Plane Solutions of Interface Cracks in Anisotropic Dissimilar Media

By Chien-Ching Ma[1] and Jyi-Jiin Luo[2]

Abstract: In this study, plane problems for bonded dissimilar half planes of anisotropic material containing an interfacial crack are considered. The problem is solved by application of a generalized Mellin transform in conjunction with the complex stress function. The dependence of the order of the stress singularity on the material constants is studied in detail. For most cases of different material combinations, the imaginary part of the stress singularity is extremely small, which indicates that the oscillatory region is very small compared to other physical dimensions. The full field solutions in the Mellin transform domain are obtained explicitly. The full field solutions for prescribed arbitrary concentrated loadings on crack faces are investigated in detail. The stress fields outside the oscillatory region are studied for dissimilar anisotropic interface crack. The stress distribution calculated from the far field, the near tip field, and the oscillatory field is studied from the numerical investigation. The dependence for the size of the oscillatory region on the oscillatory index is discussed in detail. It is shown that for most cases of the material combinations, the stress fields outside the oscillatory region and along the bonded interface are proportional to $1/\sqrt{r}$ near the crack tip, as those in homogeneous media.

INTRODUCTION

Many engineering structures are comprised of more than one material. The strength of composite materials is influenced by the orientation of existing cracks with respect to the bimaterial interface. A number of solutions for the stress and displacement fields for crack lying along bimaterial interfaces have been obtained for isotropic materials by Williams (1959), Sih and Rice (1964), Rice and Sih (1965), and Erdogan (1963, 1965). Extensions to anisotropic elasticity have been made by Gotoh (1967), Willis (1971), Bogy (1972), and Kuo and Bogy (1974a, b), and recently by Ting (1986, 1990), Qu and Bassani (1989), and Bassani and Qu (1989). All these studies of inplane problems have shown that the stresses share the inverse square-root singularity of the crack and, in addition, exhibit an oscillatory behavior as the crack tip is approached. Recently, Ma and Hour (1989, 1990) investigated the antiplane problems of two dissimilar anisotropic wedges and an inclined crack terminating at a bimaterial interface. They found that the order of the stress singularity is always real for the antiplane anisotropic problems. That is a quite different character from the in-plane case in which the complex type of stress singularity might exist.

An oscillatory singularity in the displacements and stresses at the crack tip for in-plane problems predicts the physically unrealistic phenomenon of interpenetration of the two materials and wrinkling of the crack faces. Rice (1988) indicated that this zone typically is very small under remote tensile loading, but can be large under shear loading. In an effort to remove these singularities, many studies have been made on this subject. Several modifications have been suggested to resolve this problem, such as the frictionless contact zone model of Comninou (1977), the cohesive zone model of Atkinson (1977) and the elastic-plastic analysis of Shih and Asaro (1988).

In linear-elastic fracture mechanics, stress intensity factors are usually used as the parameter in fracture criteria for homogeneous body. As long as the crack tip is embedded in a homogeneous medium, the stress state around the crack tip exhibits the standard square-root singularity. For the interface crack, the power of stress singularity is a complex number that implies oscillations in stresses and crack surface displacements. Rice and Sih (1965) analyzed the problem of a finite crack between dissimilar isotropic media and introduced a complex stress intensity factor. But the complex stress intensity factor does not have the same physical significance as those for homogeneous cracks. Rice (1988) reexamined the problem for the isotropic interface crack, and introduced definitions of stress intensity factors of classical type by using an intrinsic material length scale.

Significant progress has been made recently for a crack along an interface between dissimilar anisotropic media, which is assessed by Qu and Bassani (1989), Bassani and Qu (1989), Suo (1990), and Wu (1990). A definition of stress intensity factor is proposed by Suo (1990). Suo's definition does not recover the classical definition for homogeneous media. Based on the complex-valued stress-concentration vector introduced by Willis (1971), a new definition of real-valued stress intensity factors is introduced by Wu (1990). The definition is an extension to that for cracks in homogeneous materials and reduces to the one given by Rice (1988) for isotropic interface cracks.

In this study, in-plane problems of anisotropic interfacial cracks that are assumed to be perfectly bounded together along a common edge are considered. The full-field analytical solutions are obtained explicitly by application of Mellin transform method. Numerical results for special loadings are investigated in detail and some interesting phenomena are presented. Most of the papers on interface crack problems have focused attention on the oscillatory region. With the complete solutions in hand, we are able to investigate the problem of interface cracks in more detail. The stress fields outside the oscillatory region are studied and some interesting features are discussed. The size of the oscillatory region is strongly depend on the oscillatory index, which is determined by the material constant and will be discussed in detail. Furthermore, the stress distribution evaluated from the far field, the near tip field, and the oscillatory field is also presented in this study. It is found that the oscillatory zone is confined to a distance that is smaller than physically relevant length for small value of oscillatory index. Furthermore, we find that immediately outside this oscillatory region, the stress behaves

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in square-root singularity just like the homogeneous material. It is then proposed in this study that the classical definition of the stress intensity factor may be used as the fracture parameter for the interface crack as well, especially for the case of small value of oscillatory index.

STATEMENT OF PROBLEM AND MELLIN TRANSFORM

The two-dimensional stress-strain relations for a homogeneous anisotropic body are

$$e_{\alpha\beta} = s^{\gamma\delta}_{\alpha\beta}\sigma_{\gamma\delta} \tag{1}$$

Because of assumed elastic symmetry about $x_3 = 0$ for the plane problem, the six independent material constants are s^{11}_{11}, s^{11}_{22}, s^{11}_{12}, s^{12}_{12}, s^{12}_{22}, and s^{22}_{22}.

The solution of the two-dimensional problem has the following form in terms of complex potentials:

$$u_r + iu_\theta = e^{-i\theta} \sum_{\alpha=1,2} [\delta_\alpha \Omega'_\alpha(z_\alpha) + \rho_\alpha \bar{\Omega}'_\alpha(\bar{z}_\alpha)] \tag{2}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 4 \sum_{\alpha=1,2} [\gamma_\alpha \Omega''_\alpha(z_\alpha) + \bar{\gamma}_\alpha \bar{\Omega}''_\alpha(\bar{z}_\alpha)] \tag{3}$$

$$\sigma_{rr} - \sigma_{\theta\theta} + i2\sigma_{r\theta} = -4e^{-i2\theta} \sum_{\alpha=1,2} [\gamma^2_\alpha \Omega''_\alpha(z_\alpha) + \bar{\Omega}''_\alpha(\bar{z}_\alpha)] \tag{4}$$

where $\Omega_\alpha(\alpha = 1, 2)$ are arbitrary analytic functions of the complex variable z_α; $\bar{\Omega}_\alpha$ is complex conjugate; and primes denote derivatives with respect to the indicated arguments. The relation between z and z_α is $z = re^{i\theta}$ and $z_\alpha = z + \gamma_\alpha \bar{z}$.

The complex constants δ_α, ρ_α, and γ_α are defined in terms of the components of the elasticity tensor

$$\delta_\alpha = -2(S^{11}_{11}\gamma^2_\alpha - 2S^{11}_{12}\gamma_\alpha + S^{11}_{22})/\gamma_\alpha \tag{5}$$

$$\rho_\alpha = -2(S^{11}_{11} - 2S^{11}_{12}\bar{\gamma}_\alpha + S^{11}_{22}\bar{\gamma}^2_\alpha) \tag{6}$$

where γ_α and $1/\bar{\gamma}_\alpha$ are four roots of the equation

$$\bar{S}^{11}_{22}\gamma^4 - 4\bar{S}^{11}_{12}\gamma^3 + 2(S^{11}_{11} + 2S^{12}_{12})\gamma^2 - 4S^{11}_{12}\gamma + S^{11}_{22} = 0 \tag{7}$$

and γ_α can be selected to be those roots with modulus less than unity so that

$$|\gamma_\alpha| < 1 \tag{8}$$

in which

$$S^{11}_{11} = S^{22}_{22} = (1/4)(s^{11}_{11} + s^{22}_{22} + 4s^{12}_{12} - 2s^{11}_{22}) \tag{9a}$$

$$S^{11}_{22} = (1/4)(s^{11}_{11} + s^{22}_{22} - 4s^{12}_{12} - 2s^{11}_{22} + 4is^{11}_{12} - 4is^{12}_{22}) \tag{9b}$$

$$S^{12}_{22} = S^{11}_{12} = (1/4)(s^{11}_{11} - s^{22}_{22} + 2is^{11}_{12} + 2is^{12}_{22}) \tag{9c}$$

$$S^{12}_{12} = (1/4)(s^{11}_{11} + s^{22}_{22} + 2s^{11}_{22}) \tag{9d}$$

$$S^{22}_{11} = \bar{S}^{11}_{22}, \quad S^{22}_{12} = S^{12}_{11} = \bar{S}^{11}_{12} \tag{9e,f}$$

The solution of the problem is obtained by use of complex analogs of the standard Mellin transform. We now take Mellin transform of r^2 and r times the stress and displacement, respectively. The transforms of the two-dimensional stress $\sigma_{\alpha\beta}$ and displacement u_α are defined by

$$\hat{\sigma}_{\alpha\beta}(s, \theta) = \int_0^\infty \sigma_{\alpha\beta}(r, \theta)r^{s+1}\, dr \tag{10}$$

$$\hat{u}_\alpha(s, \theta) = \int_0^\infty u_\alpha(r, \theta)r^s\, dr \tag{11}$$

where s = complex transform parameter. The physical stress and displacement fields are recovered as follows:

$$\sigma_{\alpha\beta}(r, \theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{\sigma}_{\alpha\beta}(s, \theta)r^{-s-2}\, ds \tag{12}$$

$$u_\alpha(r, \theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{u}_\alpha(s, \theta)r^{-s-1}\, ds \tag{13}$$

where $\text{Re}(s) = c$ defines the path of integration. The choice of c has to be determined by the regularity of their integrands. Direct use of the transforms with the complex representation of the solution leads to (Bogy 1972)

$$\hat{u}_r(s, \theta) + i\hat{u}_\theta(s, \theta) = e^{-i\theta} \sum_{\alpha=1,2} \left[\frac{\delta_\alpha \phi_\alpha(s)}{(e^{i\theta} + \gamma_\alpha e^{-i\theta})^{s+1}} + \frac{\rho_\alpha \bar{\phi}_\alpha(s)}{(e^{-i\theta} + \bar{\gamma}_\alpha e^{i\theta})^{s+1}}\right] \tag{14}$$

$$\hat{\sigma}_{rr}(s, \theta) + \hat{\sigma}_{\theta\theta}(s, \theta) = -4(s + 1) \sum_{\alpha=1,2} \left[\frac{\gamma_\alpha \phi_\alpha(s)}{(e^{i\theta} + \gamma_\alpha e^{-i\theta})^{s+2}} + \frac{\bar{\gamma}_\alpha \bar{\phi}_\alpha(s)}{(e^{-i\theta} + \bar{\gamma}_\alpha e^{i\theta})^{s+2}}\right] \tag{15}$$

$$\hat{\sigma}_{\theta\theta}(s, \theta) - i\hat{\sigma}_{r\theta}(s, \theta) = -2(s + 1)e^{-i\theta} \cdot \sum_{\alpha=1,2} \left[\frac{\gamma_\alpha \phi_\alpha(s)}{(e^{i\theta} + \gamma_\alpha e^{-i\theta})^{s+1}} + \frac{\bar{\phi}_\alpha(s)}{(e^{-i\theta} + \bar{\gamma}_\alpha e^{i\theta})^{s+1}}\right] \tag{16}$$

where $\phi_\alpha(s)$ is defined as

$$\phi_\alpha(s) = \int_{0(\theta)}^\infty \Omega'_\alpha(z_\alpha)z^S_\alpha\, dz_\alpha \tag{17}$$

For convenience, define

$$H(s, \theta) = (e^{i\theta} + \gamma_\alpha e^{-i\theta})^{-s-1} \tag{18}$$

$$T(s, \theta) = [\hat{\sigma}_{\theta\theta}(s, \theta) - i\hat{\sigma}_{r\theta}(s, \theta)]/2(s + 1) \tag{19}$$

$$D(s, \theta) = \hat{u}_r(s, \theta) + i\hat{u}_\theta(s, \theta) \tag{20}$$

Then (14) and (16) can be rewritten as follows:

$$D(s, \theta) = e^{i\theta} \sum_{\alpha=1,2} [\delta_\alpha H_\alpha(s, \theta)\phi_\alpha(s) + \rho_\alpha \bar{H}_\alpha(s, \theta)\bar{\phi}_\alpha(s)] \tag{21}$$

$$T(s, \theta) = -e^{-i\theta} \sum_{\alpha=1,2} [\gamma_\alpha H_\alpha(s, \theta)\phi_\alpha(s) + \bar{H}_\alpha(s, \theta)\bar{\phi}_\alpha(s)] \tag{22}$$

EXPLICIT GENERAL SOLUTIONS IN MELLIN TRANSFORM DOMAIN

In this section, we consider an anisotropic bimaterial interface crack subjected to prescribed traction boundary conditions at the crack faces $\theta = \pm\pi$ as shown in Fig. 1.

$$\sigma_{\theta\theta}(r, \pi) - i\sigma_{r\theta}(r, \pi) = -p(r) - iq(r) \tag{23}$$

$$\sigma^*_{\theta\theta}(r, -\pi) - i\sigma^*_{r\theta}(r, -\pi) = -p^*(r) - iq^*(r) \tag{24}$$

Perfect bonding along the interface is ensured by the stress and displacement continuity conditions

$$\sigma_{\theta\theta}(r, 0) - i\sigma_{r\theta}(r, 0) = \sigma^*_{\theta\theta}(r, 0) - i\sigma^*_{r\theta}(r, 0) \tag{25}$$

$$u_r(r, 0) + iu_\theta(r, 0) = u^*_r(r, 0) + iu^*_\theta(r, 0) \tag{26}$$

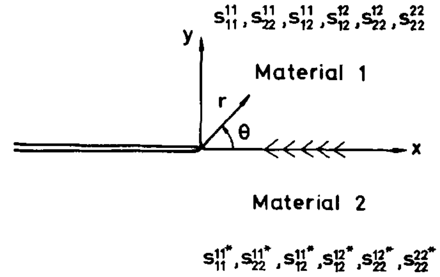

FIG. 1. Configuration of Bonded Anisotropic Dissimilar Interface Crack

Substitution (19)–(22) into the Mellin transform of (23)–(26) provides the following eight equations for the eight unknown functions $\phi_\alpha(s)$, $\bar{\phi}_\alpha(s)$, $\phi^*_\alpha(s)$, and $\bar{\phi}^*_\alpha(s)$, for $\alpha = 1, 2$:

$$\sum_{\alpha=1,2} [\gamma_\alpha H_\alpha(s, \pi)\phi_\alpha(s) + \bar{H}_\alpha(s, \pi)\bar{\phi}_\alpha(s)] = T(s) \quad (27)$$

$$\sum_{\alpha=1,2} [\gamma^*_\alpha H^*_\alpha(s, -\pi)\phi^*_\alpha(s) + \bar{H}^*_\alpha(s, -\pi)\bar{\phi}^*_\alpha(s)] = T^*(s) \quad (28)$$

$$\sum_{\alpha=1,2} [\bar{\gamma}_\alpha \bar{H}_\alpha(s, \pi)\bar{\phi}_\alpha(s) + H_\alpha(s, \pi)\phi_\alpha(s)] = \bar{T}(s) \quad (29)$$

$$\sum_{\alpha=1,2} [\bar{\gamma}^*_\alpha \bar{H}^*_\alpha(s, -\pi)\bar{\phi}^*_\alpha(s) + H^*_\alpha(s, -\pi)\phi^*_\alpha(s)] = \bar{T}^*(s) \quad (30)$$

$$\sum_{\alpha=1,2} [\gamma_\alpha H_\alpha(s, 0)\phi_\alpha(s) + \bar{H}_\alpha(s, 0)\bar{\phi}_\alpha(s)] = \sum_{\alpha=1,2} [\gamma^*_\alpha H^*_\alpha(s, 0)\phi^*_\alpha(s) + \bar{H}^*_\alpha(s, 0)\bar{\phi}^*_\alpha(s)] \quad (31)$$

$$\sum_{\alpha=1,2} [\bar{\gamma}_\alpha \bar{H}_\alpha(s, 0)\bar{\phi}_\alpha(s) + H_\alpha(s, 0)\phi_\alpha(s)] = \sum_{\alpha=1,2} [\bar{\gamma}^*_\alpha \bar{H}^*_\alpha(s, 0)\bar{\phi}^*_\alpha(s) + H^*_\alpha(s, 0)\phi^*_\alpha(s)] \quad (32)$$

$$\sum_{\alpha=1,2} [\delta_\alpha H_\alpha(s, 0)\phi_\alpha(s) + \rho_\alpha \bar{H}_\alpha(s, 0)\bar{\phi}_\alpha(s)] = \sum_{\alpha=1,2} [\delta^*_\alpha H^*_\alpha(s, 0)\phi^*_\alpha(s) + \rho^*_\alpha \bar{H}^*_\alpha(s, 0)\bar{\phi}^*_\alpha(s)] \quad (33)$$

$$\sum_{\alpha=1,2} [\bar{\delta}_\alpha \bar{H}_\alpha(s, 0)\bar{\phi}_\alpha(s) + \bar{\rho}_\alpha H_\alpha(s, 0)\phi_\alpha(s)] = \sum_{\alpha=1,2} [\bar{\delta}^*_\alpha \bar{H}^*_\alpha(s, 0)\bar{\phi}^*_\alpha(s) + \bar{\rho}^* H^*_\alpha(s, 0)\phi^*_\alpha(s)] \quad (34)$$

where

$$T(s) = T(s, \pi) = [-\hat{p}(s) - i\hat{q}(s)]/2(s + 1) \quad (35)$$

$$T^*(s) = T^*(s, -\pi) = [-\hat{p}^*(s) - i\hat{q}^*(s)]/2(s + 1) \quad (36)$$

and $\hat{p}(s)$, $\hat{q}(s)$, $\hat{p}^*(s)$, and $\hat{q}^*(s)$ denote the Mellin transforms of $p(r)$, $q(r)$, $p^*(r)$, and $q^*(r)$, respectively. This system can be solved by Cramer's rule and is simplified to yield

$$\phi_\alpha(s)H_\alpha(s, 0) = L_\alpha/L, \quad \phi^*_\alpha(s)H^*_\alpha(s, 0) = L^*_\alpha/L \quad (37a,b)$$

where

$$L = |(\gamma_1 - \gamma_2)(\gamma^*_1 - \gamma^*_2)|^2 \frac{(1 - \dot{p}^2)^2}{\dot{p}^4} (l\dot{p}^4 + m\dot{p}^2 + l) \quad (38)$$

$$L_\alpha = (-1)^\alpha(\bar{\gamma}_2 - \bar{\gamma}_1)|\gamma^*_1 - \gamma^*_2|^2 \frac{1 - \dot{p}^2}{\dot{p}^3} (GV_\alpha - HU_\alpha) \quad (39)$$

$$L^*_\alpha = (-1)^\alpha(\bar{\gamma}^*_2 - \bar{\gamma}^*_1)|\gamma_1 - \gamma_2|^2 \frac{1 - \dot{p}^2}{\dot{p}^3} (G^*V^*_\alpha - H^*U^*_\alpha) \quad (40)$$

in which

$$\dot{p} = e^{-i(s+1)\pi} \quad (41)$$

$$l = |\lambda + \lambda^*|^2 - (\eta + \xi^*)(\xi + \eta^*) \quad (42)$$

$$m = 2|\lambda + \lambda^*|^2 - (\eta + \xi^*)^2 - (\xi + \eta^*)^2 \quad (43)$$

$$\lambda = \frac{\gamma_2\delta_1 - \gamma_1\delta_2}{\gamma_1 - \gamma_2} = \frac{\rho_1 - \rho_2}{\bar{\gamma}_1 - \bar{\gamma}_2} = \frac{S^{11}_{22}}{\gamma_1\gamma_2} [\gamma_1 + \gamma_2 - \gamma_1\gamma_2(\bar{\gamma}_1 + \bar{\gamma}_2)] \quad (44)$$

$$\eta = \bar{\eta} = \frac{\delta_1 - \delta_2}{\gamma_1 - \gamma_2} = -2\left(S^{11}_{11} - \frac{S^{11}_{22}}{\gamma_1\gamma_2}\right) \quad (45)$$

$$\xi = \bar{\xi} = \frac{\gamma_2\bar{\rho}_1 - \gamma_1\bar{\rho}_2}{\gamma_1 - \gamma_2} = 2(S^{11}_{11} - S^{11}_{22}\bar{\gamma}_1\bar{\gamma}_2) \quad (46)$$

while

$$G = T(s)[\xi - \xi^* - (\xi + \eta^*)\dot{p}^2] + \bar{T}(s)[-\lambda + \lambda^* + (\lambda + \lambda^*)\dot{p}^2] + T^*(s)(\eta^* + \xi^*) + \bar{T}^*(s)(-2\lambda^*) \quad (47)$$

$$H = T(s)[\bar{\lambda} - \bar{\lambda}^* - (\bar{\lambda} + \bar{\lambda}^*)\dot{p}^2] + \bar{T}(s)[-\eta + \eta^* + (\eta + \xi^*)\dot{p}^2] + T^*(s)(2\bar{\lambda}^*) + \bar{T}^*(s)(-\eta^* - \xi^*) \quad (48)$$

$$G^* = T^*(s)[\eta + \xi^* + (\xi - \xi^*)\dot{p}^2] + \bar{T}^*(s)[-\lambda - \lambda^* + (-\lambda + \lambda^*)\dot{p}^2] + T(s)(-\eta - \xi)\dot{p}^2 + \bar{T}(s)(2\lambda)\dot{p}^2 \quad (49)$$

$$H^* = T^*(s)[\bar{\lambda} + \bar{\lambda}^* + (\bar{\lambda} - \bar{\lambda}^*)\dot{p}^2] + \bar{T}^*(s)[-\eta^* - \xi + (-\eta + \eta^*)\dot{p}^2] + T(s)(-2\bar{\lambda})\dot{p}^2 + \bar{T}(s)(\eta + \xi)\dot{p}^2 \quad (50)$$

$$U_\alpha = \sum_{\beta=1,2} (1 - \delta_{\alpha\beta})\gamma_\beta[\eta + \xi^* + (\xi + \eta^*)\dot{p}^2] - (\lambda + \lambda^*)(1 + \dot{p}^2) \quad (51)$$

$$V_\alpha = \sum_{\beta=1,2} (1 - \delta_{\alpha\beta})\gamma_\beta(\bar{\lambda} + \bar{\lambda}^*)(1 + \dot{p}^2) - [\xi + \eta^* + (\eta + \xi^*)\dot{p}^2] \quad (52)$$

$$U^*_\alpha = \sum_{\beta=1,2} (1 - \delta_{\alpha\beta})\gamma^*_\beta[\eta + \xi^* + (\xi + \eta^*)\dot{p}^2] - (\lambda + \lambda^*)(1 + \dot{p}^2) \quad (53)$$

$$V^*_\alpha = \sum_{\beta=1,2} (1 - \delta_{\alpha\beta})\gamma^*_\beta(\bar{\lambda} + \bar{\lambda}^*)(1 + \dot{p}^2) - [\xi + \eta^* + (\eta + \xi^*)\dot{p}^2] \quad (54)$$

and $\delta_{\alpha\beta}$ = Kronecker delta. From (37)–(39), it follows that

$$\phi_\alpha(s)H_\alpha(s, 0) = \frac{\dot{p}(GV_\alpha - HU_\alpha)}{(-1)^\alpha(\gamma_2 - \gamma_1)\Delta(s)} \quad (55)$$

$$\phi^*_\alpha(s)H^*_\alpha(s, 0) = \frac{\dot{p}(G^*V^*_\alpha - H^*U^*_\alpha)}{(-1)^\alpha(\gamma^*_2 - \gamma^*_1)\Delta(s)} \quad (56)$$

where

$$\Delta(s) = (1 - \dot{p}^2)(l\dot{p}^4 + m\dot{p}^2 + l) \quad (57)$$

It is worthy to note that ξ, η, ξ^*, and η^* are real values, as are l and m. The expressions for $\hat{u}_r(s, \theta)$, $\hat{u}_\theta(s, \theta)$, $\hat{\sigma}_{rr}(s, \theta)$, $\hat{\sigma}_{\theta\theta}(s, \theta)$, and $\hat{\sigma}_{r\theta}(s, \theta)$ now follow directly from the substitution of (55) and (56) into (14)–(16). This completes the formal solution for the transforms of the stress and displacement components. The physical components of stresses and displacement are then obtained from the Mellin inversion integrals (12) and (13). The similar problem for displacement prescribed on both crack faces and for traction prescribed on one face with displacements prescribed on the other is investigated in detail by Ma and Luo (1993).

The integral form of stress field given by (12), (15), (16), (55), and (56) is particularly suitable for asymptotic analysis. The choice of c in the inversion integrals in (12) satisfying

$$\text{Re}(s_1) < c < -1 \quad (58)$$

in which s_1 = zero of $\Delta(s)$ with the largest real part in the strip $-2 < \text{Re}(s) < -1$. The equation $\Delta(s) = 0$ in (57) yields

$$1 - \dot{p}^2 = 0 \quad (59)$$

or

$$\dot{p}^4 + (m/l)\dot{p}^2 + 1 = 0 \quad (60)$$

It can be shown that $(m/l)^2 - 4 > 0$, and the zeros of (59) and (60) are, respectively

$$s = n \quad (61)$$

and

$$s = n - (1/2) \pm i\beta, \quad \text{if } l < 0 \tag{62a}$$

$$s = n \pm i\beta, \quad \text{if } l > 0 \tag{62b}$$

in which n = an integer number and

$$\beta = \frac{1}{2\pi} \cosh^{-1} \left| \frac{m}{2l} \right| = \frac{1}{2\pi} \ln \left| \frac{1 + \kappa}{1 - \kappa} \right| \tag{63}$$

where

$$\kappa = \sqrt{\frac{m - 2l}{m + 2l}}$$

Using Cauchy's integral theorem, the value of the inverse integral along Γ equals the contribution from residues minus the integral along a semicircle of infinitely large radius (Γ_α^R or Γ_α^L) as shown in Fig. 2 and the latter contribution is zero. The largest contribution for the asymptotic behavior of the stress field as $r \to 0$ depends on the location of s_1, and the asymptotic expansion near the crack tip of the stress and displacement fields will be

$$\lim_{r \to 0} \sigma_{\alpha\beta}(r, \theta) = \left[\lim_{s \to -(3/2) + i\beta} \left(s + \frac{3}{2} - i\beta \right) \hat{\sigma}_{\alpha\beta}(s, \theta) \right] \cdot r^{-(1/2) - i\beta} + \left[\lim_{s \to -(3/2) - i\beta} \left(s + \frac{3}{2} + i\beta \right) \hat{\sigma}_{\alpha\beta}(s, \theta) \right] \cdot r^{-(1/2) + i\beta} + O(r^0) \tag{64}$$

$$\lim_{r \to 0} u_\alpha(r, \theta) = \left[\lim_{s \to -(3/2) + i\beta} \left(s + \frac{3}{2} - i\beta \right) \hat{u}_\alpha(s, \theta) \right] r^{(1/2) - i\beta} + \left[\lim_{s \to -(3/2) - i\beta} \left(s + \frac{3}{2} + i\beta \right) \hat{u}_\alpha(s, \theta) \right] r^{(1/2) + i\beta} + O(r) \tag{65}$$

The order of the stress singularity $\lambda = -(1/2) - i\beta$ (or $\lambda = (1/2) + i\beta$) is a complex number and the stress fields are oscillatory in the limit $r \to 0$. The magnitude of the oscillation depends on the value β, which is expressed in (63). The value of β is a nondimensional real number measuring an aspect of the dissimilarity of the two materials, and is referred to as the oscillatory index. It is worthy to note that β depends only on one material parameter κ, which is the combination of 12 material constants. From (42), (43), and (63), β can be rewritten in the following form:

$$\beta = \frac{1}{\pi} \tanh^{-1} \frac{\eta - \xi - \eta^* + \xi^*}{\sqrt{(\eta + \xi + \eta^* + \xi^*)^2 - 4|\lambda + \lambda^*|^2}} \tag{66}$$

The combination of material constants that make no oscillation in the stress field will be $\beta = 0$, or the condition that make

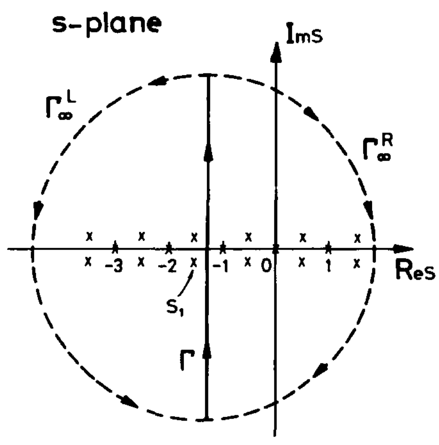

FIG. 2. Path of Integration Used in Complex s-Plane

$$\eta - \xi = \eta^* - \xi^* \tag{67}$$

Homogeneous materials obviously satisfy (67). We now examine for various special cases. For the isotropic case in plane stress, $s_{11}^{11} = s_{22}^{22} = 1/E$, $s_{22}^{11} = -\nu/E$, and $s_{12}^{12} = 1/4\mu$, which make $S_{11}^{11} = S_{22}^{22} = 1/2\mu$, $S_{12}^{12} = (1 - \nu)/2E$, and $S_{22}^{12} = S_{12}^{11} = S_{22}^{11} = S_{11}^{22} = 0$, where E = Young's modulus; ν = Poisson's ratio; and μ = shear modulus. In the isotropic case for which γ_1 and γ_2 are both zero, the corresponding results can be obtained by taking the limit $\gamma_1 \to \gamma_2 \to 0$, we have $\eta = (m - 1)/\mu$, $\xi = 1/\mu$, and $\lambda = 0$, where $m = 4/(1 + \nu)$. From the similar analysis, we have $m = 4(1 - \nu)$ for plane strain. Thus for the isotropic bimaterial interfacial crack, the singularity can be obtained from (66), and the result is

$$\beta = \frac{1}{\pi} \tanh^{-1} \frac{\mu^*(m - 2) - \mu(m^* - 2)}{\mu m^* + \mu^* m} \tag{68}$$

This result agrees with Bogy (1971) and Williams (1959). The maximum value of β can be found to be $(\ln\sqrt{3})/\pi (\approx 0.175)$.

For single anisotropic material, we have $l = 4|\lambda|^2 - (\eta + \xi)^2$ and $m = 8|\lambda|^2 - 2(\eta + \xi)^2$, hence $m = 2l$ and from (63) we obtain $\beta = 0$. Furthermore, from (47)–(54)

$$GV_\alpha - HU_\alpha = (1 + p^2)[4|\lambda|^2 - (\eta + \xi)^2] \left\{ [T^*(s) - p^2 T(s)] + \sum_{\beta=1,2} (1 - \delta_{\alpha\beta})\gamma_\beta[-\bar{T}^*(s) + p^2 \bar{T}(s)] \right\} \tag{69}$$

Finally, we get

$$\phi_\alpha(s) = (-1)^\alpha \left\{ -e^{-2is\pi} T(s) + T^*(s) + \sum_{\beta=1,2} (1 - \delta_{\alpha\beta})\gamma_\beta[e^{-2is\pi}\bar{T}(s) - \bar{T}^*(s)] \right\} \cdot [R_\alpha(\pi)]^{s+1} e^{i(s+1)\varphi_\alpha(\pi)} / 2i(\gamma_2 - \gamma_1)\sin[(s + 1)\pi] \tag{70}$$

where

$$R_\alpha(\theta) = |e^{i\theta} + \gamma_\alpha e^{-i\theta}|$$

$$\varphi_\alpha(\theta) = \arg(e^{i\theta} + \gamma_\alpha e^{-i\theta})$$

Eq. (70) is also found to agree with the result obtained by Bogy (1972) for single anisotropic materials.

SOLUTIONS FOR APPLYING ARBITRARY POINT LOADINGS ON CRACK FACES

In this section, the analytical solutions for applying arbitrary point loadings on the interfacial crack faces are obtained in a simple closed form. Some special cases that have interesting phenomena of nonoscillatory stress fields are discussed in detail. We consider two arbitrary point loadings applied on the crack faces with distance a from the crack tip as shown in Fig. 3. The traction boundary conditions are given as

$$\sigma_{\theta\theta}(r, \pi) - i\sigma_{r\theta}(r, \pi) = iR\delta(r - a) \tag{71}$$

$$\sigma_{\theta\theta}^*(r, -\pi) - i\sigma_{r\theta}^*(r, -\pi) = -iR^*\delta(r - a) \tag{72}$$

where $R = Q + iP$ and $R^* = -Q^* - iP^*$ are complex constants; and δ represents the Dirac delta function. We investigate the solutions for stresses and displacements of the material in the upper-half plane first. In this case we have

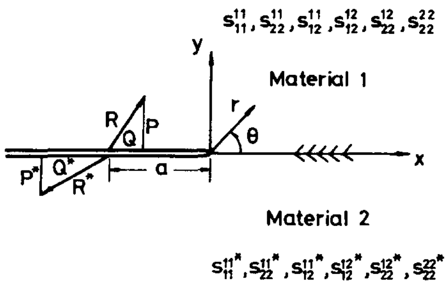

FIG. 3. Point Loadings Applied on Crack Faces of Bonded Anisotropic Dissimilar Interfacial Crack

$$\Omega''_\alpha(z_\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}[-(s+1)\phi_\alpha(s)z_\alpha^{-s-2}]\,ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \cdot \frac{-i\dot{p}(\dot{G}V_\alpha - \dot{H}U_\alpha)\zeta_\alpha^{-s-2}}{2a(-1)^\alpha(\gamma_2-\gamma_1)(1+\gamma_\alpha)(1+\dot{p}^2)(l\dot{p}^4+m\dot{p}^2+l)}\,ds \quad (73)$$

where

$$\dot{G} = R[\xi - \xi^* - (\xi+\eta^*)\dot{p}^2] - \bar{R}[-\lambda + \lambda^* + (\lambda+\lambda^*)\dot{p}^2] - R^*(\eta^*+\xi^*) + \bar{R}^*(-2\lambda^*)$$

$$\dot{H} = R[\bar{\lambda} - \bar{\lambda}^* - (\bar{\lambda}+\bar{\lambda}^*)\dot{p}^2] - \bar{R}[-\eta + \eta^* + (\eta+\xi^*)\dot{p}^2] - R^*(2\bar{\lambda}^*) + \bar{R}^*(-\eta^*-\xi^*)$$

and

$$\zeta_\alpha = \frac{z+\gamma_\alpha\bar{z}}{a(1+\gamma_\alpha)}$$

We can now use the inversion integrals in (73) to obtain $\Omega''_\alpha(z_\alpha)$ and the full field solutions of stresses can then be obtained. There are several methods that can be considered for the determination of the solutions. One method would be the direct numerical integration and the second possibility is to make use of the theory of residues. The second method is chosen in this study because it has the very obvious advantage of showing the form of the resulting expressions and interesting features. Applying Cauchy's integral theorem, the value of the inversion integral along Γ equals the residues minus the integral along a semicircle of infinitely large radius (Γ_α^R or Γ_α^L) as shown in Fig. 2, and the latter contribution is zero. It is clear from (73) that the poles can occur only at the zeros of the characteristic function $(1+\dot{p}^2)(l\dot{p}^4+m\dot{p}^2+l) = 0$. The zeros of the characteristic function have been discussed in detail and expressed earlier. If $|\zeta_\alpha| < 1$, the poles that should be taken into account are those that come from the left-hand side of the integral path Γ

$$\Omega''_\alpha(z_\alpha) = \sum_{n=1}^{\infty}\left[\operatorname*{Res}_{s=-n-1} + \operatorname*{Res}_{s=-n-(1/2)+i\beta} + \operatorname*{Res}_{s=-n-(1/2)-i\beta}\right] \cdot[-(s+1)\phi_\alpha(s)z_\alpha^{-s-2}] = \frac{1}{a}\sum_{n=1}^{\infty}(-1)^{n-1}[A_\alpha\zeta_\alpha^{n-1} + B_\alpha^+\zeta_\alpha^{n-(3/2)-i\beta} + B_\alpha^-\zeta_\alpha^{n-(3/2)+i\beta}] \quad (74)$$

where

$$A_\alpha = \frac{(\dot{G}V_\alpha - \dot{H}U_\alpha)_{\dot{p}^2=1}}{(-1)^\alpha 4\pi(1+\gamma_\alpha)(\gamma_2-\gamma_1)(2l+m)} \quad (75)$$

$$B_\alpha^+ = \frac{(\dot{G}V_\alpha - \dot{H}U_\alpha)_{\dot{p}^2=-e^{2\pi\beta}}}{(-1)^\alpha 4\pi i(1+\gamma_\alpha)(\gamma_2-\gamma_1)e^{\pi\beta}(1+e^{2\pi\beta})(-2le^{2\pi\beta}+m)} \quad (76)$$

$$B_\alpha^- = \frac{(\dot{G}V_\alpha - \dot{H}U_\alpha)_{\dot{p}^2=-e^{-2\pi\beta}}}{(-1)^\alpha 4\pi i(1+\gamma_\alpha)(\gamma_2-\gamma_1)e^{-\pi\beta}(1+e^{-2\pi\beta})(-2le^{-2\pi\beta}+m)} \quad (77)$$

The infinite series expressed in (74) can be obtained in a simple closed form as follows:

$$\Omega''_\alpha(z_\alpha) = \frac{1}{a(1+\zeta_\alpha)}[A_\alpha + B_\alpha^+\zeta_\alpha^{-(1/2)-i\beta} + B_\alpha^-\zeta_\alpha^{-(1/2)+i\beta}] \quad (78)$$

The constant value of A_α comes from the contribution of the integer poles (i.e., $s = -n-1$), and B_α^+ and B_α^- are from the contribution of the complex conjugate poles (i.e., $s = -n - 1/2 \pm i\beta$). If $|\zeta_\alpha| > 1$ and follows the similar procedure as discussed for $|\zeta_\alpha| < 1$, we get the same result as shown in (78). Therefore (78) is valid of the full field on the upper plane. For the displacement field, we need $\Omega'_\alpha(z_\alpha)$, which can be expressed as follows:

$$\Omega'_\alpha(z_\alpha) = (1+\gamma_\alpha)[A_\alpha\ln(1+\zeta_\alpha) + B_\alpha^+\omega^+(\zeta_\alpha) + B_\alpha^-\omega^-(\zeta_\alpha)] \quad (79)$$

For $|\zeta_\alpha| < 1$

$$\omega^+(\zeta_\alpha) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\zeta_\alpha^{n-(1/2)-i\beta}}{n-(1/2)-i\beta} \quad (80)$$

$$\omega^-(\zeta_\alpha) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\zeta_\alpha^{n-(1/2)+i\beta}}{n-(1/2)+i\beta} \quad (81)$$

For $|\zeta_\alpha| > 1$

$$\omega^+(\zeta_\alpha) = \sum_{n=1}^{\infty}(-1)^{n+1} \cdot\left[\frac{4(2n-1)}{(2n-1)^2+4\beta^2} + \frac{\zeta_\alpha^{-n+(1/2)-i\beta}}{-n+(1/2)-i\beta}\right] \quad (82)$$

$$\omega^-(\zeta_\alpha) = \sum_{n=1}^{\infty}(-1)^{n+1} \cdot\left[\frac{4(2n-1)}{(2n-1)^2+4\beta^2} + \frac{\zeta_\alpha^{-n+(1/2)+i\beta}}{-n+(1/2)+i\beta}\right] \quad (83)$$

The solutions of stress and displacement fields in the lower half planes are found to be

$$\Omega_\alpha^{*\prime\prime}(z_\alpha^*) = \frac{1}{a(1+\zeta_\alpha^*)}[A_\alpha^* + B_\alpha^{*+}\zeta_\alpha^{*-(1/2)-i\beta} + B_\alpha^{*-}\zeta_\alpha^{*-(1/2)+i\beta}] \quad (84)$$

$$\Omega_\alpha^{*\prime}(z_\alpha^*) = (1+\gamma_\alpha^*)\{A_\alpha^*.\ln(1+\zeta_\alpha^*) + B_\alpha^{*+}\omega^+(\zeta_\alpha^*) + B_\alpha^{*-}\omega^-(\zeta_\alpha^*)] \quad (85)$$

where

$$A_\alpha^* = \frac{(\dot{G}^*V_\alpha^* - \dot{H}^*U_\alpha^*)_{\dot{p}^2=1}}{(-1)^\alpha 4\pi(1+\gamma_\alpha^*)(\gamma_2^*-\gamma_1^*)(2l+m)} \quad (86)$$

$$B_\alpha^{*+} = \frac{(\dot{G}V_\alpha^* - \dot{H}^*U_\alpha^*)_{\dot{p}^2=-e^{2\pi\beta}}}{(-1)^\alpha 4\pi i(1+\gamma_\alpha^*)(\gamma_2^*-\gamma_1^*)e^{\pi\beta}(1+e^{2\pi\beta})(-2le^{2\pi\beta}+m)} \quad (87)$$

$$B_\alpha^{*-} = \frac{(\dot{G}V_\alpha^* - \dot{H}^*U_\alpha^*)_{\dot{p}^2=-e^{-2\pi\beta}}}{(-1)^\alpha 4\pi i(1+\gamma_\alpha^*)(\gamma_2^*-\gamma_1^*)e^{-\pi\beta}(1+e^{-2\pi\beta})(-2le^{-2\pi\beta}+m)} \quad (88)$$

$$z_\alpha^* = z + \gamma_\alpha^*\bar{z} \quad (89)$$

$$\zeta_\alpha^* = \frac{z + \gamma_\alpha^* \bar{z}}{a(1 + \gamma_\alpha^*)} \quad (90)$$

The full field solutions for stresses and displacements in a two-dimensional anisotropic interfacial crack can be obtained from (78), (79), (84), (85), and (2)–(4).

Due to the presence of the oscillatory complex types of singularities for the interfacial crack problems, the classical definition of the stress intensity factor in homogeneous materials can not be used in this case. Some researches are made to study the definition of the stress intensity factor in interfacial crack, e.g., Rice and Sih (1965), Hutchinson et al. (1987), Rice (1988), Wu (1990), and Suo (1990). The definition of real-valued stress intensity factors proposed by Wu (1990) is adopted in this study. This definition is an extension to that for cracks in homogeneous materials and reduces to the one given by Rice (1988) for isotropic interface cracks. The stress intensity factors K for interface cracks are defined as follows:

$$K(\hat{r}) = \sqrt{2\pi\hat{r}}(\sigma_{\theta\theta} - i\sigma_{r\theta})|_{\theta=0, r=\hat{r}\to 0} = \sqrt{2\pi\hat{r}}2 \sum_{\alpha=1,2} [\gamma_\alpha(1 + \gamma_\alpha)\Omega_\alpha''(z_\alpha) + (1 + \bar{\gamma}_\alpha)\bar{\Omega}_\alpha''(\bar{z}_\alpha)]|_{r=\hat{r}\to 0} \quad (91)$$

Thus, the stress intensity factors for applying point loads on crack faces investigated in this paper can be represented as

$$K(\hat{r}) = \frac{2\sqrt{2\pi}}{\sqrt{a}} \sum_{\alpha=1,2} \{\gamma_\alpha(1 + \gamma_\alpha)[B_\alpha^+(\hat{r}/a)^{-i\beta} + B_\alpha^-(\hat{r}/a)^{i\beta}] + (1 + \bar{\gamma}_\alpha)[\bar{B}_\alpha^+(\hat{r}/a)^{i\beta} + \bar{B}_\alpha^-(\hat{r}/a)^{-i\beta}]\} \quad (92)$$

The material length $\hat{r}$ is introduced following the concepts of Rice (1988). In fact, if $\beta = 0$, the usual definition of the stress intensity factors is recovered. The necessary and sufficient condition for $\beta = 0$ is shown to be $\eta - \xi = \eta^* - \xi^*$.

There are some special cases where the stress fields have interesting phenomena. For a material combination such that the oscillatory index $\beta = 0$ and with a pair of self-equilibrium loadings (i.e., $R^* = -R$) applied on the interface crack faces, the full field solutions of stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ in the upper half plane can be reduced to the following simple expression:

$$\sigma_{\theta\theta} - i\sigma_{r\theta} = \frac{e^{-i\theta}}{2\pi a} \sum_{\alpha=1,2} [\gamma_\alpha(e^{i\theta} + \gamma_\alpha e^{-i\theta})\Psi_\alpha + (e^{-i\theta} + \bar{\gamma}_\alpha e^{i\theta})\bar{\Psi}_\alpha] \quad (93)$$

where

$$\Psi_1 = \frac{i(R + \gamma_2\bar{R})\zeta_1^{-1/2}}{(1 + \zeta_1)(1 + \gamma_1)(\gamma_2 - \gamma_1)} \quad (94)$$

$$\Psi_2 = \frac{i(R + \gamma_1\bar{R})\zeta_2^{-1/2}}{(1 + \zeta_2)(1 + \gamma_2)(\gamma_1 - \gamma_2)} \quad (95)$$

It can be seen very clearly from the solutions (93) that the stresses on the upper medium depend only on the material constants of the upper medium and are the same as the solutions for homogeneous anisotropic medium of identical material properties with crack. Furthermore, along the interface line $\theta = 0$, the stresses are

$$\sigma_{\theta\theta} = \frac{P}{\pi\sqrt{ar}(1 + r/a)} \quad (96)$$

$$\sigma_{r\theta} = \frac{Q}{\pi\sqrt{ar}(1 + r/a)} \quad (97)$$

$$\sigma_{rr} = \frac{1}{\pi\sqrt{ar}(1 + r/a)} \cdot \left[\frac{(\gamma_1\gamma_2\bar{R} - R)i}{(1 + \gamma_1)(1 + \gamma_2)} - \frac{(\bar{\gamma}_1\bar{\gamma}_2R - \bar{R})i}{(1 + \bar{\gamma}_1)(1 + \bar{\gamma}_2)} - P\right] \quad (98)$$

The stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ along the interface line are independent of material constants and are the same as the homogeneous isotropic material, while σ_{rr} depends on material constants. The stress intensity factors are also the same as the homogeneous isotropic crack, which are

$$K_I = P\sqrt{2/\pi a} \quad (99)$$

$$K_{II} = Q\sqrt{2/\pi a} \quad (100)$$

Now, we consider the special case of the single homogeneous anisotropic crack. The results expressed in (78) and (79) can be reduced to the following simple closed form:

$$\Omega_1''(z_1) = \frac{1}{8\pi a(1 + \gamma_1)(\gamma_2 - \gamma_1)} \left[\frac{R + R^* + \gamma_2(\bar{R} + \bar{R}^*)}{1 + \zeta_1} + \frac{R - R^* + \gamma_2(\bar{R} - \bar{R}^*)}{1 + \zeta_1} i\zeta_1^{-1/2}\right] \quad (101)$$

$$\Omega_2''(z_2) = \frac{1}{8\pi a(1 + \gamma_2)(\gamma_1 - \gamma_2)} \left[\frac{R + R^* + \gamma_1(\bar{R} + \bar{R}^*)}{1 + \zeta_2} + \frac{R - R^* + \gamma_1(\bar{R} - \bar{R}^*)}{1 + \zeta_2} i\zeta_2^{-1/2}\right] \quad (102)$$

$$\Omega_1'(z_1) = \frac{1}{8\pi(\gamma_2 - \gamma_1)} \{[R + R^* + \gamma_2(\bar{R} + \bar{R}^*)]\ln(1 + \zeta_1) + [R - R^* + \gamma_2(\bar{R} - \bar{R}^*)]2i \tan^{-1}\sqrt{\zeta_1}\} \quad (103)$$

$$\Omega_2'(z_2) = \frac{1}{8\pi(\gamma_1 - \gamma_2)} \{[R + R^* + \gamma_1(\bar{R} + \bar{R}^*)]\ln(1 + \zeta_2) + [R - R^* + \gamma_1(\bar{R} - \bar{R}^*)]2i \tan^{-1}\sqrt{\zeta_2}\} \quad (104)$$

The stresses along $\theta = 0$ can be obtained as follows:

$$\sigma_{\theta\theta} = \frac{P + P^*}{2\pi\sqrt{ra}(1 + r/a)} \quad (105)$$

$$\sigma_{r\theta} = \frac{Q + Q^*}{2\pi\sqrt{ra}(1 + r/a)} \quad (106)$$

$$\sigma_{rr} = \frac{1}{2\pi a(1 + r/a)} \left[\frac{\gamma_1\gamma_2(\bar{R} + \bar{R}^*) - (R + R^*)}{(1 + \gamma_1)(1 + \gamma_2)} + \frac{\bar{\gamma}_1\bar{\gamma}_2(R + R^*) - (\bar{R} + \bar{R}^*)}{(1 + \bar{\gamma}_1)(1 + \bar{\gamma}_2)}\right] + \frac{i}{2\pi\sqrt{ra}(1 + r/a)} \cdot \left[\frac{\gamma_1\gamma_2(\bar{R} - \bar{R}^*) - (R - R^*)}{(1 + \gamma_1)(1 + \gamma_2)} - \frac{\bar{\gamma}_1\bar{\gamma}_2(R - R^*) - (\bar{R} - \bar{R}^*)}{(1 + \bar{\gamma}_1)(1 + \bar{\gamma}_2)}\right] - \frac{P + P^*}{2\pi\sqrt{ra}(1 + r/a)} \quad (107)$$

As shown in (105) and (106), stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ along $\theta = 0$ are independent of material constants and are the same as isotropic materials. These results do not restrict the applied loadings to be self-equilibrium. Stress σ_{rr} depends on material constants as expressed in (107).

NUMERICAL RESULTS

In this section, we shall see some numerical results of the full field solutions calculated based on the analytical solutions

TABLE 1. Material Constants for Oak and Spruce

Material constant (1)	s^{11}_{11} (2)	s^{22}_{22} (3)	s^{11}_{22} (4)	s^{12}_{12} (5)	γ_1 (6)	γ_2 (7)
Oak	10.15	1.72	−0.87	3.2	0.395	0.026
Spruce	15.5	0.587	−0.33	2.875	0.608	0111

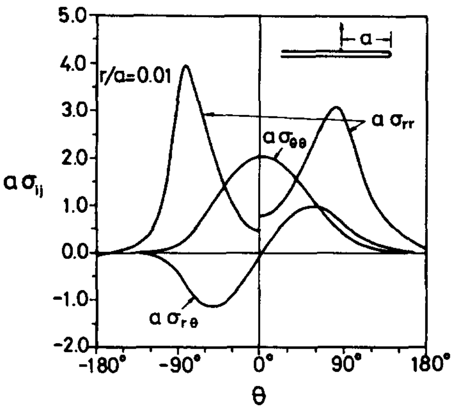

FIG. 4. Angular Distribution of Stresses near Crack Tip for Applying Point Loading on Crack Faces

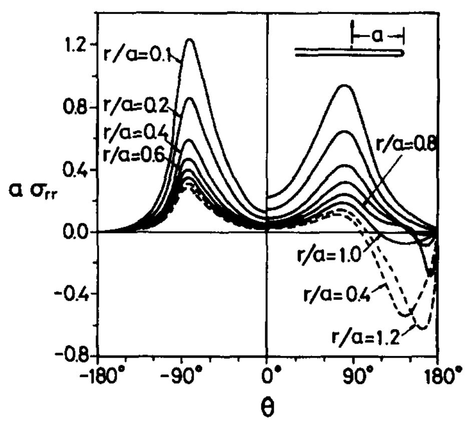

FIG. 5. Full Field Solution of Stress σ_{rr} for fixed r

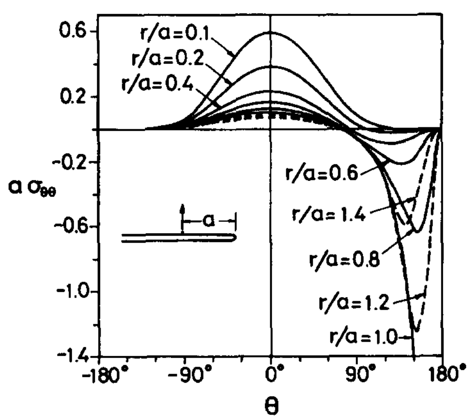

FIG. 6. Full Field Solution of Stress $\sigma_{\theta\theta}$ for Fixed r

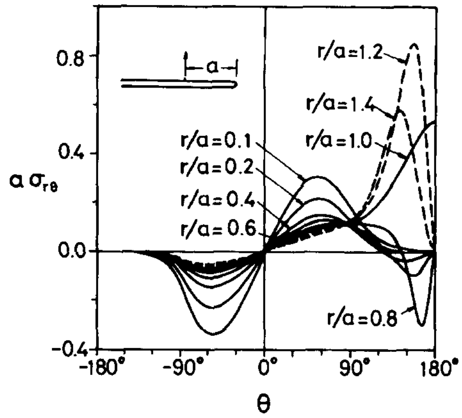

FIG. 7. Full Field Solution of Stress $\sigma_{r\theta}$ for Fixed r

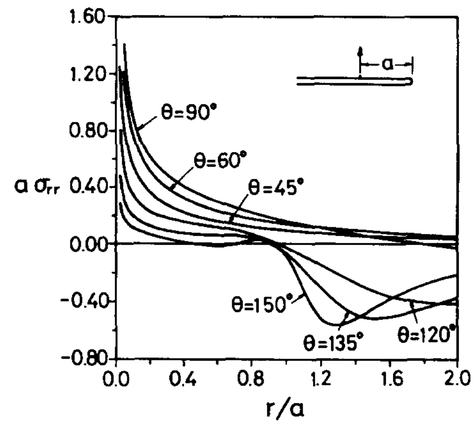

FIG. 8. Full Field Solution of Stress σ_{rr} for Fixed θ

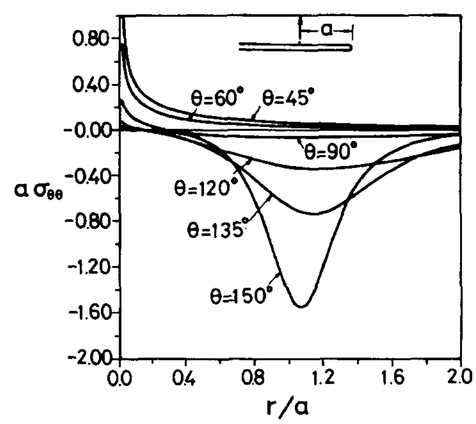

FIG. 9. Full Field Solution of Stress $\sigma_{\theta\theta}$ for Fixed θ

obtained in the previous sections. The materials chosen for the numerical investigation are oak in the upper half plane and spruce in the lower half plane. These materials possess a very high degree of aeolotropy. The elastic constants for these two materials are expressed in Table 1. The values of γ_1 and γ_2 are all real. This material combination will make the oscillatory index $\beta = 0.012$. For a point loading applied at the upper crack face, the stresses near the crack tip ($r/a = 0.01$) are plotted in Fig. 4. Figs. 5–7 show the angular variation of stresses for fixed r/a. The stress components $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ are continuous at $\theta = 0$, while σ_{rr} is, in general, discontinuous there. Figs. 8–10 show the stresses for fixed θ. Because the oscillatory region is so small for β equal to 0.012, it does not show the oscillatory character of the stresses in these figures. Fig. 11 is the displacement on the crack faces for applying a pair of concentrated point loading ($P = P^* = 0.001$ and $Q = Q^* = 0$) on crack faces. Solid lines represent the displacement fields calculated from exact solutions expressed in (79)–(83), and dashed lines are obtained from the near tip fields.

Next, we consider two composite materials, *E*-glass/epoxy and stainless steel/aluminum bonded together with an interfacial crack. The material properties of these two composite

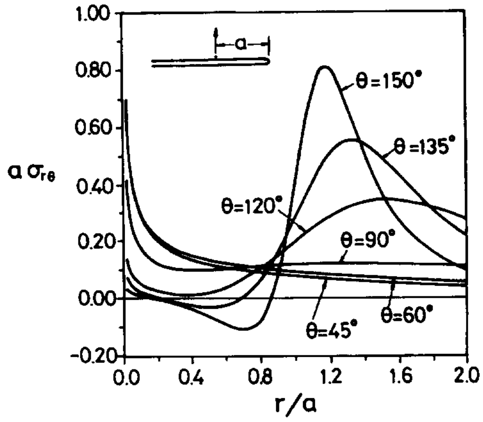

FIG. 10. Full Field Solution of Stress $\sigma_{r\theta}$ for Fixed θ

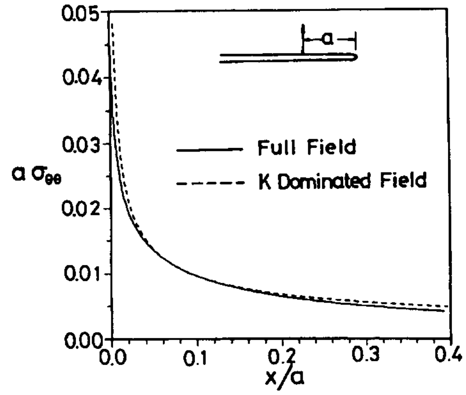

FIG. 13. Comparison of Solution of Stress $\sigma_{\theta\theta}$ Immediately outside Oscillatory Region and K-Dominate Field (β = 0.171)

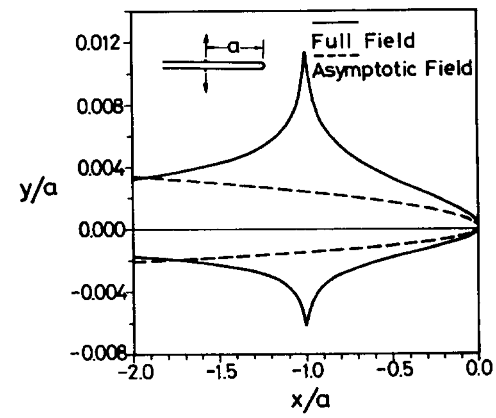

FIG. 11. Crack Face Deformation by Applying Symmetric Concentrated Loadings

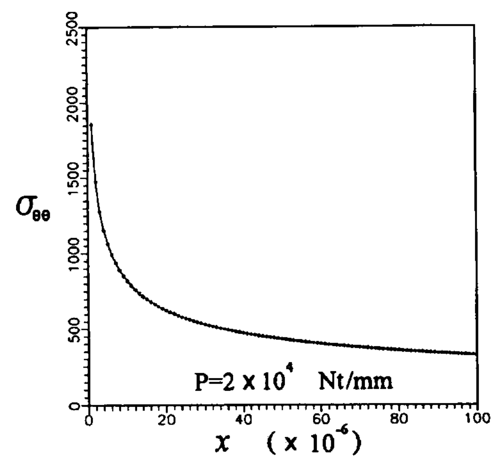

FIG. 14. Stress Distribution in Singular Field along Interface

TABLE 2. Material Constants for E-Glass/Epoxy and Stainless Steel/Aluminum

Material constant (1)	E_{11} (GPa) (2)	E_{22} (GPa) (3)	μ_{12} (GPa) (4)	ν_{12} (5)
E-Glass/Epoxy	37.92	9.17	3.45	0.28
Stainless Steel/Aluminum	136.0	107.0	40.5	0.32

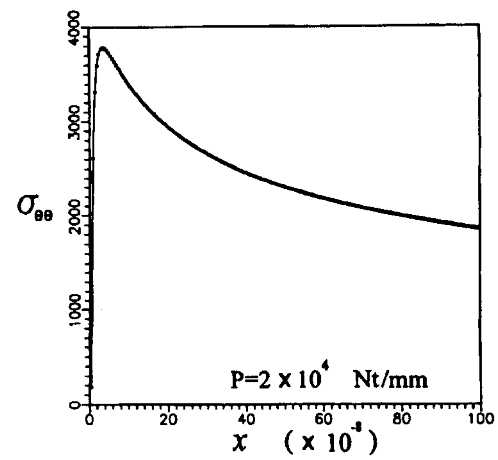

FIG. 15. Stress Distribution near Oscillatory Field along Interface

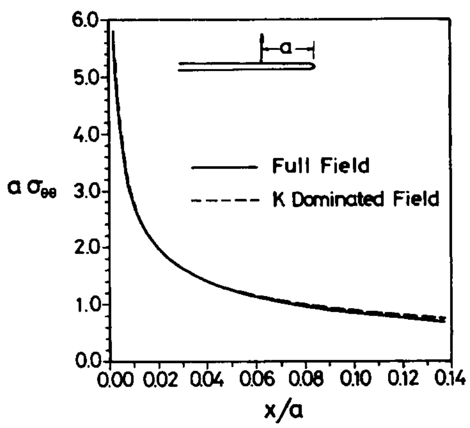

FIG. 12. Comparison of Solution of Stress $\sigma_{\theta\theta}$ Immediately outside Oscillatory Region and K-Dominate Field (β = 0.08)

materials are indicated in Table 2. These material combinations are chosen in order to have large β value of 0.08. The value of β is usually very small [usually under 0.05; see also Suo (1990)] for most material combinations so that the region of oscillation is also very small. The predicted zone of overlapping of crack faces is confined to a distance that is smaller than physically relevant length scales. The linear elasticity and continuum theory are not applicable in this region practically. Thus we can consider this oscillatory region as the nonlinear zone just like the plastic zone or process zone. If this region is extremely small and outside this zone there does indeed exist an ordinary stress intensity factor field, then the classical concept for stress intensity factor in homogeneous body may be applied for the interfacial crack problems as well. In Fig. 12, the solid line represents the exact full field solution of stress $\sigma_{\theta\theta}$ along the crack tip line, and the dashed line indicates K field from the ordinary definition. This figure shows perfect match for these two lines, which indicates the fact that immediately outside the oscillatory region, the stress behaves in square-root singularity just like the homogeneous crack. Thus the use of the classical definition of the stress intensity factor in homogeneous material as a possible fracture parameter in interface crack may be a worthy direction of research. To obtain an extremely large value of β, two materials with very large difference in material properties are chosen: E_{11} = 100 GPa; E_{22} = 99.9 GPa; μ_{12} = 33.3 GPa; and ν_{12}^* = 0.001. This extreme difference in material combination will induce oscillatory index β = 0.171. Fig. 13 shows the exact full field solution of stress $\sigma_{\theta\theta}$ along the crack-tip line and the ordinary definition for stress intensity factor field. We can see that the stress near the crack tip for the large value of β is much smaller than that for a small value of β.

With the complete full field solution at hand, the dependence of the size for the oscillatory region on the oscillatory index will be discussed numerically in detail. If the characteristic length of the size for the oscillatory region is denoted as R, then we have $R = O(10^{-69}a)$ for $\beta = 0.01$, $R = O(10^{-23}a)$ for $\beta = 0.03$, $R = O(10^{-14}a)$ for $\beta = 0.05$, $R = O(10^{-10}a)$ for $\beta = 0.07$, $R = O(10^{-8}a)$ for $\beta = 0.09$, $R = O(10^{-7}a)$ for $\beta = 0.1$, $R = O(10^{-6}a)$ for $\beta = 0.12$, $R = O(10^{-5}a)$ for $\beta = 0.14$, and $R = O(10^{-4}a)$ for $\beta = 0.17$. Here we can see very clearly that the oscillatory region is very small: we have $O(10^{-10}a)$ for $\beta \leq 0.07$, and for most material combinations the oscillatory index β is usually under 0.05. It is also indicated in Fig. 12 that outside this oscillatory region and near the crack tip there does exist a stress field with square-root singularity. Because this oscillatory region is confined in an extremely small area, it is then suggested in this study that the classical definition of stress intensity factor may be used as the fracture parameter for small value of β ($\beta \leq 0.07$).

Finally, we would like to show the stress distribution evaluated from the far field, the near tip singular field, and the region near the oscillatory field. The materials chosen for numerical investigation are both isotropic and the material constants are μ = 5 GPa, ν = 0.3, μ^* = 1 GPa, and ν^* = 0.3. This material combination will make the oscillatory index β = 0.075. The distance from the symmetrical applied loads ($Q = Q^* = 0$) is set to be 5 mm. Fig. 14 shows the singular field of $\sigma_{\theta\theta}$ along the interface for x is small. As the stress approach to the oscillatory region, the stress begins to oscillate in the order of $10^{-9}a$ and the result is shown in Fig. 15.

CONCLUSIONS

The problem of plane deformation for dissimilar anisotropic interface cracks is solved by application of the Mellin transform. The explicit general expressions of solutions for stresses and displacements in the Mellin transform domain are obtained. The dependence of the order of stress singularity on material constants is expressed in explicit closed form. In this study, the full field solutions of stresses for applying point loadings on anisotropic interfacial crack faces are used to study the features of the interface crack problems. It is shown that stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ along the crack line are independent of material constants for self-equilibrium loadings when the oscillatory index is equal to zero. The imaginary part of the stress singularity or the oscillatory index for anisotropic interface crack is very small for most cases of material combinations. This indicates that the oscillation region is extremely small compared to other physical dimensions. It is also found that if the oscillatory index is not large, at a distance r that is large compared to the size of the oscillation region but small compared to the crack length, the stresses show $1/\sqrt{r}$ dependence like those near the crack tip in homogeneous media. Based on this finding, it seems possible that the classical definition of stress intensity factor for square-root singularity can also be applied in the interfacial crack problem. The validity of using the classical stress intensity factor as a proper controlling parameter for interfacial fracture is under investigation and will be reported in future communications.

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