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中文摘要：

我們順利解決本計劃研究之問題，證明 2 維或 3 維的浸入緊緻特殊拉格拉奇子流形，若是只有孤立自交點，則一定是一系列嵌入特殊拉格拉奇子流形的極限。這個結果在更高維時，一般是不對，但我們證明在某些特殊情況，定理依然成立。這個研究成果，已經寫成學術文章，投稿於雜誌審稿中。在此計劃中，我們不但學會了一些很重要的分析方法，文章同時引起國內外這方面專家的注意及興趣，也繼續發現一些需要進一步發展及探討的問題。

關鍵詞：緊緻、特殊拉格拉奇子流形、浸入、嵌入

Abstract

In this project, we proof the following theorems:

Theorem *Every compact, connected, and immersed special Lagrangian submanifold, which has only isolated transversal self-intersection points in a compact 2 or 3 dimensional Calabi-Yau manifold, is the limit of a family of embedded special Lagrangian submanifolds.*

Remark: The theorem is not expected to hold in general when the dimension n is bigger than 3. Never the less, we show that if the tangent planes at the self-intersection point satisfy an angle condition, then the theorem holds for any dimension (as follows). One can also try to do the connected sum of two special Lagrangian submanifolds. However, it is easy to see that this will not work by simply counting the dimension of local deformations of a special Lagrangian submanifold.

Theorem *Suppose that L is a compact, connected, and immersed special Lagrangian submanifold in a compact n-dimensional Calabi-Yau manifold, $n > 3$. Moreover, assume that L has only isolated transversal self-intersection of two sheets and the two tangent planes at each intersection point satisfy the angle condition $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$. Then L is the limit of a family of embedded special Lagrangian submanifolds.*

Keywords: special Lagrangian, Calabi-Yau manifold.

Embedded Special Lagrangian Submanifolds in Calabi-Yau Manifolds

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In this paper, we will proof the following theorem:

Theorem *Every compact, connected, and immersed special Lagrangian submanifold, which has only isolated transversal self-intersection points in a compact 2 or 3 dimensional Calabi-Yau manifold, is the limit of a family of embedded special Lagrangian submanifolds.*

Remark: N.C. Leung points out that the theorem is not expected to hold in general when the dimension n is bigger than 3. We will discuss this point on section 2. Never the less, we show that if the tangent planes at the self-intersection point satisfy an angle condition, then the theorem holds for any dimension (see Theorem 4). One can also try to do the connected sum of two special Lagrangian submanifolds. However, it is easy to see that this will not work by simply counting the dimension of local deformations of a special Lagrangian submanifold [9]. This is pointed out by R. Schoen.

The method in this paper can also be used to deform a special Lagrangian submanifold with singularities. This will be discussed in a future paper. To make our presentation less messy, the constant C in the paper may change in different contexts. Its dependency will be specified whenever it is essential. This work is independent and different from A. Butscher's in [2] (see also [3]), but the technique is similar. We have had some discussions after the work was finished. Both the author and A. Butscher benefited from the discussions and made some changes in each other's work. We basically use the same setting as Butscher's in [2] and omit some computation to avoid repetition. However, there are still a few differences in the treatment. Some is due to the nature of the problem and some is for clarity and correctness. I also should point out that in [3], A. Butscher only considers $n \geq 3$. However, it is easy to see that the results quoted work for $n = 2$ and there is no such restriction in [2]. When the arguments depend on the dimension, I will indicate clearly and discuss separately.

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1 Preliminaries

Calibrated geometry and the notion of special Lagrangian submanifold were developed by R. Harvey and H. B. Lawson in [6]. We refer to their paper for a detailed discussion of this subject. The followings are some basic definitions:

Definition 1 *A closed, differential p-form ϕ on a Riemannian manifold N is called a calibration if its comass is 1. That is, $\phi(e_1, \cdots, e_p) \leq 1$ for any oriented, orthonormal p-frame on TN and the equality holds at some place.*

Definition 2 *A submanifold M of N is calibrated by ϕ, if $\phi|_M = dV_M$, where dV_M is the induced volume form on M.*

A very useful property of calibrated submanifolds is illustrated in the next proposition.

Proposition 1 *[6] If M is calibrated by ϕ, then M has the least volume among all representatives in its homology class.*

For instance, a p-dimensional complex submanifold in a Kähler manifold N is calibrated by $\frac{1}{p!}\omega^p$, where ω is the Kähler form on N, and hence is volume minimizing. R. Harvey and H. B. Lawson showed that $Re\, dZ$ in R^{2n}, where $dZ = dz_1 \wedge \cdots \wedge dz_n$, is a calibration. The corresponding calibrated submanifolds are called special Lagrangian. The form $Re\,(e^{i\theta}dZ)$, where θ is a constant, is also a calibration, and its corresponding calibrated submanifolds are called special Lagrangian of phase θ. In a Calabi-Yau manifold N, there exists a parallel holomorphic $(n, 0)$ form Ω which is of unit length. The n-form $Re\,\Omega$ is a calibration and a Lagrangian submanifold in N is called special Lagrangian if it is calibrated by $Re\,\Omega$. Recall that a Lagrangian submanifold is a real n-dimensional submanifold on which the restriction of ω vanishes, where $2n$ is the real dimension of N.

G. Lawlor [8] modified an example of R. Harvey and H. B. Lawson [6] and defined the following submanifolds, which will be called Lawlor necks in this paper:

Assume that $a_1, \cdots, a_n$, $n \geq 2$, are n positive real numbers and $a = (a_1, \cdots, a_n)$. Set

$$\theta_k(a, \mu) = \int_0^\mu \frac{a_k\, ds}{(1 + a_k s^2)\sqrt{P(s)}} \quad \text{for} \quad \mu \geq 0,$$

where

$$P(s) = \frac{(1 + a_1 s^2) \cdots (1 + a_n s^2) - 1}{s^2}.$$

One can extend $\theta_k(a,\mu)$ to negative μ by $\theta_k(a,-\mu) = -\theta_k(a,\mu)$. Define $\Phi_a : R \times S^{n-1} \to R^{2n}$ by

$$\Phi_a(\mu, x_1, \cdots, x_n) = (z_1 x_1, \cdots, z_n x_n),$$

where

$$x_1^2 + \cdots + x_n^2 = 1 \quad \text{and} \quad z_k = \sqrt{\frac{1}{a_k} + \mu^2}\, e^{i\theta_k(a,\mu)}.$$

Note that

$$\Phi_{\frac{a}{t^2}}(\mu, x_1, \cdots, x_n) = t\,\Phi_a(\frac{\mu}{t}, x_1, \cdots, x_n) \quad \text{for} \quad t > 0.$$

Hence we can assume $\inf_{k=1,\cdots,n} a_k = 1$. Denote

$$\theta_k(a) = \int_0^\infty \frac{a_k\, ds}{(1 + a_k s^2)\sqrt{P(s)}}, \quad \text{for} \quad k = 1, \cdots, n.$$

One can prove that $\theta_1(a) + \cdots + \theta_n(a) = \frac{\pi}{2}$. By an argument in [8], there is a bijection between positive $\theta_1, \cdots, \theta_n$ satisfying $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$ and $a_1, \cdots, a_n$ satisfying $\inf_{k=1,\cdots,n} a_k = 1$. Moreover, G. Lawlor proved that the image of Φ_a, which is denoted by M_a, is embedded, calibrated by $Im\, dZ$, and asymptotic to P_θ and $P_{-\theta}$, where P_θ is the plane

$$P_\theta = \{\, (t_1 e^{i\theta_1(a)}, \cdots, t_n e^{i\theta_n(a)}) : t_j \in R,\ j = 1, \cdots, n \,\}.$$

Note that M_a, P_θ and $-P_{-\theta}$ are special Lagrangian of phase $\frac{\pi}{2}$. By moving these spaces by a phase, we can always make them special Lagrangian. We thus will not specify the phase any more. But when we talk about special Lagrangian submanifolds in this paper, we do mean that they are calibrated by the *same* form, i.e. they are of the same phase. (see [2], [5], [7], [8]).

A. Butscher [2] studies carefully the asymptotic behavior of the above Lawlor neck. We summarize some of his results here for completeness. He proves that $|\theta_k(a,\mu) - \theta_k(a)| \leq \frac{1}{n|\mu|^n}$. Moreover, there exists a positive real number R_0 so that $M_a \setminus B_{R_0}(0)$ can be written as the graph of the gradient of a function

$$\Psi : P_i \setminus B_{R_0}(0) \to R, \quad i = 1, 2.$$

Here we split R^{2n} as $P_i \times P_i^\perp$ to write the graph. The function Ψ has the properties that

$$|\Psi(x)| \leq \frac{C}{|x|^{n-2}}, \quad |\nabla\Psi(x)| \leq \frac{C}{|x|^{n-1}}, \quad |\nabla^2\Psi(x)| \leq \frac{C}{|x|^n},$$

$$|\nabla^3\Psi(x)| \leq \frac{C}{|x|^{n+1}}, \quad \text{and} \quad |\nabla^4\Psi(x)| \leq \frac{C}{|x|^{n+2}}$$

for $x \in P_i$ with $|x| \geq R_0$. The constant C depends only on a and n. The scaled manifold

$$\varepsilon(M_a \setminus B_{R_0}(0)) = \varepsilon M_a \setminus B_{\varepsilon R_0}(0), \quad \varepsilon > 0,$$

is the graph of the gradient of a function

$$\Psi_\varepsilon : P_i \setminus B_{\varepsilon R_0}(0) \to R, \quad i = 1, 2.$$

The function $\Psi_\varepsilon(x) = \varepsilon^2\Psi(\frac{x}{\varepsilon})$ satisfies

$$|\Psi_\varepsilon(x)| \leq \frac{C\varepsilon^n}{|x|^{n-2}}, \quad |\nabla\Psi_\varepsilon(x)| \leq \frac{C\varepsilon^n}{|x|^{n-1}}, \quad |\nabla^2\Psi_\varepsilon(x)| \leq \frac{C\varepsilon^n}{|x|^n},$$

$$|\nabla^3\Psi_\varepsilon(x)| \leq \frac{C\varepsilon^n}{|x|^{n+1}}, \quad \text{and} \quad |\nabla^4\Psi_\varepsilon(x)| \leq \frac{C\varepsilon^n}{|x|^{n+2}}$$

for $x \in P_i$ with $|x| \geq \varepsilon R_0$.

2 Local model

Assume that p is a self-intersection point and locally it is the transversal intersection of two sheets. We would like to use the Lawlor neck as a local model. So we first need to find a Lawlor neck which is asymptotic to the two tangent planes at p. Then cut off a small ball at p on each sheet, glue in a scaled Lawlor neck, and connect it to the original submanifold outside the balls. However, there is a condition $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$ for the planes which the Lawlor neck can be asymptotic to. In this section, we will discuss how the condition affects the application. In particular, we show that this condition is always satisfied for our situation in dimension 2 and 3, but it is not true when $n \geq 4$. Hence when $n \geq 4$, we need to add the angle condition $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$ in Theorem 4. We will also discuss why the assertion cannot hold in general if $n \geq 4$.

Recall that a Lagrangian plane (which is always assumed to contain the origin) in R^{2n} is the image of the real $x_1, \cdots, x_n$ plane by a linear transformation $A \in U(n)$. Thus the set of Lagrangian planes can be identified with $U(n)/SO(n)$ [6]. Given a pair of Lagrangian planes P_1 and P_2, we claim that in suitable coordinates, one can make P_1 to be the $x_1, \cdots, x_n$ plane and P_2 to be of the form $\{ (t_1 e^{i\omega_1}, \cdots, t_n e^{i\omega_n}) : t_j \in R,\ j = 1, \cdots, n \}$. This is because the Lie algebra $u(n)$ of $U(n)$ is decomposed into the direct sum of S and $so(n)$, where S is the set of pure imaginary symmetric matrices and $so(n)$ is the set of real skew symmetric matrices. The subalgebra S and $so(n)$ corresponds to the -1 eigenspace and 1 eigenspace of the involution $\tau : u(n) \to u(n)$ respectively, where $\tau(y) = -y^t$. Since one can diagonalize a real symmetric matrix, it follows that $S = \cup\, k\, T\, k^{-1}$, where T is a pure imaginary diagonal matrix and k is in $SO(n)$. The symmetric space $U(n)/SO(n)$ is exactly $\exp S$. The claim is thus proved. We like to thank C.L. Terng's discussion on this observation. Furthermore, if we denote $|\omega_j|$ by β_j, we can assume that

$$0 \le \beta_1 \le \beta_2 \le \cdots \le \beta_{n-1} \le \frac{\pi}{2} \quad \text{and} \quad \beta_{n-1} \le \beta_n \le \pi - \beta_{n-1}.$$

They are exactly the characterizing angles between P_1 and P_2 as defined in [8]. Note that one has $0 \le \sum_{j=1}^{n} \beta_j \le \frac{n\pi}{2}$. Now Suppose that P_1 and P_2 are two special Lagrangian planes which intersect only at the origin. Then $\beta_1 > 0$ and when $n = 2$ or 3, one has $\sum_{j=1}^{n} \omega_j = 0$. It implies that $\beta_1 = \beta_2$ in the case $n = 2$ and $\beta_1 + \beta_2 = \beta_3$ in the case $n = 3$. If we change the orientation on P_2, which is denoted by $-P_2$, then its characterizing angles with P_1 satisfy $\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 = \pi$ in the case $n = 3$. Change the coordinates such that $P_1 = P_\theta$ and $-P_2 = P_{-\theta}$, where $\theta = (\frac{\bar{\beta}_1}{2}, \frac{\bar{\beta}_1}{2}, \frac{\bar{\beta}_1}{2})$. We thus can find a Lawlor neck which is asymptotic to P_1 and $-P_2$. When $n = 2$, one can also obtain the same conclusion. But this is not true when $n \ge 4$. For example, the $x_1, \cdots, x_4$ plane and $y_1, \cdots, y_4$ plane in R^8 are both calibrated by $Re\, dZ$ and contain the origin. However, all the characterizing angles between these two planes are $\frac{\pi}{2}$. Hence the sum of the angles is 2π and there does not exist a Lawlor neck which is asymptotic to the $x_1, \cdots, x_4$ plane and $y_1, \cdots, y_4$ plane.

The geometric obstruction for finding a Lawlor neck in $n \ge 4$ comes from the followings: There is an angle criterion which says that the nonzero sum (oriented union) $P_1 + P_2$ is area minimizing if and only if the characterizing angles between them satisfy the inequality $\beta_n \le \beta_1 + \cdots + \beta_{n-1}$. (See [5], [8], [11].) Suppose that P_1 and P_2 are two special Lagrangian planes. By the property of calibration, we know that $P_1 + P_2$ is area minimizing. Assume that their characterizing angles satisfy $\beta_n < \beta_1 + \cdots + \beta_{n-1}$ and there exists a special Lagrangian L asymptotic to P_1 and P_2. A Lawlor neck has the property that it is the union of compact hypersurfaces in a family of Lagrangian planes. Assume that L has the same property. We first find two Lagrangian planes P_1' and P_2' near P_1 and P_2, which are not special Lagrangian and whose characterizing angles $\{\beta_j'\}$, $j = 1, \cdots, n$, still satisfy $\beta_n' < \beta_1' + \cdots + \beta_{n-1}'$. If the intersection of L and $P_1' + P_2'$ is a compact hypersurface in $P_1' + P_2'$, then the intersection will be the boundary of a compact portion of L and also be the boundary of a compact subset $E_1 + E_2$ in $P_1' + P_2'$. By the special Lagrangian condition on L and applying the angle criterion to P_1' and P_2', we know that both sets are volume minimizing with the same boundary. It follows that they are calibrated by the same form, which is a contradiction because P_1' and P_2' are chosen to be not special Lagrangian. Thus we cannot have a Lawlor neck to approximate such a pair. Can we find local models of different nature to resolve the isolated self-intersection point in general? The answer is very likely still no. This is observed by N.C. Leung and the reason will be explained in next paragraph.

One can consider complex Lagrangian submanifolds in a hyperkähler manifold. Recall that there is a S^2 family of compatible complex structures in a hyperkåhler manifold. A complex Lagrangian submanifold is a complex submanifold with respect to one of the compatible complex structures, and is special Lagrangian with respect to another compatible complex structure. By the property of calibration, any subspace (even singular) which presents the homology class of a complex Lagrangian submanifold and is volume minimizing in the class, must be calibrated by the same form, and hence also be complex Lagrangian. Thus all special Lagrangian submanifolds in the homology class of a complex Lagrangian submanifold are complex Lagrangian. It means that we must do the connected sum in the complex category. This is known to be impossible in general when the complex dimension is bigger than one. In particular, when we add a handle ($\cong S^{n-1} \times R$) to the original submanifold, it will increase the dimension of the first homology group by one. If the original submanifold is complex Lagrangian, then this new topology cannot be complex Lagrangian because it does not satisfy a necessary condition of a Kähler manifold (the first homology group is even dimension). The upshot for the above observation is that either the theorem does not hold in general when $n > 3$, or one cannot find a compact, connected, complex Lagrangian submanifold in a hyperkähler manifold which has only isolated transversal self-intersection points.

3 Approximate submanifolds

Suppose that L is a compact, connected, and immersed special Lagrangian which has only isolated transversal self-intersection points in a compact n-dimensional Calabi-Yau manifold N, where $n \geq 2$. Without loss of generality, we can assume that there is only one self-intersection point p on L and locally it is the transversal intersection of two sheets of L. In a small neighborhood of p, the metric in N is equivalent to the Euclidean metric in R^{2n}. Thus for simplicity, the distance and norm in the following construction of approximate submanifolds in this section are all with respect to the Euclidean metric unless specified explicitly. Assume that the ball of radius r_0 at p in N, which is denoted by B_{r_0}, is both a Darboux neighborhood and a normal neighborhood near p. That is, we can choose coordinates $x_1, \cdots, x_n, y_1, \cdots, y_n$ such that p is the origin and for $q \in B_{r_0}$:

1. the Kähler form satisfies $\omega(q) = \sum_{i=1}^{i=n} dx_i \wedge dy_i$,
2. the metric ds^2 satisfies $|ds^2(q) - ds_0^2| \leq C|q|^2$, where $ds_0^2 = dx_i^2 + dy_i^2$,

3. the complex structure J satisfies $|J(q) - J_0| \le C|q|^2$, where J_0 is the standard complex structure in R^{2n},

4. the parallel holomorphic $(n, 0)$ form Ω satisfies $|\Omega(q) - dZ| \le C|q|^2$, where $dZ = dz_1 \wedge \cdots \wedge dz_n$ and $z_j = x_j + iy_j$, $j = 1, \cdots, n$.

Assume that the two tangent planes at p are P_1 and P_2 respectively. Then P_1 and P_2 are special Lagrangian and $L \cap B_{r_0}$ is Lagrangian with respect to the standard symplectic structure in R^{2n}. It follows that $L \cap B_{r_1}$ can be written as the graph of the gradient of a function

$$\psi : P_i \cap B_{r_1} \to R, \quad i = 1, 2,$$

for some $r_1 < r_0$. Moreover, we can choose ψ satisfying

$$|\psi(x)| \le K|x|^3, \quad |\nabla\psi(x)| \le K|x|^2, \quad |\nabla^2\psi(x)| \le K|x|, \quad |\nabla^3\psi(x)| \le K,$$

and $|\nabla^4\psi(x)| \le C_K$ for $x \in P_i$ with $|x| \le r_1$, where K is a constant depending on the curvature of L in B_{r_1} and C_K depends on the derivative of K. There exists a Lawlor neck in R^{2n} with suitable $a = (a_1, \cdots, a_n)$ that is asymptotic to P_1 and P_2, when $P_1 = P_\theta$, $-P_2 = P_{-\theta}$, and $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$. By the discussion in last section, this condition is always satisfied when $n = 2$ or 3. From now on, we focus on the situations in $n \ge 2$ where we can find a Lawlor neck to approximate the pair of tangent planes. We first scale the Lawlor neck M_a by ε. Outside a small ball $B_{\varepsilon R_0}$, the manifold εM_a can be written as the graph of the gradient of Ψ_ε over P_1 and P_2. To match ψ and Ψ_ε together, A. Butscher has the following estimate:

Lemma 1 *[2] There exist constants α_0 and c depending on L only, such that if $0 < \alpha < \alpha_0$, $r = \frac{\alpha}{K}$, and $\varepsilon < c\,\alpha^{1+\frac{1}{n}}$, then $|\nabla^2\psi(x)| \le \alpha$ and $|\nabla^2\Psi_\varepsilon(x)| \le \alpha$ for any $x \in P_i$ with $\frac{r}{2} \le |x| \le r$.*

Roughly speaking, we want the approximate submanifolds to be εM_a near p, and to be L outside a neighborhood of p. We also want to require the interpolation to be Lagrangian. Recall that the graph of the gradient of a function on a Lagrangian plane is always Lagrangian. Hence the following combination of ψ and Ψ_ε is a good candidate for our purpose. First, assume that η is a smooth function on R^n satisfying $\eta(x) \equiv 1$ when $|x| \leq \frac{r}{2}$ and $\eta(x) \equiv 0$ when $|x| \geq \frac{3r}{4}$. Moreover, it also satisfies

$$0 \leq \eta(x) \leq 1, \quad |\nabla \eta(x)| \leq \frac{C}{r}, \quad |\nabla^2 \eta(x)| \leq \frac{C}{r^2}, \quad |\nabla^3 \eta(x)| \leq \frac{C}{r^3},$$

and $|\nabla^4 \eta(x)| \leq \dfrac{C}{r^4}$ for every x. Next define the interpolation to be the graph

$$T_i = \{(x, \nabla[(1-\eta)\psi + \eta\Psi_\varepsilon](x)) \in P_i \times P_i^\perp, \ \frac{r}{2} \leq |x| \leq r\}, \quad i = 1, 2.$$

It is easy to check that

$$|\nabla[(1-\eta)\psi + \eta\Psi_\varepsilon]| < Cr^2, \quad \text{for} \ \ \frac{r}{2} \leq |x| \leq r.$$

Denote

$$B'_r = B_r^{P_1} \times R^n \cap B_r^{P_2} \times R^n \subset P_1 \times P_1^\perp \cap P_2 \times P_2^\perp,$$

where $B_r^{P_i} = B_r \cap P_i, \ i = 1, 2$. We then define the approximate submanifold to be

$$M_\alpha = (\varepsilon M_a \cap B'_{\frac{r}{2}}) \cup T_1 \cup T_2 \cup (L \setminus B'_r).$$

The approximate submanifold is Lagrangian and satisfies the following properties:

$$\begin{cases} |H(q)| \leq C\,|q| & \text{for} \ \ q \in \varepsilon M_a \cap B'_{\frac{r}{2}} \\ |H(q)| \leq C & \text{for} \ \ q \in T_1 \cup T_2 \\ |H(q)| = 0 & \text{for} \ \ q \in L \setminus B'_r \end{cases},$$

where H is the mean curvature vector of M_α in N. One also has

$$\begin{cases} |Im\,\Omega|_{M_\alpha}(q)| \leq C\,|q|^2 & \text{for} \quad q \in \varepsilon M_\alpha \cap B'_{\frac{r}{2}} \\ |Im\,\Omega|_{M_\alpha}(q)| \leq C\,\alpha & \text{for} \quad q \in T_1 \cup T_2 \\ |Im\,\Omega|_{M_\alpha}(q)| = 0 & \text{for} \quad q \in L \setminus B'_r \end{cases}.$$

The situation in R^{2n} is computed in [2]. Because $|\Omega(q) - dZ| \leq C|q|^2$ and $|H(q) - H_0(q)| \leq C|q|$, where H_0 is the mean curvature vector of $M_\alpha \cap B'_r$ in R^{2n} with the Euclidean metric, we thus obtain the above estimates.

We now investigate some properties of the approximate submanifolds M_α. From the construction, it is easy to see that they are embedded Lagrangian submanifolds with area uniformly bounded from above and below. Moreover, because M_α converges to L in Hausdorff distance, we have $\int_{M_\alpha} Im\,\Omega = 0$. For a submanifold $M^n \subset R^l$, J. H. Michael and L. Simon [10] proved the Sobolev inequality:

$$\left(\int_M h^{\frac{n}{n-1}}\,dV\right)^{\frac{n-1}{n}} \leq C(n) \int_M (|\nabla^M h| + h|\bar{H}|)\,dV,$$

where $h > 0$ is a C^1 function on M with compact support and $\bar{H}$ is the mean curvature of M in R^l. By embedding N isometrically in R^l, the corresponding mean curvature $\bar{H}_\alpha$ of M_α in R^l is uniformly bounded. Thus the Sobolev constant on M_α is uniformly bounded. (See the discussion in the Appendix.) We then can prove the following estimate concerning the first eigenvalue.

Theorem 1 *When α is small enough, the first eigenvalue $\lambda_1(M_\alpha)$ for the Laplace operator on M_α is bounded below by $\frac{1}{4}\lambda_1(L)$.*

Proof. Suppose that f_α is the first eigenfuction for the Laplace operator on M_α satisfying

$$\int_{M_\alpha} f_\alpha \, dV = 0, \quad \int_{M_\alpha} f_\alpha^2 \, dV = 1, \quad \textit{and} \quad \int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2 \, dV = \lambda_1(M_\alpha).$$

Because $\Delta_{M_\alpha} f_\alpha = -\lambda_1(M_\alpha) f_\alpha$, one has

$$\Delta_{M_\alpha} f_\alpha^2 = -2\lambda_1(M_\alpha) f_\alpha^2 + |\nabla^{M_\alpha} f_\alpha|^2 \geq -2\lambda_1(M_\alpha) f_\alpha^2.$$

Assume that the theorem is not true. Then there exists a subsequence $\{\alpha_j\}$ which tends to zero, such that $\lambda_1(M_{\alpha_j}) < \frac{1}{4}\lambda_1(L)$. By Lemma 5 in the Appendix, it follows that

$$f_{\alpha_j}^2 \leq C \int_{M_\alpha} f_{\alpha_j}^2 \, dV \leq C.$$

Since $\lambda_1(M_{\alpha_j})$ and $Vol(M_{\alpha_j})$ are bounded uniformly, the constant C is independent of j.

When $n > 2$, let φ_δ be a nonnegative function in N satisfying $\varphi_\delta \equiv 1$ on $N \setminus B_\delta$, $\varphi_\delta \equiv 0$ on $B_{\frac{\delta}{2}}$, $0 \leq \varphi_\delta \leq 1$ on $B_\delta \setminus B_{\frac{\delta}{2}}$, and $|\nabla^N \varphi_\delta| \leq \frac{3}{\delta}$. A direct computation shows that

$$\begin{aligned}
& \textstyle\int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}|^2 \, dV \\
= \; & \textstyle\int_{M_{\alpha_j}} (|\nabla^{M_{\alpha_j}} \varphi_\delta|^2 f_{\alpha_j}^2 + \varphi_\delta^2 |\nabla^{M_{\alpha_j}} f_{\alpha_j}|^2 + 2\varphi_\delta f_{\alpha_j} \nabla^{M_{\alpha_j}} \varphi_\delta \cdot \nabla^{M_{\alpha_j}} f_{\alpha_j}) \, dV \\
\leq \; & 2 \textstyle\int_{M_{\alpha_j}} \varphi_\delta^2 |\nabla^{M_{\alpha_j}} f_{\alpha_j}|^2 \, dV + 2 \int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta|^2 f_{\alpha_j}^2 \, dV \\
\leq \; & 2 \textstyle\int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} f_{\alpha_j}|^2 \, dV + 2 \int_{M_{\alpha_j} \cap B_\delta \setminus B_{\frac{\delta}{2}}} |\nabla^{M_{\alpha_j}} \varphi_\delta|^2 f_{\alpha_j}^2 \, dV \\
\leq \; & 2\lambda_1(M_{\alpha_j}) + \tfrac{C}{\delta^2} Vol(M_{\alpha_j} \cap B_\delta \setminus B_{\frac{\delta}{2}}) \\
\leq \; & 2\lambda_1(M_{\alpha_j}) + C\delta^{n-2}.
\end{aligned}$$

In the above estimates, we use $|\nabla^{M_{\alpha_j}}\varphi_\delta| \leq |\nabla^N \varphi_\delta|$ and $Vol(M_{\alpha_j} \cap B_\delta) \leq C\delta^n$ by monotonicity formula [14]. We also have

$$\int_{M_{\alpha_j}} (\varphi_\delta f_{\alpha_j})^2\, dV \geq 1 - \int_{M_{\alpha_j} \cap B_\delta} f_{\alpha_j}^2\, dV \geq 1 - C\, Vol(M_{\alpha_j} \cap B_\delta) \geq 1 - C\delta^n,$$

and

$$\Big(\int_{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}\, dV\Big)^2 = \Big(\int_{M_{\alpha_j} \cap B_\delta} (1 - \varphi_\delta) f_{\alpha_j}\, dV\Big)^2 \leq C\, Vol(M_{\alpha_j} \cap B_\delta) \leq C\delta^n.$$

Recall that M_{α_j} is the same as L in $N \setminus B_{\frac{\delta}{2}}$ for $\alpha_j \leq \frac{K\delta}{2}$. Therefore,

$$\begin{aligned} \frac{\int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}|^2\, dV}{\int_{M_{\alpha_j}} (\varphi_\delta f_{\alpha_j})^2\, dV - (\int_{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}\, dV)^2} &= \frac{\int_L |\nabla^L \varphi_\delta f_{\alpha_j}|^2\, dV}{\int_L (\varphi_\delta f_{\alpha_j})^2\, dV - (\int_L \varphi_\delta f_{\alpha_j}\, dV)^2} \\ &\geq \lambda_1(L). \end{aligned}$$

On the other hand, it follows from the above estimates that

$$\frac{\int_{M_{\alpha_j}} |\nabla^{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}|^2\, dV}{\int_{M_{\alpha_j}} (\varphi_\delta f_{\alpha_j})^2\, dV - (\int_{M_{\alpha_j}} \varphi_\delta f_{\alpha_j}\, dV)^2} \leq \frac{2\lambda_1(M_{\alpha_j}) + C\delta^{n-2}}{1 - C\delta^n}.$$

Choose δ small enough so that $C\delta^{n-2} < \min(\frac{\lambda_1(L)}{4}, \frac{1}{4})$. Then by combining the two inequalities, one gets $\lambda_1(M_{\alpha_j}) > \frac{1}{4}\lambda_1(L)$ when $\alpha_j \leq \frac{K\delta}{2}$, which is a contradiction. Thus the theorem is proved in the case $n > 2$.

When $n = 2$, we need to modify the function φ_δ as follows:

$$\varphi_\delta(x) = \begin{cases} 0 & |x| < \delta^2 \\ \left(\dfrac{\log \frac{|x|}{\delta^2}}{\log \frac{1}{\delta}}\right) & \delta^2 \leq |x| \leq \delta\ . \\ 1 & |x| > \delta \end{cases}$$

A direct computation gives

$$\int_{M_{\alpha_j}\cap B_\delta\setminus B_{\delta^2}} |\nabla^{M_{\alpha_j}}\varphi_\delta|^2\, dV \le \frac{C}{|\log\delta|}.$$

Recall that M_{α_j} is the same as L in $N\setminus B_{\delta^2}$ for $\alpha_j \le K\delta^2$. Similar arguments as in the case $n > 2$ lead to

$$\lambda_1(L) \le \frac{2\lambda_1(M_{\alpha_j}) + \frac{C}{|\log\delta|}}{1 - C\delta^n}.$$

Choose δ small enough so that $\frac{C}{|\log\delta|} < \frac{\lambda_1(L)}{4}$ and $C\delta^n \le \frac{1}{4}$. One will get $\lambda_1(M_{\alpha_j}) > \frac{1}{4}\lambda_1(L)$ when $\alpha_j \le K\delta^2$, which is a contradiction. This completes the proof of the theorem.

Q.E.D.

Remark: It is easy to see from the proof that the lower bound can be improved and the estimate also works for other singularities. Because the submanifold L is compact and connected, its first eigenvalue $\lambda_1(L)$ is a positive number.

4 Perturbation

There exists a constant c_1 such that the exponential map from the normal bundle $T^\perp M_\alpha$ into N is an embedding in the $c_1\varepsilon$ neighborhood of M_α. Choose a smooth function η_α such that $\eta_\alpha(s) \equiv 1$ when $|s| \le \frac{c_1\varepsilon}{2}$, and $\eta_\alpha(s) \equiv 0$ when $|s| \ge \frac{3c_1\varepsilon}{4}$. Moreover, it also satisfies

$$0 \le \eta_\alpha(s) \le 1, \quad |\nabla\eta_\alpha(s)| \le \frac{C}{\varepsilon}, \quad |\nabla^2\eta_\alpha(s)| \le \frac{C}{\varepsilon^2}, \quad \text{and} \quad |\nabla^3\eta_\alpha(s)| \le \frac{C}{\varepsilon^3}$$

for every s. Given a $C^{2,\beta}$ function u on M_α, $0 \le \beta \le 1$, we can extend it into a $C^{2,\beta}$ function U on N by defining $U(\exp(x,v)) = \eta_\alpha(|v|)u(x)$ for $x \in M_\alpha$ and $v \in T_x^\perp M_\alpha$. We then solve the Hamiltonian flow:

$$\frac{\partial \phi(t,q)}{\partial t} = J\nabla^N U(\phi(t,q)) \quad \text{and} \quad \phi(0,q) = q \quad \text{for} \quad q \in N.$$

There exists a unique solution $C^{1,\beta}$ for small t. Note that if $\phi_U(t,q)$ is a solution defined by U, then $\phi_U(st,q)$ is a solution defined by sU. Denote

$$\phi_u(x) = \phi_U(1,x) \quad \text{for} \quad x \in M_\alpha.$$

The map ϕ_u can be defined for u in a neighborhood of the zero function. In particular, it is defined when $\|(\nabla^N)^2 U\|_{0,N} < 1$. Because $\phi_U(1,q)$ is a symplectic map, the image $\phi_u(M_\alpha)$ is Lagrangian. Moreover, the family of maps $\phi_{tu}, 0 \le t \le 1$, is a homotopy between ϕ_u and ϕ_0. Define a $C^{0,\beta}$ function on M_α by $\mathcal{F}_\alpha(u)(x) = \star\phi_u^*(Im\,\Omega)(x)$, where $\star$ is the star operator with respect to the induced metric on M_α. If we can find a function u such that ϕ_u is an embedding and satisfies $\mathcal{F}_\alpha(u) = 0$, then $\phi_u(M_\alpha)$ will be an embedded special Lagrangian submanifold. Therefore, the goal is to find the zero set of $\mathcal{F}_\alpha$. The differential of $\mathcal{F}_\alpha$ at the zero function is

$$D\mathcal{F}_\alpha(0)(u) = \star\phi_0^*(d\, i_{J\nabla^N U} Im\,\Omega)(x).$$

Because M_α is Lagrangian, there exists a function $\theta(x)$ (mod 2π) on M_α, such that

$$\Omega|_{M_\alpha} = e^{i\theta(x)}\,\omega_1 \wedge \cdots \wedge \omega_n,$$

where $\omega_1 \cdots \omega_n$ is a local orthonormal basis on the cotangent bundle $T^* M_\alpha$ [13]. Note that

$$\begin{aligned} &\phi_0^*(i_{J\nabla^N U} Im\,\Omega) \\ &= Im \textstyle\sum_{\beta=1}^n e^{i\theta(x)} \left[i\,(J\nabla^N U)^{n+\beta} \omega_1 \wedge \cdots \wedge \check{\omega}_\beta \cdots \wedge \omega_n \right. \\ &\qquad\qquad \left. + (J\nabla^N U)^\beta \omega_1 \wedge \cdots \wedge \check{\omega}_\beta \cdots \wedge \omega_n \right] \\ &= \cos\theta(x) \star du, \end{aligned}$$

where $\check{\omega}_\beta$ means that ω_β does not appear and the last equality follows from the definition of U. Because $H = J\nabla^{M_\alpha}\theta$ [13], we thus have

$$D\mathcal{F}_\alpha(0)(u) = \cos\theta(x)\Delta_{M_\alpha} u - \sin\theta(x) < H, J\nabla^{M_\alpha} u > .$$

It will be denoted by $\mathcal{L}u$ for simplicity. Because $|\sin\theta| = |\phi_0^*(Im\,\Omega)| \leq C\,\alpha$, it follows that $|\theta(x)| \leq C\,\alpha$. One then can show

Proposition 2 *When α is small, the operator $\mathcal{L}$ is an elliptic operator and its kernel consists of the constant functions. Moreover, the first eigenvalue $\lambda_1(M_\alpha, \mathcal{L})$ for the operator $\mathcal{L}$ on M_α has a uniform positive lower bound.*

Proof. When α is small, $\cos\theta(x)$ is close to 1 and hence $\mathcal{L}$ is an elliptic operator. Constants are clearly in the kernel of $\mathcal{L}$. Suppose that $\mathcal{L}u = 0$ and $\int_{M_\alpha} u\,dV = 0$ (i.e., normalize u such that it is perpendicular to constants). Because $\cos\theta(x)$ is nonzero, $\mathcal{L}u = 0$ is equivalent to

$$\Delta_{M_\alpha} u - \tan\theta(x) < H, J\nabla^{M_\alpha} u >= 0.$$

Multiply u on both sides, and integrate over M_α. We then get

$$\begin{aligned}
\textstyle\int_{M_\alpha} |\nabla^{M_\alpha} u|^2\, dV &= -\textstyle\int_{M_\alpha} u \Delta_{M_\alpha} u\, dV \\
&= -\textstyle\int_{M_\alpha} u \tan\theta(x) < H, J\nabla^{M_\alpha} u > dV \\
&\leq C \max_{M_\alpha} (\tan\theta(x)) \textstyle\int_{M_\alpha} |u||\nabla^{M_\alpha} u|\, dV \\
&\leq C \max_{M_\alpha} (\tan\theta(x)) (\textstyle\int_{M_\alpha} |u|^2\, dV)^{\frac{1}{2}} (\textstyle\int_{M_\alpha} |\nabla^{M_\alpha} u|^2\, dV)^{\frac{1}{2}}.
\end{aligned}$$

We use the fact that $|H|$ is bounded in the first inequality above. When α tends to zero, the number $\max_{M_\alpha} (\tan\theta(x))$ also tends to zero. On the other hand, the first eigenvalue $\lambda_1(M_\alpha)$ for the Laplace operator on M_α is bounded below by $\frac{1}{4}\lambda_1(L)$ from Theorem 1. It implies that u is identically zero when α is sufficiently small. Thus the kernel of $\mathcal{L}$ consists of only constant solutions.

We now estimate $\lambda_1(M_\alpha, \mathcal{L})$. Suppose that f_α is the first eigenfuction of $\mathcal{L}$, which satisfies

$$\int_{M_\alpha} f_\alpha\, dV = 0, \quad \int_{M_\alpha} f_\alpha^2\, dV = 1, \quad \textit{and} \quad \mathcal{L} f_\alpha = -\lambda_1(M_\alpha, \mathcal{L}) f_\alpha.$$

By choosing α small enough, we can assume that $\cos\theta(x) > \frac{1}{2}$. Multiply both sides of the equation by $-\dfrac{f_\alpha}{\cos\theta(x)}$ and integrate over M_α. We have

$$\begin{aligned}
&-\textstyle\int_{M_\alpha} f_\alpha \Delta_{M_\alpha} f_\alpha\, dV + \textstyle\int_{M_\alpha} f_\alpha \tan\theta(x) < H, J\nabla^{M_\alpha} f_\alpha > dV \\
= \quad & \lambda_1(M_\alpha, \mathcal{L}) \int_{M_\alpha} \frac{f_\alpha^2}{\cos\theta(x)}\, dV.
\end{aligned}$$

A direct computation shows that

$$\begin{aligned}
& |\textstyle\int_{M_\alpha} f_\alpha \tan\theta(x) < H, J\nabla^{M_\alpha} f_\alpha > dV| \\
\leq \quad & C \max_{M_\alpha} (\tan\theta(x)) (\textstyle\int_{M_\alpha} |f_\alpha|^2\, dV)^{\frac{1}{2}} (\textstyle\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2\, dV)^{\frac{1}{2}} \\
\leq \quad & C \max_{M_\alpha} (\tan\theta(x)) (\textstyle\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2\, dV)^{\frac{1}{2}}.
\end{aligned}$$

Plugging this into the above equality, we will get

$$
\begin{aligned}
& 2\lambda_1(M_\alpha, \mathcal{L}) \\
\geq\ & \lambda_1(M_\alpha, \mathcal{L}) \textstyle\int_{M_\alpha} \frac{f_\alpha^2}{\cos\theta(x)}\, dV \\
\geq\ & \textstyle\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2\, dV - C \max\limits_{M_\alpha} (\tan\theta(x)) (\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2\, dV)^{\frac{1}{2}} \\
\geq\ & \tfrac{1}{2} \textstyle\int_{M_\alpha} |\nabla^{M_\alpha} f_\alpha|^2\, dV \\
\geq\ & \tfrac{1}{2}\lambda_1(M_\alpha),
\end{aligned}
$$

when α is sufficiently small. This completes the proof of the proposition.

Q.E.D.

5 The theorem

We first set some notation which will be used in the rest of this paper. Assume that u is a function on M_α. We denote

$$\|u\|_{0,M_\alpha} = \sup_{M_\alpha} |u|,$$

$$[u]_{\beta,M_\alpha} = \sup_{x,x' \in M_\alpha} \frac{|u(x) - u(x')|}{dist(x,x')^\beta}, \quad 0 < \beta < 1,$$

and

$$\|u\|_{L^p} = \left(\int_{M_\alpha} u^p\, dV\right)^{\frac{1}{p}}.$$

We can embed N isometrically into R^l and set

$$[(\nabla^{M_\alpha})^k u]_{\beta,M_\alpha} = \sup_{x,x' \in M_\alpha} \frac{|(\nabla^{M_\alpha})^k u(x) - (\nabla^{M_\alpha})^k u(x')|}{dist(x,x')^\beta},$$

where k is a positive integer.

When α tends to zero, the neck on M_α will shrink to p. Thus we need to introduce a weighted norm to do the estimates. Roughly speaking, we want to choose the weight function $\rho(x)$ on M_α such that $\rho(x)$ is less than the radius of a normal ball at $x \in M_\alpha$. More precisely, we can choose that $\rho(x)$ is of the form [2]:

$$\rho(x) = \begin{cases} c\,\varepsilon & \text{for } \; x \in M_\alpha \cap B_{\varepsilon r_2} \\ \text{interpolation} & \text{for } \; x \in M_\alpha \cap B_{r_2} \setminus B_{\varepsilon r_2} \\ R_2 & \text{for } \; x \in M_\alpha \setminus B_{r_2} \end{cases}$$

for some constants r_2 and R_2. In addition, we can also require $\rho(x)$ to satisfy the following properties:

1. $\|\nabla^{M_\alpha}\rho\|_{0,M_\alpha} \leq C$,
2. $c\,\alpha \leq \rho(x) \leq C\,\alpha$ for $x \in T_1 \cup T_2$,
3. $\|\rho^{-1}\|_{L^p} \leq C$ for $p < n$.

Definition 3 *Let u be a $C^{k,\beta}$ function on M_α, where k is an integer and $0 < \beta < 1$. The ρ-weighted (k,β) norm $\|u\|_{C_\rho^{k,\beta}(M_\alpha)}$ of u is defined as the sum:*

$$\|u\|_{0,M_\alpha} + \|\rho|\nabla^{M_\alpha}u|\,\|_{0,M_\alpha} + \cdots + \|\rho^k|(\nabla^{M_\alpha})^k u|\,\|_{0,M_\alpha} + [\rho^{k+\beta}(\nabla^{M_\alpha})^k u]_{\beta,M_\alpha}.$$

Proposition 3 *The operator $\mathcal{L}$ is a bounded operator between the Banach space $C^{2,\beta}(M_\alpha)$ with norm $\|\cdot\|_{C_\rho^{2,\beta}(M_\alpha)}$ and the Banach space $C^{0,\beta}(M_\alpha)$ with norm $\|\rho^2\cdot\|_{C_\rho^{0,\beta}(M_\alpha)}$.*

Proof. Note that

$$\begin{aligned} & \|\rho^2\mathcal{L}u\|_{C_\rho^{0,\beta}(M_\alpha)} \\ \leq\; & \|\rho^2\cos\theta\Delta_{M_\alpha}u\|_{C_\rho^{0,\beta}(M_\alpha)} + \|\rho^2\sin\theta < H, J\nabla^{M_\alpha}u > \|_{C_\rho^{0,\beta}(M_\alpha)}. \end{aligned}$$

A direct computation gives

$$\|\rho^2 \cos\theta \Delta_{M_\alpha} u\|_{C^{0,\beta}_\rho(M_\alpha)}$$

$$\leq \|\rho^2 \Delta_{M_\alpha} u\|_{0,M_\alpha} + [\rho^{2+\beta}\Delta_{M_\alpha} u]_{\beta,M_\alpha} + [\cos\theta]_{\beta,M_\alpha}\|\rho^\beta\|_{0,M_\alpha}\|\rho^2\Delta_{M_\alpha} u\|_{0,M_\alpha}$$

$$\leq C\|u\|_{C^{2,\beta}_\rho(M_\alpha)}.$$

We also have

$$\|\rho^2 \sin\theta < H, J\nabla^{M_\alpha} u > \|_{C^{0,\beta}_\rho(M_\alpha)}$$

$$\leq \|\rho\sin\theta\,|H|\,\|_{0,M_\alpha}\|\rho\,|\nabla^{M_\alpha}u|\,\|_{0,M_\alpha} + \|\rho\sin\theta\,|H|\,\|_{0,M_\alpha}[\rho^{1+\beta}\nabla^{M_\alpha}u]_{\beta,M_\alpha}$$

$$+[\rho\sin\theta H]_{\beta,M_\alpha}\|\rho^\beta\|_{0,M_\alpha}\|\rho\,|\nabla^{M_\alpha}u|\,\|_{0,M_\alpha}.$$

Using the fact that the mean curvature is zero outside a small ball and the properties of ρ and $\sin\theta$, it follows that

$$\|\rho^2 \sin\theta < H, J\nabla^{M_\alpha} u > \|_{0,M_\alpha} \leq C\,\alpha^2\|\rho\,|\nabla^{M_\alpha}u|\,\|_{0,M_\alpha}.$$

Moreover, when we estimate $\|\rho^2 \sin\theta < H, J\nabla^{M_\alpha} u > \|_{C^{0,\beta}_\rho(M_\alpha)}$, all the suprema involved can be taken only over the small ball. When $dist(x,x') \geq \alpha$, one has that

$$\frac{|\rho\sin\theta H(x) - \rho\sin\theta H(x')|}{dist(x,x')^\beta} \leq C\,\alpha^{2-\beta}.$$

Note that $\|(\nabla^{M_\alpha})^2 \sin\theta\|_{0,M_\alpha} \leq C\,\alpha^{-1}$. Hence when $dist(x,x') \leq \alpha$, one has

$$\frac{|\rho\sin\theta H(x) - \rho\sin\theta H(x')|}{dist(x,x')^\beta} \leq C\,\alpha\,\alpha^{1-\beta} = C\,\alpha^{2-\beta}.$$

Therefore,

$$\|\rho^2 \sin\theta < H, J\nabla^{M_\alpha} u > \|_{C^{0,\beta}_\rho(M_\alpha)} \leq C\,\alpha^2\|u\|_{C^{1,\beta}_\rho(M_\alpha)}.$$

We thus have

$$\|\rho^2 \mathcal{L}u\|_{C^{0,\beta}_\rho(M_\alpha)} \leq C\|u\|_{C^{2,\beta}_\rho(M_\alpha)}.$$

Q.E.D.

By the elliptic Schauder estimate [2] for the ρ-weighted (k,β) norms, one can prove that

$$\|u\|_{C^{2,\beta}_\rho(M_\alpha)} \le C\varepsilon^{-\beta}(\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)} + \|u\|_{0,M_\alpha}).$$

In the Appendix, we show that $\|u\|_{0,M_\alpha} \le C\varepsilon^{-\nu}\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)}$ for u satisfying $\int_{M_\alpha} u\,dV = 0$. We thus have

$$\|u\|_{C^{2,\beta}_\rho(M_\alpha)} \le C\varepsilon^{-(\beta+\nu)}\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)}.$$

In the next Lemma, we bound $\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)}$ by $\|\rho^2\mathcal{L}u\|_{C^{0,\beta}_\rho(M_\alpha)}$ and hence obtain

Lemma 2 *Suppose that u is a $C^{2,\beta}$ function on M_α, $0<\beta<1$, which satisfies $\int_{M_\alpha} u\,dV = 0$. Then when α is small, one has that*

$$\|u\|_{C^{2,\beta}_\rho(M_\alpha)} \le C\varepsilon^{-(\beta+\nu)}\|\rho^2\mathcal{L}u\|_{C^{0,\beta}_\rho(M_\alpha)},$$

where ν is any positive number. The constant C depends on ν, but is independent of α.

Proof. Note that

$$\|\rho^2\mathcal{L}u\|_{C^{0,\beta}_\rho(M_\alpha)} \ge \|\rho^2\cos\theta\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)} - \|\rho^2\sin\theta < H, J\nabla^{M_\alpha}u > \|_{C^{0,\beta}_\rho(M_\alpha)},$$

and

$$\|\rho^2\cos\theta\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)} = \|\rho^2\cos\theta\Delta_{M_\alpha}u\|_{0,M_\alpha} + [\rho^{2+\beta}\cos\theta\Delta_{M_\alpha}u]_{\beta,M_\alpha}.$$

When α is small, we have

$$\|\rho^2\cos\theta\Delta_{M_\alpha}u\|_{0,M_\alpha} \ge \frac{1}{2}\|\rho^2\Delta_{M_\alpha}u\|_{0,M_\alpha}$$

and

$$[\rho^{2+\beta}\cos\theta\Delta_{M_\alpha}u]_{\beta,M_\alpha} \geq \frac{1}{2}[\rho^{2+\beta}\Delta_{M_\alpha}u]_{\beta,M_\alpha} - C\alpha\|\rho^2\Delta_{M_\alpha}u\|_{0,M_\alpha}.$$

Hence

$$\|\rho^2\cos\theta\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)} \geq \frac{1}{3}\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)}.$$

On the other hand, we have

$$\begin{aligned}\|\rho^2\sin\theta < H, J\nabla^{M_\alpha}u > \|_{C^{0,\beta}_\rho(M_\alpha)} &\leq C\,\alpha^2\|u\|_{C^{1,\beta}_\rho(M_\alpha)} \\ &\leq C\,\alpha^2\,\varepsilon^{-(\beta+\nu)}\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)}.\end{aligned}$$

Putting all these estimates together, we get

$$\|\rho^2\mathcal{L}u\|_{C^{0,\beta}_\rho(M_\alpha)} \geq \frac{1}{4}\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)}$$

and the proposition is therefore proved.

Q.E.D.

Remark: Note that in the proof we first need to fix ν and then choose α small enough.

Denote the Banach space of $C^{2,\beta}$ functions on M_α which satisfies $\int_{M_\alpha} u\,dV = 0$ with norm $\|\cdot\|_{C^{2,\beta}_\rho(M_\alpha)}$ by $\mathcal{B}_1$ and the Banach space of $C^{0,\beta}$ functions on M_α which satisfies $\int_{M_\alpha} u\,dV = 0$ with norm $\|\rho^2\cdot\|_{C^{0,\beta}_\rho(M_\alpha)}$ by $\mathcal{B}_2$. Because $\int_{M_\alpha} Im\,\Omega = 0$ and the family of maps $\phi_{tu}, 0 \leq t \leq 1$, is a homotopy between ϕ_u and ϕ_0, it follows that $\int_{M_\alpha} \mathcal{F}_\alpha(u)\,dV = 0$. Thus we can restrict $\mathcal{F}_\alpha$ as a map from $\mathcal{B}_1$ into $\mathcal{B}_2$. A direct computation shows that the operator $\mathcal{L}$ is self-adjoint. By Proposition 2, we consequently have:

Proposition 4 *The operator $\mathcal{L}$ from $\mathcal{B}_1$ into $\mathcal{B}_2$ is injective and surjective.*

We will apply the following version of inverse function theorem to $\mathcal{F}_\alpha$.

Theorem 2 *[1] Let $\mathcal{F} : \mathcal{B} \to \mathcal{B}'$ be a C^1 map between Banach spaces and suppose that the differential $D\mathcal{F}(0)$ of $\mathcal{F}$ at 0 is an isomorphism. Moreover, suppose that $\mathcal{F}$ satisfies the estimates:*

1. $\|D\mathcal{F}(0)x\|_{\mathcal{B}'} \geq C_L\|x\|_{\mathcal{B}}$ *for any* $x \in \mathcal{B}$,
2. $\|D\mathcal{F}(0)y - D\mathcal{F}(x)y\|_{\mathcal{B}'} \leq C_N\|x\|_{\mathcal{B}}\|y\|_{\mathcal{B}}$ *for all x sufficiently near 0 and for any* $y \in \mathcal{B}$,

where C_L and C_N are constants independent of x and y. Then there exist neighborhoods $\mathcal{U}$ of 0 and $\mathcal{V}$ of $\mathcal{F}(0)$ so that $\mathcal{F} : \mathcal{U} \to \mathcal{V}$ is a C^1 diffeomorphism and $\mathcal{V}$ contains the ball $B_{\frac{C_L}{2}r}(\mathcal{F}(0))$, where $r \leq \frac{C_L}{2C_N}$. Furthermore, the image of the ball $B_r(0)$ under $\mathcal{F}$ contains the ball $B_{\frac{C_L}{2}r}(\mathcal{F}(0))$.

We already get an estimate on C_L in Lemma 2 and still need an estimate on C_N to apply Theorem 2.

Lemma 3 *Assume that $v \in \mathcal{B}_1$ and is sufficiently near 0. The differential of $\mathcal{F}_\alpha$ at v satisfies the following estimate:*

$$\|\rho^2(D\mathcal{F}_\alpha(v)(u) - D\mathcal{F}_\alpha(0)(u))\|_{C^{0,\beta}_\rho(M_\alpha)} \leq C\varepsilon^{-\beta}\|v\|_{C^{2,\beta}_\rho(M_\alpha)}\|u\|_{C^{2,\beta}_\rho(M_\alpha)}$$

for all $u \in \mathcal{B}_1$.

Remark: The lemma is also proved in [3]. But the proof and the estimate obtained are slightly different.

Proof. Suppose that the Riemannian metric on N is g. Define a conformal metric $g' = s^{-2}g$, where s is a constant. Assume that the Hamiltonian flow determined by v with respective to the metric g is the same as the Hamiltonian flow determined by a function v_s with respect to the metric g', which is denoted by ϕ'_{tv_s}. Then

$$J\nabla^N v = \frac{d\phi_{tv}}{dt} = \frac{d\phi'_{tv_s}}{dt} = J\nabla^{(N,g')} v_s = s\, J\nabla^N v_s,$$

where $\nabla^{(N,g')}$ is the covariant derivative with respect to the metric g'. It implies that $v_s = s^{-1}v$. Denote $s^{-n}\,\Omega$ by Ω', which is a unit length holomorphic $(n, 0)$ form on (N, g'). Define $G_\alpha(v_s) = \star' (\phi'_{v_s})^*(Im\,\Omega')$, where $\star'$ is the star operator with respect to the metric on M_α induced from $(N,\, g')$. Since

$$\begin{aligned}
\star' (\phi'_{v_s})^*(Im\,\Omega') &= \star' (\phi'_{v_s})^*(s^{-n} Im\,\Omega) \\
&= \star' s^{-n} \phi_v^*(Im\,\Omega) \\
&= \star\, \phi_v^*(Im\,\Omega),
\end{aligned}$$

one has $G_\alpha(v_s) = \mathcal{F}_\alpha(v)$. Assume that $K_1 \geq \frac{1}{2}$ is an upper bound of $\|\nabla^{M_\alpha}\rho\|_{0,M_\alpha}$. Choose $x \in M_\alpha$ and let $s = \rho(x)$. Then the function ρ satisfies $\frac{s}{2} \leq \rho \leq \frac{3s}{2}$ in the ball $B_{\frac{s}{2K_1}}(x)$ and the induced metric are bounded uniformly in this ball. Denote the norm with respect to the metric g' by $\|\cdot\|^{g'}$. We have

$$\begin{aligned}
&\|\rho^2\big(\frac{d\mathcal{F}_\alpha(v+tu)}{dt}|_{t=0} - \frac{d\mathcal{F}_\alpha(tu)}{dt}|_{t=0}\big)\|_{0,B_{\frac{s}{2K_1}}(x)} \\
\leq\ & \tfrac{9s^2}{4}\|D\mathcal{F}_\alpha(v)(u) - D\mathcal{F}_\alpha(0)(u)\|_{0,B_{\frac{s}{2K_1}}(x)} \\
=\ & \tfrac{9s^2}{4}\|DG_\alpha(v_s)(u_s) - DG_\alpha(0)(u_s)\|^{g'}_{0,B_{\frac{1}{2K_1}}(x)} \\
\leq\ & Cs^2\|u_s\|^{g'}_{C^2(B_{\frac{1}{2K_1}}(x))}\|v_s\|^{g'}_{C^2(B_{\frac{1}{2K_1}}(x))},
\end{aligned}$$

where in the last inequality we use the fact that M_α with the metric induced from (N, g') has uniformly bounded geometry in $B_{\frac{1}{2K_1}}(x)$. Because

$$\begin{aligned} & \|u_s\|^{g'}_{C^2(B_{\frac{1}{2K_1}}(x))} \\ = \ & \|s^{-1}u\|^{g'}_{0,B_{\frac{1}{2K_1}}(x)} + \|\nabla^{g'} s^{-1}u\|^{g'}_{0,B_{\frac{1}{2K_1}}(x)} + \|(\nabla^{g'})^2 s^{-1}u\|^{g'}_{0,B_{\frac{1}{2K_1}}(x)} \\ = \ & s^{-1}\|u\|_{0,B_{\frac{s}{2K_1}}(x)} + \|\nabla u\|_{0,B_{\frac{s}{2K_1}}(x)} + s\|\nabla^2 u\|_{0,B_{\frac{s}{2K_1}}(x)} \\ \leq \ & Cs^{-1}\|u\|_{C^2_\beta(M_\alpha)}, \end{aligned}$$

it follows that

$$\|\rho^2(D\mathcal{F}_\alpha(H)(u) - D\mathcal{F}_\alpha(0)(u))\|_{0,B_{\frac{s}{2K_1}}(x)} \leq C\|H\|_{C^2_\beta(M_\alpha)}\|u\|_{C^2_\beta(M_\alpha)}.$$

We next need to estimate the following quantity (A):

$$\frac{|\rho^{2+\beta}(D\mathcal{F}_\alpha(H)(u) - D\mathcal{F}_\alpha(0)(u))(x) - \rho^{2+\beta}(D\mathcal{F}_\alpha(H)(u) - D\mathcal{F}_\alpha(0)(u))(x')|}{dist(x, x')^\beta}.$$

When $dist(x, x') \leq \frac{s}{2K_1}$, we have

$$(A) \leq Cs^2\|u_s\|^{g'}_{C^{2,\beta}(B_{\frac{1}{2K_1}}(x))}\|H_s\|^{g'}_{C^{2,\beta}(B_{\frac{1}{2K_1}}(x))}.$$

Since

$$\begin{aligned} \|u_s\|^{g'}_{C^{2,\beta}(B_{\frac{1}{2K_1}}(x))} \ & \leq s^{-1}(\|u\|_{0,B_{\frac{s}{2K_1}}(x)} + s\|\nabla u\|_{0,B_{\frac{s}{2K_1}}(x)} \\ & \quad + s^2\|\nabla^2 u\|_{0,B_{\frac{s}{2K_1}}(x)} + s^{2+\beta}[\nabla^2 u]_{\beta,B_{\frac{s}{2K_1}}(x)}) \end{aligned}$$

and

$$\frac{s^{2+\beta}|\nabla^2 u(x) - \nabla^2 u(x')|}{dist(x,x')^\beta}$$

$$\leq C\frac{|\rho^{2+\beta}(x)\nabla^2 u(x) - \rho^{2+\beta}(x)\nabla^2 u(x')|}{dist(x,x')^\beta}$$

$$\leq C\frac{|\rho^{2+\beta}(x)\nabla^2 u(x) - \rho^{2+\beta}(x')\nabla^2 u(x') + \rho^{2+\beta}(x')\nabla^2 u(x') - \rho^{2+\beta}(x)\nabla^2 u(x')|}{dist(x,x')^\beta}$$

$$\leq C([\rho^{2+\beta}\nabla^2 u]_{\beta,B_{\frac{s}{2K_1}}(x)} + s^2\|\nabla^2 u\|_{0,B_{\frac{s}{2K_1}}(x)}),$$

we thus have

$$\|u_s\|^{g'}_{C^{2,\beta}(B_{\frac{1}{2K_1}}(x))} \leq Cs^{-1}\|u\|_{C^{2,\beta}_\rho(M_\alpha)}.$$

When $dist(x,x') \geq \frac{s}{2K_1}$, it is easy to see that

$$(A) \leq Cs^{-\beta}\|H\|_{C^2_\beta(M_\alpha)}\|u\|_{C^2_\rho(M_\alpha)}.$$

Putting the estimates together, we therefore get

$$\|\rho^{2+\beta}(D\mathcal{F}_\alpha(H)(u) - D\mathcal{F}_\alpha(0)(u))\|_{\beta,M_\alpha} \leq C\varepsilon^{-\beta}\|H\|_{C^{2,\beta}_\rho(M_\alpha)}\|u\|_{C^{2,\beta}_\rho(M_\alpha)}.$$

Hence it follows that

$$\|\rho^2(D\mathcal{F}_\alpha(H)(u) - D\mathcal{F}_\alpha(0)(u))\|_{C^{0,\beta}_\rho(M_\alpha)} \leq C\varepsilon^{-\beta}\|H\|_{C^{2,\beta}_\rho(M_\alpha)}\|u\|_{C^{2,\beta}_\rho(M_\alpha)}.$$

Q.E.D.

We can choose $\nu = \beta$ in Lemma 6. Then choose $C_L = \frac{1}{C}\varepsilon^{2\beta}$ by Lemma 2 and $C_N = C\varepsilon^{-\beta}$ by Lemma 3. Applying Theorem 2, we therefore conclude that the image of the ball $B_r(0)$ under $\mathcal{F}_\alpha$ contains the ball $B_{\frac{C_L}{2}r}(\mathcal{F}_\alpha(0))$, where $r \leq \frac{\varepsilon^{3\beta}}{2C^2}$.

Lemma 4 *The zero function lies in the ball $B_{\frac{C_L}{2}\varepsilon^{2+\beta}}(\mathcal{F}_\alpha(0))$.*

Proof. Denote $E = \star\phi_0^*(Im\,\Omega) = \mathcal{F}_\alpha(0)$. Recall that $|E(x)| \leq C\alpha$ and $E(x) = 0$ for $x \in M_\alpha \setminus B_{\frac{\alpha}{K}}$. This together with the properties of ρ thus imply that

$$\|\rho^2 E\|_{0,M_\alpha} \leq C\alpha^3.$$

Moreover, we have

$$|\nabla^{M_\alpha} E| = |\cos\theta \nabla^{M_\alpha}\theta| \leq |H| \leq C.$$

Therefore, when $dist(x, x') \leq \varepsilon$, it follows that

$$\frac{|\rho^{2+\beta}E(x) - \rho^{2+\beta}E(x')|}{dist(x,x')^\beta} \leq C\alpha^{2+\beta}\varepsilon^{1-\beta}.$$

When $dist(x, x') \geq \varepsilon$, it follows that

$$\frac{|\rho^{2+\beta}E(x) - \rho^{2+\beta}E(x')|}{dist(x,x')^\beta} \leq C\alpha^{3+\beta}\varepsilon^{-\beta}.$$

Since $\varepsilon = \alpha^{\frac{n+1}{n}}$, we thus obtain

$$\|\rho^2 E\|_{C^{0,\beta}_\rho(M_\alpha)} \leq C\alpha^{3-\beta} \leq \frac{C_L}{2}\varepsilon^{2+\beta}$$

when β and ε are small enough.

Q.E.D.

The extension function U satisfies

$$\|\nabla^N U\|_{0,N} \leq C\varepsilon^{-1}(\|u\|_{0,M_\alpha} + \varepsilon\|\nabla^{M_\alpha} u\|_{0,M_\alpha}) \leq C\varepsilon^{-1}\|u\|_{C^1_\rho(M_\alpha)},$$

$$\|(\nabla^N)^2 U\|_{0,N} \leq C\varepsilon^{-2}\|u\|_{C^2_\rho(M_\alpha)}.$$

When $\|(\nabla^N)^2 U\|_{0,N} \leq \frac{1}{2}$, or $\|u\|_{C^2_\beta(M_\alpha)} \leq \frac{\varepsilon^2}{2C}$, the map ϕ_u is defined. When $\|\nabla^N U\|_{0,N} \leq c_1\varepsilon$, or $\|u\|_{C^1_\beta(M_\alpha)} \leq \frac{c_1\varepsilon^2}{C}$, then $\phi_u(M_\alpha)$ is embedded in a $c_1\varepsilon$ neighborhood of M_α. Choose $r = \varepsilon^{2+\beta} \leq \frac{\varepsilon^{3\beta}}{2C^2}$ in Theorem 2, we knows that there exists a function $u \in \mathcal{B}_1$ with $\|u\|_{C^{2,\beta}_\beta(M_\alpha)} \leq \varepsilon^{2+\beta}$ such that $\mathcal{F}_\alpha(u) = 0$. It follows that $\phi_u(M_\alpha)$ is an embedded special Lagrangian submanifold. We hence prove the main theorem of the paper:

Theorem 3 *Every compact, connected, and immersed special Lagrangian, which has only isolated transversal self-intersection points in a compact 2 or 3 dimensional Calabi-Yau manifold, is the limit of a family of embedded special Lagrangian submanifolds.*

Theorem 4 *Suppose that L is a compact, connected, and immersed special Lagrangian submanifold in a compact n-dimensional Calabi-Yau manifold,* $n > 3$. *Moreover, assume that L has only isolated transversal self-intersection of two sheets and the two tangent planes at each intersection point satisfy the angle condition* $\theta_1 + \cdots + \theta_n = \frac{\pi}{2}$ *(see section 2). Then L is the limit of a family of embedded special Lagrangian submanifolds.*

Appendix : Supremun Estimate

The author would like to thank A. Butscher for informing her of the useful references [12], [15], and showing her a basic argument [4] for the De Giorgi-Nash estimates in this Appendix. We modify these arguments and present the material here for the reader's reference and completeness.

For a submanifold $M^n \subset R^l$, J. H. Michael and L. Simon [10] proved the

Sobolev inequality:

$$(\int_M h^{\frac{n}{n-1}}\,dV)^{\frac{n-1}{n}} \le C(n)\int_M (|\nabla^M h| + h|\bar{H}|)\,dV,$$

where $h > 0$ is a C^1 function on M with compact support and $\bar{H}$ is the mean curvature of M in R^l. When $n > 2$, the inequality can be converted easily into

$$(\int_M h^{\frac{2n}{n-2}}\,dV)^{\frac{n-2}{n}} \le C(n)\,(\int_M |\nabla^M h|^2\,dV + \int_M h^2|\bar{H}|^2\,dV).$$

Or write as

$$(\int_M h^{\frac{2n}{n-2}}\,dV)^{\frac{n-2}{n}} \le C(n)\,(\int_M |\nabla^M h|^2\,dV + Vol(M)^{-\frac{2}{n}}\int_M h^2\,dV), \quad (1)$$

where we absorb $\sup|\bar{H}|^2$ with $Vol(M)^{-\frac{2}{n}}$ to make the expression scaling invariant. When $n = 2$, the Sobolev inequality implies

$$(\int_M h^{\frac{2\kappa}{\kappa-2}}\,dV)^{\frac{\kappa-2}{\kappa}} \le C(\kappa)\;Vol(M)^{\frac{\kappa-2}{\kappa}}(\int_M |\nabla^M h|^2\,dV + Vol(M)^{-1}\int_M h^2\,dV), \quad (2)$$

for any $\kappa > 2$. Because both $\bar{H}$ and $Vol(M)$ are uniformly bounded in our cases, we thus omit the dependency of the constants on $\sup|H|^2\,Vol(M)^{\frac{2}{n}}$. By the above inequality, we have the following estimate:

Lemma 5 *Suppose u is a positive sub-solution of the equation $\Delta_M u \ge gu$ on a closed manifold M, where g is a L^1 function satisfying the estimate $\|g\|_{L^{\frac{r}{2}}} \le \bar{c}\;Vol(M)^{\frac{2}{r}-\frac{2}{n}}$ for some $r > n$. Then $\|u\|_{0,M} \le C_p\;Vol(M)^{-\frac{1}{p}}\|u\|_{L^p}$ for $p > 0$. The constant C_p depends on n, r, $\bar{c}$ and p.*

Proof. Multiply both sides of the inequality by u^{q-1}, $q \ge 2$, and then integrate over M. One thus has

$$\int_M u^{q-1}\Delta_M u\,dV \ge \int_M g u^q\,dV,$$

or

$$-(q-1)\int_M u^{q-2}|\nabla^M u|^2\,dV \geq \int_M g u^q\,dV.$$

By rewriting the left hand side and using Hölder inequality, this leads to

$$\frac{4(q-1)}{q^2}\int_M |\nabla^M u^{\frac{q}{2}}|^2\,dV \leq \|g\|_{L^{\frac{r}{2}}}\|u^q\|_{L^{\frac{r}{r-2}}}.$$

Plug this into the Sobolev inequality. For $n>2$, one gets

$$\begin{aligned}(\textstyle\int_M u^{\frac{q}{2}\cdot\frac{2n}{n-2}}\,dV)^{\frac{n-2}{n}} &\leq C(n)\,(\textstyle\int_M |\nabla^M u^{\frac{q}{2}}|^2\,dV + Vol(M)^{-\frac{2}{n}}\int_M u^q\,dV)\\ &\leq C(n)\,(cq\|g\|_{L^{\frac{r}{2}}}\|u^q\|_{L^{\frac{r}{r-2}}} + Vol(M)^{\frac{2}{r}-\frac{2}{n}}\|u^q\|_{L^{\frac{r}{r-2}}})\\ &\leq Cq\;Vol(M)^{\frac{2}{r}-\frac{2}{n}}\|u^q\|_{L^{\frac{r}{r-2}}}.\end{aligned}$$

The constant C depends on n and $\bar{c}$. For $n=2$, we can choose $\kappa=\frac{r+2}{2}$ and similarly get

$$(\int_M u^{\frac{q}{2}\cdot\frac{2\kappa}{\kappa-2}}\,dV)^{\frac{\kappa-2}{\kappa}} \leq Cq\;Vol(M)^{\frac{2}{r}-\frac{2}{\kappa}}\|u^q\|_{L^{\frac{r}{r-2}}}.$$

The constant C depends on κ and $\bar{c}$. Denote $\hat{n}=n$ for $n>2$ and $\hat{n}=\kappa$ for $n=2$. Thus one has

$$(\int_M u^{\frac{q\hat{n}}{\hat{n}-2}}\,dV)^{\frac{\hat{n}-2}{q\hat{n}}} \leq (Cq\;Vol(M)^{\frac{2}{r}-\frac{2}{\hat{n}}})^{\frac{1}{q}}\|u\|_{L^{\frac{qr}{r-2}}}.$$

Denote $\Psi(x)=(\int_M u^x dV)^{\frac{1}{x}}$. Then the inequality can be written as

$$\Psi(qk) \leq (Cq\;Vol(M)^{\frac{2}{r}-\frac{2}{\hat{n}}})^{\frac{1}{q}}\,\Psi(qs),$$

where $k=\frac{\hat{n}}{\hat{n}-2}$ and $s=\frac{r}{r-2}$. Because r is greater than $\hat{n}$, thus $\gamma=\frac{k}{s}$ is greater than one, and

$$\Psi(\gamma x) \leq (C\frac{x}{s}\,Vol(M)^{\frac{2}{r}-\frac{2}{\hat{n}}})^{\frac{s}{x}}\,\Psi(x),$$

for any $x \geq 2s$. Choose $x = \gamma^{m-1}p$ and $p \geq 2s$. One then has

$$
\begin{aligned}
\Psi(p\gamma^m) &\leq (C \tfrac{p\gamma^{m-1}}{s} \, Vol(M)^{\frac{2}{r}-\frac{2}{\hat{n}}})^{\frac{s}{p\gamma^{m-1}}} \, \Psi(p\gamma^{m-1}) \\
&\leq (\tfrac{Cp}{s} \, Vol(M)^{\frac{2}{r}-\frac{2}{\hat{n}}})^{\frac{s}{p}\sum_{i=0}^{m-1}\frac{1}{\gamma^i}} \, \gamma^{\sum_{i=0}^{m-1}\frac{i}{\gamma^i}} \, \Psi(p).
\end{aligned}
$$

Let m go to infinity and notice that

$$
\sum_{i=0}^{\infty} \frac{1}{\gamma^i} = \frac{k}{k-s} \quad \textit{and} \quad -\frac{1}{k} + \frac{1}{s} = \frac{2}{\hat{n}} - \frac{2}{r}.
$$

One then gets

$$
\|u\|_{0,M} \leq C \; Vol(M)^{-\frac{1}{p}} \|u\|_{L^p}, \quad \text{for } p \geq 2s. \tag{3}
$$

The constant C depends on $\hat{n}$, r and $\bar{c}$. For general p, first recall that one has

$$
\begin{aligned}
(\textstyle\int_M u^{qk} \, dV)^{\frac{1}{k}} &\leq Cq \; Vol(M)^{\frac{2}{r}-\frac{2}{\hat{n}}} \, (\textstyle\int_M u^{qs} \, dV)^{\frac{1}{s}} \\
&= Cq \; Vol(M)^{\frac{1}{k}-\frac{1}{s}} \, (\textstyle\int_M u^{qs} \, dV)^{\frac{1}{s}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& Vol(M)^{-\frac{1}{kq}} \, (\textstyle\int_M u^{qk} \, dV)^{\frac{1}{kq}} \\
\leq \; & (Cq)^{\frac{1}{q}} [Vol(M)^{-\frac{1}{\lambda}} (\textstyle\int_M u^{qs(1-\varepsilon)\lambda} \, dV)^{\frac{1}{\lambda}} \, Vol(M)^{-\frac{1}{\mu}} (\textstyle\int_M u^{qs\varepsilon\mu} \, dV)^{\frac{1}{\mu}}]^{\frac{1}{sq}},
\end{aligned}
$$

where $1 > \varepsilon > 0$ and $\frac{1}{\lambda} + \frac{1}{\mu} = 1$. If we choose λ satisfying $qs(1-\varepsilon)\lambda = qk$, then it follows that

$$
Vol(M)^{-\frac{\varepsilon}{kq}} (\int_M u^{kq} \, dV)^{\frac{\varepsilon}{kq}} \leq (Cq)^{\frac{1}{q}} \, Vol(M)^{-\frac{1}{\mu qs}} (\int_M u^{\mu qs\varepsilon} \, dV)^{\frac{1}{\mu qs}}.
$$

That is,

$$
Vol(M)^{-\frac{1}{kq}} (\int_M u^{kq} \, dV)^{\frac{1}{kq}} \leq (Cq)^{\frac{1}{q\epsilon}} \, Vol(M)^{-\frac{1}{\mu qs\varepsilon}} (\int_M u^{\mu qs\varepsilon} \, dV)^{\frac{1}{\mu qs\varepsilon}}.
$$

Let $q = 2$ and $p = 2s\varepsilon\mu = \frac{2ks\varepsilon}{k-s+s\varepsilon}$, then

$$Vol(M)^{-\frac{1}{2k}}(\int_M u^{2k}\, dV)^{\frac{1}{2k}} \leq (2C)^{\frac{1}{2\varepsilon}}\, Vol(M)^{-\frac{1}{p}}(\int_M u^p\, dV)^{\frac{1}{p}}. \qquad (4)$$

By varying ε, we can choose p to be any positive number. Combine (3) and (4), and one gets

$$\|u\|_{0,M} \leq C_p\, Vol(M)^{-\frac{1}{p}}\|u\|_{L^p}, \quad \text{for } p > 0.$$

Q.E.D.

Remark: The constant on the right hand side of the inequality (1) or (2) is called the Sobolev constant on M. The quantity can be defined in a general Riemannian manifold, which is not necessarily a submanifold of R^l. We only use (1) or (2) to derive the estimate. Thus the lemma holds in general and the constant C_p depends on the Sobolev constant on M, r, $\bar{c}$ and p.

From Lemma 5, We can get the following supremun estimate:

Theorem 5 *Suppose that u is a $W^{1,2}$ weak solution for $\Delta_M u = f$ on a closed Riemannian manifold M, where f satisfies $\|f\|_{L^{\frac{r}{2}}} < \infty$, for some $r > n$. Then*

$$\|u\|_{0,M} \leq C\,(Vol(M)^{-\frac{1}{2}}\|u\|_{L^2} + Vol(M)^{\frac{2}{n}-\frac{2}{r}}\|f\|_{L^{\frac{r}{2}}}),$$

where C depends only on the Sobolev constant on M and r.

Proof. Define $\beta = Vol(M)^{\frac{2}{n}-\frac{2}{r}}\|f\|_{L^{\frac{r}{2}}}$ and $\xi = \frac{1}{2}[(\frac{u}{\beta})^2 + 1]$. It is easy to see that ξ is a weak solution for

$$\Delta_M \xi \geq \frac{f}{\beta}\frac{u}{\beta} = \frac{f}{\beta}\frac{u}{\beta}\frac{1}{\xi}\xi.$$

Denote $g = \frac{f}{\beta}\frac{u}{\beta}\frac{1}{\xi}$. Because $|\frac{u}{\beta\xi}| < 2$, one then has

$$\|g\|_{L^{\frac{2}{r}}} \leq \frac{2}{\beta}\|f\|_{L^{\frac{2}{r}}} \leq 2\,Vol(M)^{\frac{2}{r}-\frac{2}{n}}.$$

By Lemma 5, it follows that

$$\xi \leq C\,Vol(M)^{-1}\|\xi\|_{L^1}.$$

Hence

$$|\frac{u}{\beta}|^2 \leq C\,Vol(M)^{-1}[\frac{1}{2}\|(\frac{u}{\beta})^2\|_{L^1} + Vol(M)].$$

Therefore,

$$\begin{aligned}
|u| &\leq C\beta\sqrt{1 + Vol(M)^{-1}\|(\tfrac{u}{\beta})^2\|_{L^1}} \\
&\leq C\sqrt{\beta^2 + Vol(M)^{-1}\|u^2\|_{L^1}} \\
&\leq C(\beta + Vol(M)^{-\frac{1}{2}}\|u\|_{L^2}) \\
&= C(Vol(M)^{\frac{2}{n}-\frac{2}{r}}\|f\|_{L^{\frac{r}{2}}} + Vol(M)^{-\frac{1}{2}}\|u\|_{L^2}).
\end{aligned}$$

Again, the constant C may change slightly in different places.

Q.E.D.

Suppose that M_α and ρ are as defined in section 3 and section 5. We need the following estimate in weighted norm to prove the main theorem.

Lemma 6 *[3] Suppose that u is a $C^{2,\beta}$ function on M_α, $0 < \beta < 1$, which satisfies $\int_{M_\alpha} u\,dV = 0$. Then one has $\|u\|_{0,M_\alpha} \leq C\varepsilon^{-\nu}\|\rho^2\Delta_{M_\alpha}u\|_{C^{0,\beta}_\rho(M_\alpha)}$ for α small enough, where ν is any positive number. The constant C depends on ν, but is independent of α.*

Proof. Assume that the Lemma does not hold. Then there exists a sequence $\alpha_j \to 0$, its corresponding ε_j, ρ_j, and $u_j \in C^{2,\beta}(M_{\alpha_j})$ which satisfies $\int_{M_{\alpha_j}} u_j \, dV = 0$ and

$$\|u_j\|_{0,M_{\alpha_j}} \geq j\, \varepsilon_j^{-\nu} \|\rho_j^2 \, \Delta_{M_{\alpha_j}} u_j\|_{C^{0,\beta}_{\rho_j}(M_{\alpha_j})}.$$

We can normalize u_j such that $\|u_j\|_{0,M_{\alpha_j}} = 1$. It then follows that

$$\|\rho_j^2 \, \Delta_{M_{\alpha_j}} u_j\|_{C^{0,\beta}_{\rho_j}(M_{\alpha_j})} \leq \frac{1}{j}\, \varepsilon_j^{\nu}.$$

On the other hand, by Theorem 4 we have

$$\|u_j\|_{0,M_{\alpha_j}} \leq C\, [\, Vol(M_{\alpha_j})^{-\frac{1}{2}} \|u_j\|_{L^2} + Vol(M_{\alpha_j})^{\frac{2}{n}-\frac{2}{r}} \|\Delta_{M_{\alpha_j}} u_j\|_{L^{\frac{r}{2}}}].$$

Because u_j satisfies $\int_{M_{\alpha_j}} u_j \, dV = 0$, one has

$$\lambda_1(M_{\alpha_j}) \int_{M_{\alpha_j}} u_j^2 \, dV \leq -\int_{M_{\alpha_j}} < \Delta_{M_{\alpha_j}} u_j, u_j > \, dV \leq \|\Delta_{M_{\alpha_j}} u_j\|_{L^1}.$$

Remember that the Sobolev constant on M_{α_j} is bounded uniformly, the volume $Vol(M_{\alpha_j})$ is bounded uniformly from above and below, and $\lambda_1(M_{\alpha_j})$ is bounded below by $\frac{1}{4}\lambda_1(L)$. Thus when r satisfies $-r + \frac{\nu r}{2} \geq -n$,

$$\begin{aligned} 1 = \|u_j\|_{0,M_{\alpha_j}} &\leq C_r(\|\Delta_{M_{\alpha_j}} u_j\|_{L^1} + \|\Delta_{M_{\alpha_j}} u_j\|_{L^{\frac{r}{2}}}) \\ &\leq C_r[\textstyle\int_{M_{\alpha_j}} \rho_j^{-r} (\rho_j^2 \, \Delta_{M_{\alpha_j}} u_j)^{\frac{r}{2}} \, dV]^{\frac{2}{r}} \\ &\leq C_r[\textstyle\int_{M_{\alpha_j}} \rho_j^{-r} \, (\frac{1}{j}\, \varepsilon_j^{\nu})^{\frac{r}{2}} \, dV]^{\frac{2}{r}} \\ &\leq \frac{C_r}{j} (\textstyle\int_{M_{\alpha_j}} \rho_j^{-r+\frac{\nu r}{2}} \, dV)^{\frac{2}{r}} \\ &\leq \frac{C_r}{j}. \end{aligned}$$

The constant C_r may change in different places. Given $\nu > 0$, we first find r which satisfies $-r + \frac{\nu r}{2} \geq -n$. Because the constant C_r is independent of j, the above inequality then leads to a contradiction. Hence the lemma must hold.

Q.E.D.

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