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Radii, Surface Diffuseness, and Binding Energies of Atomic Nuclei

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In efforts for seeking for a theoretical basis of the previously proposed rules for *calculating* nuclear radii, the procedure of Green is reversed to determine a family of nuclear potentials. These potentials have flat bases and exponential tails. The radius of inner region and surface diffuseness of the potential are firstly determined for the entire range of mass number A from the nuclear radii predicted by the proposed rules. The depth is then adjusted so as to give the correct trends of the binding energies of the last nucleons. To test the depth and the diffuseness thus determined, the location of the 3s and 4s maxima in the neutron cross section is determined and is compared with the experimental data on the s-wave strength function. For proton potentials, 3s maximum locates at A=55 and 4s maximum at $A = 162$, which agrees with the prediction of most theories using spherical nuclei..

I. INTRODUCTION

In the previous paper[1] two simple rules have been proposed to calculate the nuclear radii through the entire mass range. They arc:

(1) ***The closed shell nuclei of proton number (or of neutron number) 2, 8, 20, 28, 50, 82, 126···have the radii of proton (or neutron) density 2.5, 3.5, 4.5, 5.0, 6.0, 7.0, 8.0···fm respectively.***

(2) The radii of proton (or neutron j density of other nuclei can be found by an interpolation between the above set of numbers 2.5, 3.5, 4.5, 5.0, 6.0···fm.

The results have been in good conformity with the empirical data[1] and with the prediction of the Elton's semi-empirical formula. In efforts for seeking for the theoretical basis of these rules, a calculation[2] has been made for the radii of the magic nuclei using the Green's independent particle model of nucleus.[3),4),5)] For both 'of the proton and neutron densities, apart from the very heavy nuclei, the rules have been verified satisfactorily. However, for the very heavy nuclei, especially, for the neutron densities the agreement has not been so satisfactory on account of the inherent simplicity of the nuclear potentials assumed in the Green's model.

1) J. L. Hwang, Chin. J. Phys. **1,** 24 (1963).
2) J. L. Hwang, and S. H. Yee, Chin. J. Pbys. **1,** 28 (1963).
3) A. E. S. Green, and K. Lee, Phys. Rev. 99, 772 (1955). This paper will he referred to as GI.
4) A. E.S. Green, Phys. Rev. 102, 1325 (1955). This paper will be referred to as GII.
5) A. E. S. Green, Phys. Rev. 104, 1617 (1956). This paper will be referred to as GIII.

In the present work the procedure of Green is reversed to find a family of single particle static potentials which reproduce the same systematics of the nuclear radii as predicted by the proposed rules. These potentials are then examined to see whether or not they are in conflict with the established facts such as the binding energies of the last nucleon and the location of the low velocity maxima of the neutron scattering cross section.

II. THE INNER POTENTIAL RADIUS AND SURFACE DIFFUSENESS

The potentials studied by Green and his collaborators are defined by

$$V(r) = -V_0, \qquad r \leq a$$
$$V(r) = -V_0 \exp\left[-(r-a)/d\right] \qquad r \geq a, \tag{1}$$

with a term about 45 times as large as the Thomas-Frenkel spin-orbit term and with the parameters

$$V_0 = 40 \text{ Mev}, \quad d = 1 \text{ fm}$$

and

$$a = 1.32A^{1/3} - 0.8 \text{ fm}.$$

The eigenvalues and approximate analytic solutions of radial wave equations for these potentials were obtained, and from them various quantities were computed and illustrated in diagrams as functions of the mass number A. Most part of the present work proceeds with aids of these diagrams. Since in the proposed rules the nuclear radii are expressed not in terms of A, but rather of the neutron number N or the proton number Z, either N or Z should be converted into A and *vice versa* by means of the empirical neutron excess for the β-stable nuclei,[6]

$$N - Z = 0.4A^2/(A + 200), \tag{3}$$

and

$$N + Z = A. \tag{3}'$$

The present work starts from Eqs. (3) and (3)'. For a given nucleus of mass number A, the neutron number N and the proton number Z are found from Eqs. (3) and (3)', and the radius of its neutron density $R(N)$ and that of proton density $R(Z)$ are computed by the proposed rules.

On the other hand, the values of radii predicted by the Green theory, denoted by $R_r(N)$ and $R_G(Z)$, can be obtained from the Fig 6 of GIII and the relations[5]

$$R_r(N) = (5/3)^{1/2} a \left[N^{-1} \sum_i \langle \rho^2 \rangle_i\right]^{1/2} \tag{4}$$

and

$$R_r(Z) = (5/3)^{1/2} a \left[Z^{-1} \sum_i \langle \rho^2 \rangle_i\right]^{1/2}, \tag{4}'$$

where the N neutrons (or--Z-protons-)--should be- placed in the lowest states which are permitted by the exclusion principle. The computations are done seperately for $R_G(N)$

6) A. E. S. Green, *Nuclear Physics* (McGraw-Hill Book Company, Inc., New York. 1955), Chaps. 8 and 9.

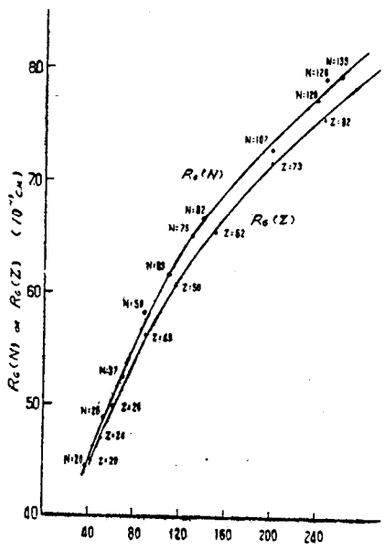

Fig. 1 Radii of proton densities and neutron densities for P-stable nuclei, calculated from the Green's theory.

and $R_G(Z)$ of some typical nuclei, and are illustrated in Fig 1 together with their smoothed results. This figure is then used to find the corresponding mass numbers which would give the same values as those of ***R(N)*** and ***R(Z)*** calculated formerly from ther proposed rules. These mass numbers will be referred to as ***A,(N)*** and $A_G(Z)$ respectively. $A_G(N)$ and $A_G(Z)$ thus determined are illustrated in Fig. 2 as a fumction of ***A.*** All diagrams given in GII and GIII may be used immediately as far as $A_G(N)$ or $A_G(Z)$ is used instead of the original ***A.***

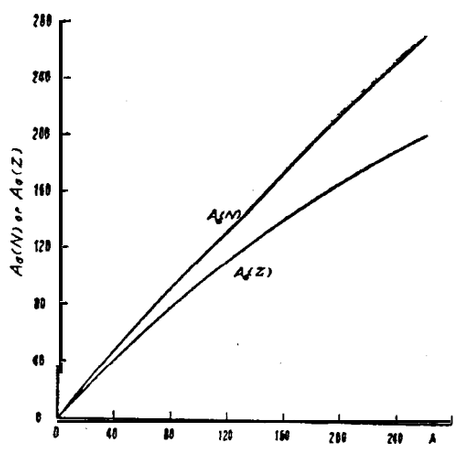

Fig. 2 Diagrams in the Green's paper may be used in the present work merely replacing the given mass number A by $A_G(N)$ or $A_G(Z)$.

The dimension of the static potentials which reproduce the same magnitude of nuclear radii as given by the proposed rules may be found with use of these $A,(N)$ and $A_G(Z)$. The radius of the inner uniform region of potential is readily given by Fig. 2 of GII, or analytically by Eq. (2) as

$$\text{a(N)} = 1.32A_G^{1/3}(N) - 0.8 \quad \text{for neutron potentials,} \tag{5}$$

$$a(Z) = 1.32A_G^{1/3}(Z) - 0.8 \quad \text{for proton potentials.} \tag{5}'$$

The diffuseness δ defined by $d/a, d$ being the universal surface extension (1 fm for all nuclei), is also given by the same figure, or analytically by

$$\delta(N) = \text{l/a(N)} \quad \text{for neutron potentials,} \tag{6}$$

$$\delta(Z) = \text{l/a(Z)} \quad \text{for proton potentials.} \tag{6}'$$

The derived u(N), $a(Z), \delta(N)$ and $\delta(Z)$ arc given in Fig. 3 directly in terms of A (not of $A_G(N)$ nor $A_G(Z)$).

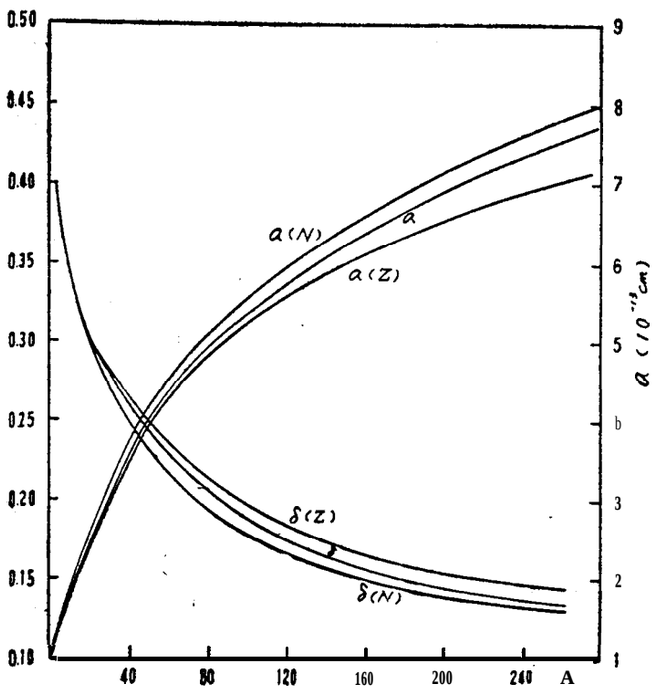

Fig. 3 Modified values of inner potential radius a and surface diffuseness δ. a(N) and $\delta(N)$ are those for the neutron potentials, and a(Z) and 6(Z) those for the proton potentials.

III. THE BINDING ENERGY OF THE LAST NUCLEON WITH V_0=40 MEV

In the Green's work the depth of the potentials V_0 has been fixed to 40 Mev for all nuclei; This value together with the modified values of a and δ given in Eqs. (5), (5)′, (6) and (6)′, however, yields larger binding energy for the last neutron and

smaller binding energy for the last proton. This fact is illustrated in this section, and the correct values of V_0, which variate with different A, is derived in the next section.

The binding enorgy of the last neutron (or the last proton) may be read from the diagram of eigenvalues vs. A, Fig. 6 of GIII. The neutrons (or protons) are put on the energy levels in such a manner that they can constitute the lowest energy, state as permitted by the Pauli exclusion principle. The energy of the occupied highest level corresponding to the shifted A *i. e.* $A_G(N)$ (or $A_G(Z)$)is read and plotted. The effect of the attractive proton potential anomaly V_a should be taken into consideration; for convenience sake, it is chosen to bc one-half thc magnitude of thc Coulomb valuc, that is,

$$V_a = \frac{1}{2} U_c = \frac{(Z-1)e^2}{2(a+d)}. \tag{7}$$

Thc binding cnergies thus determined are plotted in Fig. 4 and Fig. 5 for some typical nuclei. The empirical values

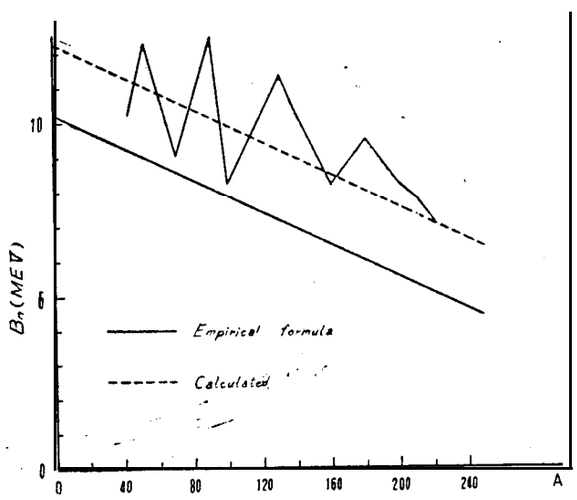

Fig. 4 Binding. energies- of- the. last-neutrons for $V_0=40$ Mev. The general trend of calculated values is 2.0 Mev larger than that of the empirical values.

$$B_n = 10.235 - 0.01862A \text{ (Mev)}, \tag{8}$$

and

$$B_p = 9.453 - 0.01862A \text{ (Mevj)}, \tag{8}'$$

are illustrated for comparison. For the last neutron, the general trend of the determined values is 2 Mev larger than the empirical one. However, for the proton, it is 1.2 Mev smaller. The fixed value of the potential depth $V_0 = 40$ Mev is therefore thought to bc not so rcasonablc.

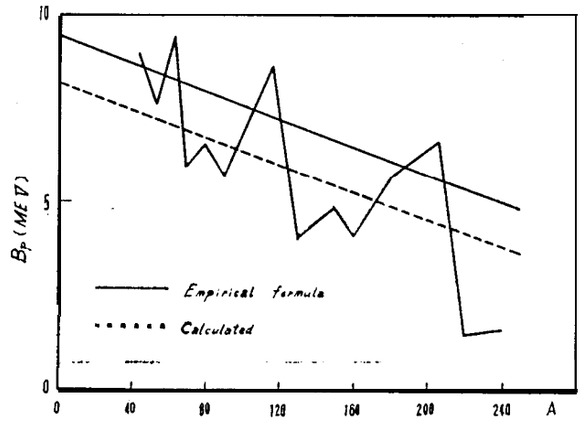

Fig. 5 Binding energies of the last protons for $V_0=40$ Mev plus one-half of the Coulomb potential energy. The general trend of calculated values is 1.2 Mev smaller than that of empirical values.

VI. MODIFICATION OF THE POTENTIAL DEPTH

Suppose the nucleus may be treated as the thomas-Fermi model. Then the number of particles is given by

$$N(or\ Z) = \frac{8}{3}\pi P_0^3 \Omega / h^3, . \tag{9}$$

where P_0 is the maximum momentum and Ω is the nuclear volume. Remembering that the maximum kinetic energy* $P_0^2/2m$ is equal to $-(W-V_0)$ and the nuclear volume is approximately equal to $(4/3)\ \pi a^3$ we obtain from Eq. (9) a relation connecting a, V_0 and W,

$$a^2(\ W-\ V_0) = \text{const.}, \tag{10}$$

for any nucleus 'of constant N(or Z). When the depth V_0 and the radius a of the potential change to V_0' and a' at a fixed N (or Z), and the energy of the top level W thus becomes W', then the relation

$$a^2(W-V_0) = a'^2(W'-V'_0) \tag{11}$$

holds. Let us assume that Eq. (11) also holds' for the diffuse potentials which are being studied here. For the present purpose Eq. (11) gives

$$V_0' = W' + (a^2/a'^2)\,(V_0 - W). \tag{12}$$

Letting $V_0 = 40$ Mev and inserting the general trend of the binding energy of the last neutron determined by Green into,, W and the empirical value into W', the revised value of the potential depth may readily be obtained. Since in the Green's paper these two quantities conform quite well, we have $W = W.'$ If a(N) is used for a', V_0' gives the modified depth of the neutron potential V,(N). Similarly, for protons, if $V_0 = 40 + V_a$

* W denotes the absolute value of the binding energy of the last nucleon.

and a(2) are used, V_0' then gives the modified depth of the effective proton potential $(V_0(Z))_{eff}$ in which the anamalous attractive potential $V_a=\frac{1}{2}U_c$ participates. The modified depth of the pure- proton potential $V_0(Z)$ is obtained merely by subtracting $\frac{1}{2}U_c$. The net potential acting upon a proton is given by the sum of the effective potential (attractive) and the classical Coulomb potential (repulsive). Therefore its depth is found to be

$$(V_0(Z))_{net}=V_0(Z)-\frac{1}{2}U_c.$$

As shown in Fig. 6, the variation of $V_0(N)$ and $(V_0(Z))_{net}$ with the change of A show

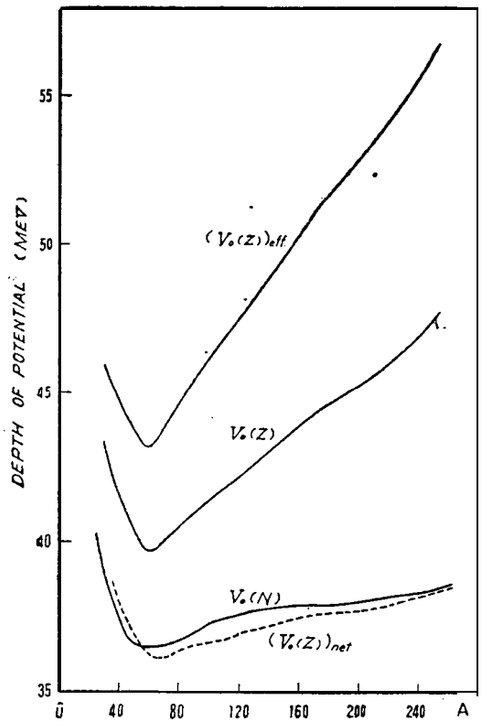

Fig. 6 Modified well strength for the neutron potentials $V_0(N)$, pure potentials $V_0(Z)$, effective proton potentials $(V_0(Z))_{eff}=V_0(Z)+V_a=V_0(Z)+\frac{1}{2}U_c$, and for the net proton potentials $(V_0(Z))_{net}=(V_0(Z))_{eff}-U_c=V_0(Z)-\frac{1}{2}U_c$. In each case, it is no longer constant, but varies with A.

the same behavior, except for a slight difference in the magnitude. $V_0(N)$ has a minimum at N= 28 and $V_0(Z)$ at $Z=28$.

V. LOW VELOCITY 3s AND 4s MAXIMA

The low velocity maxima in the total neutron cross section can be interpreted in terms of the critical ε_0 values at which various s states of binding set in. These critical ε_0 values have been calculated by Green and Lee[3] for the appropriate diffuseness parameters 6. Two of them, those for $3s$ and $4s$ maxima, are cited here (Fig. 7).

On the other hand, following, the definition

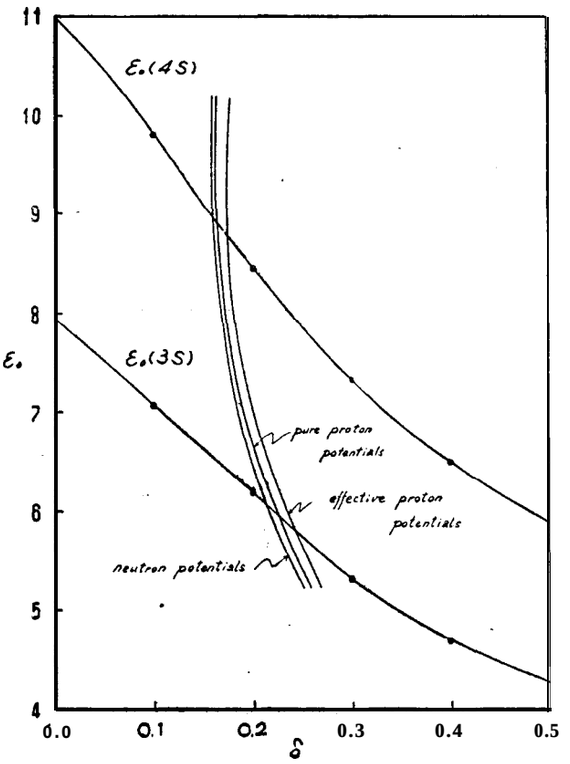

Fig. 7 Well strength *vs* diffuseness parameter ii, (i) for **3s** and 4s maxima in the neutron total cross section, and (ii) for the realistic nuclei. From the intersections of (i) and (ii) we can determine the location of A's at which 3s and 42 maxima occur. The effective proton potentials cannot give the correct location.

$$\varepsilon_0 = \left(\frac{2m}{\hbar^2} V_0 a^2\right)^{1/2} = \left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{V_0}{\delta}^{1/2},$$

where V_0 and δ are the functions of $A_G(N)$ (or $A_G(Z)$), the values of ε_0 are calculated for different $A_G(N)$ (or $A_G(Z)$) and are plotted against 6(N) (or $\delta(Z)$), as shown in Fig. 7. The $A_G(N)'s$ (or $A_G(Z)$) corresponding to the intersections with the Green-Lee curves represent the nuclei at which 3s and $4s$ maxima occur. For the neutron potential, the mass number A- 57 corresponds to the 3s maximam and A = 172 to the $4s$ maximum. For pure proton potential 3s maximum occurs at $A = 55$ and $4s$ maximum at A- 162. In contrast with this, for the effective proton potential $(V_0(Z))_{eff} = V_0(Z) + V_a = V_0(Z) + \frac{1}{2}U_c$, $3s$ maximum locates at A -46 and $4s$ maximum at A- 130. In the Green's paper A= 55 and A= 170 have been taken as the location of the $3s$ and $4s$ maxima.

The experimental data[7] on the s-wave strength function show obvious peaks at

7) J. A. Harvey, Invited Paper to Symposium on Neutron Time-of-Flight Methods, Saclay, France, July 25-27, 1961.

mass numbers 55, 147 and 184. Chase, Wilets and Edmonds[8] have explained these peaks relevantly in terms of a spheroidal nucleus with a deformation. However, the peaks at $A=147$ and $A=187$ join to one at $A=158$, remaining another peak at $A=55$ unchanged, if the nucleus is spherical and with a diffuse surface.[8),9)] The foregoing calculation, especially for the case of pure proton potentials, is in good agreement with this theoretical prediction. The failure of the effective proton potential means that the Coulomb potential has no influence on the neutron scattering phenomena.

VI. CONCLUDING REMARKS

The problem of probing the theoretical basis of nuclear size is essentially equivalent to that of inferring the origin of the nuclear potentials. And since the origin of the nuclear potential should be explained from the nucleon-nucleon interaction within the finite nuclei, this problem becomes much involved. Tauber and Wu[10] used the modified Ritz method to determine the single particle wave functions from a phenomenological nucleon-nucleon interaction. These wave functions were then used in the Hartree-Fock equation to calculate the binding energies and the effective central field for the individual particle in the various shells. The application of this orthodox method to heavy nuclei, however, seems to be rarely done.

Recently, Brueckner, Lockett and Rotenberg[11] have solved the complete self-constitent field problem for the finite nuclei with use of the so called K-matrix. Rozsnai[12] has also carried out the calculation by a simplified method starting from the Hamiltonian proposed by Duerr. Although both theories have achieved considerable success, a complete quantitative agreement has not been found as yet. Before a complete success for understanding the structure of finite nuclei through the many body problem approach is achieved, it is still believed worthwhile to treat the problem in such a manner as has been done in the present paper.

The family of potentials derived in the foregoing sections from the proposed rules of nuclear radii is thought to be more realistic than the original Green's one. This fact, in turn, assures the truth of the proposed rules, and constitutes an indirect proof for them. An elegant and direct proof is still difficult because of the present situation described in the former paragraphs.

The type of potentials adopted here is in many respects similar to the Wood-Saxon potential which is widely used now in the optical model. It has a theoretical basis on

8) D. M. Chase, L. Wilets and A. R. Edmonds, Phys. Rev. 110, 1080 (1958).
9) E. J. Camphell, H. Feshbach, C. E. Porter and V. F. Weisskopf, Technical Report, No. 73 (Laboratory for Nuclear Science, Massachusetts Institute of Technology, 1960).
10) G. E. Tauber and T. Y. Wu, Phys. Rev. 105, 1772 (1957).
11) K. A. Brueckner, A. M. Lockett and M. Rotenberg, Phys. Rev. 121; 255 (1961).
12) B. Rozsnyai, Phys. Rev. 124, 860 (1961).

a nonlinear theory of scalar mesons.[13] It has, in addition, an outstanding advantage that the Wood-Saxon potential does not possess; that is the convenient analytical properties of the wave functions.

There is always a feeling that only one kind of potentials and therefore one kind of nuclear radii are appropriate for studying the nuclear phenomena. Many investigators merely used the proton potential and the radius of the proton density. If the basis for the theory of such type[14] is solidified, our proposed rules of nuclear radii and the argument made in the foregoing sections must be modified. Obviously such modification is very easy for our case.

13) B. J. Malenka, Phys. Rev. 86, 68 (1952).
14) L. R. B. Elton, Nuclear Physics 23, 631 (1961).