf{x},t), u_j(\mathbf{z},t)\}_L$. However, the reversible processes can indirectly produce $\delta\varepsilon/\delta\bar{x}_n$ and $\delta\varepsilon/\delta\bar{u}_n$ by transforming the energy density $\varepsilon$ to the entropy density $s$ through the equilibrium relations $s = \tilde{s}(\varepsilon, \rho, \hat{\mathbf{C}})$, which implies

$$\frac{\delta s}{\delta\bar{x}_n} = \frac{\partial\tilde{s}}{\partial\varepsilon}\frac{\delta\varepsilon}{\delta\bar{x}_n} + \frac{\partial\tilde{s}}{\partial\rho}\frac{\delta\rho}{\delta\bar{x}_n} + \frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\frac{\delta\hat{C}_{KL}}{\delta\bar{x}_n}, \qquad \frac{\delta s}{\delta\bar{u}_n} = \frac{\partial\tilde{s}}{\partial\varepsilon}\frac{\delta\varepsilon}{\delta\bar{u}_n} + \frac{\partial\tilde{s}}{\partial\rho}\frac{\delta\rho}{\delta\bar{u}_n} + \frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\frac{\delta\hat{C}_{KL}}{\delta\bar{u}_n}.$$

In view of $\delta s/\delta\bar{x}_n$, $\delta\rho/\delta\bar{x}_n$, $\delta\hat{C}_{KL}/\delta\bar{x}_n$, $\delta s/\delta\bar{u}_n$, $\delta\rho/\delta\bar{u}_n$, and $\delta\hat{C}_{KL}/\delta\bar{u}_n$ in (14), it can be shown that

$$\frac{\delta\varepsilon}{\delta\bar{x}_n}=\frac{1}{(\partial\tilde{s}/\partial\varepsilon)}\left(\left(s_0-\frac{\partial\tilde{s}}{\partial\rho}\rho_0-\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\rho_0 C_{KL}\right)\frac{\partial\delta^3[\bar{\mathbf{x}}-\mathbf{x}]}{\partial\bar{x}_n}\right.$$
$$\left.+\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\frac{\partial}{\partial X_M}(\rho_0(\bar{x}_{n,L}\delta_{KM}+\bar{x}_{n,K}\delta_{LM})\delta^3[\bar{\mathbf{x}}-\mathbf{x}])\right), \tag{94}$$

$$\frac{\delta\varepsilon}{\delta\bar{u}_n}=\frac{1}{(\partial\tilde{s}/\partial\varepsilon)}\left(\frac{\delta s}{\delta\bar{u}_n}-\frac{\partial\tilde{s}}{\partial\rho}\frac{\delta\rho}{\delta\bar{u}_n}-\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\frac{\delta\hat{C}_{KL}}{\delta\bar{u}_n}\right)=0. \tag{95}$$

Using the following calculation of

$$\begin{aligned}\{\varepsilon(\mathbf{x},t),u_j(\mathbf{z},t)\}_L&=\int_\Omega\left(\frac{\delta\varepsilon}{\delta\bar{x}_n}\frac{\delta u_j}{\delta\bar{u}_n}-\frac{\delta u_j}{\delta\bar{x}_n}\frac{\delta\varepsilon}{\delta\bar{u}_n}\right)\mathrm{d}^3X\\&=\frac{1}{(\partial\tilde{s}/\partial\varepsilon)}\left[\left(s(\mathbf{z},t)-\frac{\partial\tilde{s}}{\partial\rho}\rho(\mathbf{z},t)-\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\hat{C}_{KL}(\mathbf{z},t)\right)\frac{\partial\delta^3[\mathbf{z}-\mathbf{x}]}{\partial z_j}\right.\\&\qquad\left.+\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\delta^3[\mathbf{z}-\mathbf{x}]\frac{\partial}{\partial z_m}(\rho(z_{m,K}z_{j,L}+z_{m,L}z_{j,K}))\right],\end{aligned}$$

the fourth bracket $\{F,G\}_E^{\varepsilon}$ in Eq. (92) can be derived in the form of

$$\int_{\Omega'}\left(\frac{1}{(\partial\tilde{s}/\partial\varepsilon)}\frac{\delta F}{\delta\varepsilon}\left(-\left(\frac{\delta G}{\delta u_j}s\right)_{,j}+\frac{\partial\tilde{s}}{\partial\rho}\left(\frac{\delta G}{\delta u_j}\rho\right)_{,j}+\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\left(\frac{\delta G}{\delta u_j}\hat{C}_{KL}\right)_{,j}\right.\right.$$
$$\left.\left.-\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}\left(\frac{\delta G}{\delta u_k}\right)_{,l}\rho(x_{l,K}x_{k,L}+x_{l,L}x_{k,K})\right)-(F\Longleftrightarrow G)\right)\mathrm{d}^3x. \tag{96}$$

Now let the Hamiltonian functional $H$ be

$$H[\rho,u_i,\varepsilon]=\int_{\Omega'}\left(\frac{u_iu_i}{2\rho}+\varepsilon\right)\mathrm{d}^3x, \tag{97}$$

where the internal energy density $\varepsilon$ is treated as a state variable such that the functional $H$ is not explicitly dependent on the variable $\hat{C}_{KL}$. Substituting the final expression of the Poisson bracket into the equation $\mathrm{d}F/\mathrm{d}t=\{F,H\}_E$ produces the evolution equations for $(\rho, u_i, \varepsilon, \hat{C}_{KL})$:

$$\frac{\partial\rho}{\partial t}=-\left(\frac{\delta H}{\delta u_j}\rho\right)_{,j}=-(v_j\rho)_{,j}, \tag{98}$$

$$\frac{\partial u_i}{\partial t}=-(\rho v_jv_i)_{,j}+\tau_{ki,k}, \tag{99}$$

$$\frac{\partial\varepsilon}{\partial t}=\left(\frac{\partial\tilde{s}}{\partial\varepsilon}\right)^{-1}\left(\frac{\partial\tilde{s}}{\partial\hat{C}_{KL}}((v_j\hat{C}_{KL})_{,j}-\rho v_{k,l}(x_{l,K}x_{k,L}+x_{l,L}x_{k,K}))-(v_js)_{,j}+\frac{\partial\tilde{s}}{\partial\rho}(v_j\rho)_{,j}\right), \tag{100}$$

$$\frac{\partial\hat{C}_{KL}}{\partial t}=-(v_k\hat{C}_{KL})_{,k}+\rho v_{k,l}(x_{k,L}x_{l,K}+x_{k,K}x_{l,L}), \tag{101}$$

while stress tensor $\tau_{ki}$ are

$$\tau_{ki}=-\left(\frac{s}{\partial\tilde{s}/\partial\varepsilon}-\rho\frac{\partial\tilde{s}/\partial\rho}{\partial\tilde{s}/\partial\varepsilon}-\hat{C}_{KL}\frac{\partial\tilde{s}/\partial\hat{C}_{KL}}{\partial\tilde{s}/\partial\varepsilon}-\varepsilon\right)\delta_{ki}-\frac{\partial\tilde{s}/\partial\hat{C}_{KL}}{\partial\tilde{s}/\partial\varepsilon}\rho(x_{k,L}x_{i,K}+x_{k,K}x_{i,L}). \tag{102}$$

Eqs. (98), (99), and (101) are the same as Eqs. (22), (23), and (25), and Eq. (100) can be further simplified with the help of the proposition of the relation $s=\tilde{s}(\varepsilon,\rho,\hat{\mathbf{C}})$. The substitution of the spatial differentiation $s_{,j}=\rho_{,j}(\partial\tilde{s}/\partial\rho)+\varepsilon_{,j}(\partial\tilde{s}/\partial\varepsilon)+(\hat{C}_{KL})_{,j}(\partial\tilde{s}/\partial\hat{C}_{KL})$ into Eq. (100) and the definitions of pressure and stress tensor in Eq. (102) lead to

$$\frac{\partial \varepsilon}{\partial t} = -\varepsilon_{,j} v_j + v_{j,j}\left(\frac{-s}{\partial\tilde{s}/\partial\varepsilon} + \rho\frac{\partial\tilde{s}/\partial\rho}{\partial\tilde{s}/\partial\varepsilon} + \hat{C}_{KL}\frac{\partial\tilde{s}/\partial\hat{C}_{KL}}{\partial\tilde{s}/\partial\varepsilon}\right) - \frac{\partial\tilde{s}/\partial\hat{C}_{KL}}{\partial\tilde{s}/\partial\varepsilon}\rho v_{k,l}(x_{k,L}x_{l,K} + x_{k,K}x_{l,L}), \tag{103}$$

which is identical to the internal energy equation (30).

The reason why Poisson bracket (92), characterizing the kinematics for an elastic solid with state variables $(\rho, \mathbf{u}, \varepsilon, \hat{\mathbf{C}})$, can be established in this way is founded on the existence of the relation $s = \tilde{s}(\rho, \varepsilon, \hat{\mathbf{C}})$. This relation is valid only under the local equilibrium hypothesis (cf. Edwards and Beris, 1998; Beris, 2001).

## 6. Summary and conclusions

This paper discusses the construction of noncanonical Poisson brackets for elastic solids and micromorphic elastic solids using the LE transformation method. Lagrangian canonical variables and Eulerian state variables are correlated by the Dirac delta function, and noncanonical Poisson brackets are obtained directly from the corresponding canonical brackets. The mass density, momentum density, entropy density, and the Green deformation density tensor are the state variables for elastic solids. The mass density, momentum density, entropy density, the deformation density tensor, the microdeformation density tensor, and the wryness density tensor are the state variables for micromorphic elastic solids. Specifying the Hamiltonian functionals in the Poisson brackets and considering the independence of the states variables allow the construction of the evolution equations for these variables.

Elastic solids use different kinds of elastic strain tensors, such as the Green deformation tensor, the Cauchy deformation tensor, and the higher-order deformation tensor, as strain variables in the Poisson bracket formalism. A common feature of these variables is that they are all composed of the deformation gradient. For micromorphic solids, the deformable directors and the deformation gradient constitute the basic units of the three elastic strain density measures. The evolution equations for these strain measures can be checked by taking the material time derivatives of these strains.

Furthermore, this paper discusses variable transformation technique, taking into account a nonconservative state variable. This paper also reconsiders elastic solids, in which the internal density $\varepsilon$ in the set of the state variables $(\rho, \mathbf{u}, \varepsilon, \hat{\mathbf{C}})$ is not conserved with the motion of material element. Employing the variable transformation, the functional derivatives associated with $\varepsilon$ can be transformed into those with the conserved quantities $\rho$, $s$, and $\hat{\mathbf{C}}$.

There are generally two approaches in standard continuum mechanics to finding the evolution of a system. The first approach is to give the balance equations, and the constitutive equations are followed by the thermodynamical theory (cf. Muschik et al., 2001). The boundary conditions in this approach can be obtained by applying the global-form balance equations to the boundary surface. The second approach is the presentation of a variational principle such as the method of virtual power (Germain, 1973; Maugin, 1980). Using the variational approach, the balance equations and boundary conditions can be simultaneously obtained in a single energy equation. In this paper, we adopt Hamilton's method as a third alternative. The benefit of Poisson bracket formalism is that the conservative part of a system's evolution can be obtained from a single equation, $\mathrm{d}F/\mathrm{d}t = \{F, H\}$, as soon as the Poisson bracket and the Hamiltonian functional $H$ are given. However, it should be noted that there are two differences between the method of noncanonical Poisson bracket and the other two methods. The first difference is that boundary conditions cannot be directly generated by this formalism. This is because all of the noncanonical Poisson brackets originate from their discrete counterparts, which are devised to formulate the evolution equations of a system. The second difference is that the method of noncanonical Poisson bracket mixes balance and constitutive equations. This is in contrast to the Coleman-Noll continuum thermomechanics approach for which constitutive functions are introduced after the formulation of the balance equations. The second difference is due to the fact that the method of noncanonical Poisson bracket first selects the state variables and then introduces the other quantities through constitutive relations. In the case of an elastic solid, the state variables are the mass density $\rho$, momentum density $\mathbf{u}$, entropy density $s$, and strain measure $\hat{\mathbf{C}}$. The Hamiltonian helps to determine the velocity $\mathbf{v}$, pressure $p$, and stress tensor $\boldsymbol{\tau}$.

This study makes it clear that the conservative part of the evolution of a system is associated with the motion of the system and the Poisson brackets of the continuous elastic systems – elastic solids, microcontinuum solids, and higher gradient continuum can be constructed with the help of LE transformation. As for the dissipative part of a system, which is beyond the scope of this study, the specific material property should be introduced to construct the dissipative bracket.

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