VARIETIES WITH VANISHING HOLOMORPHIC EULER CHARACTERISTIC II

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ABSTRACT. We continue our study on smooth complex projective varieties X of maximal Albanese dimension and of general type satisfying $\chi(X, \mathscr{O}_X) = 0$. We formulate a conjectural characterization of such varieties and prove this conjecture when the Albanese variety has only three simple factors.

1. INTRODUCTION

The purpose of this paper is to study the birational geometry of varieties X of general type and of maximal Albanese dimension with $\chi(\omega_X) = 0$.

In recent years, these varieties have attracted considerable attention. Green and Lazarsfeld showed in [GL1] that a variety of maximal Albanese dimension satisfies $\chi(X, \omega_X) \geq 0$. It was conjectured by Kollár [K3, 18.12.1] that a variety of general type and maximal Albanese dimension would satisfy $\chi(\omega_X) > 0$. A couple years later, Ein-Lazarsfeld disprove the conjecture by providing an example of threefold of general type and maximal Albanese dimension with $\chi(\omega_X) = 0$.

In fact, in the recent studies on the structure of the pluricanonical maps and of the Iitaka map, it has been realized that the case $\chi(\omega_X) = 0$ is usually the hardest case. For example, it was shown in [CH1] that the tricanonical map is birational for varieties of general type and maximal Albanese dimension with $\chi(\omega_X) > 0$. However, if we assume $\chi(\omega_X) = 0$ instead, then it is more difficult to prove that the tricanonical map is birational (see [JLT]).

It is thus natural and important to characterize or classify varieties of general type and maximal Albanese dimension with $\chi(\omega_X) = 0$ explicitly. Our previous joint work with Olivier Debarre was the starting point toward this direction, in which we prove a characterization in dimension three. Since the characterizing properties are preserved under birational maps and finite étale maps, it was shown that the Albanese variety has at least three simple factors and the example of Ein and Lazarsfeld is the only possibility in dimension three.

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The main result in this article is to prove a similar characterization in higher dimensions assuming that the Albanese variety has three simple factors.

Theorem 1.1. *Let X be a variety of general type and of maximal Albanese dimension and assume that A_X has only three simple factors. If $\chi(X, \omega_X) = 0$, there exist simple abelian varieties K_1, K_2, K_3, double coverings from normal varieties $F_i \to K_i$ with associated involution τ_i, and an isogeny $\eta\colon K_1 \times K_2 \times K_3 \to A_X$, such that the base change $\widetilde{X}$ is birational to*

$$(F_1 \times F_2 \times F_3)/\langle\sigma\rangle$$

where $\sigma = \tau_1 \times \tau_2 \times \tau_3$ is the diagonal involution. That is, we have the following commutative diagram:

$$\begin{tikzcd}
 & (F_1 \times F_2 \times F_3)/\langle\sigma\rangle \arrow[dr, two heads] & \\
\widetilde{X} \arrow[ur, two heads, "\varepsilon"] \arrow[rr, two heads, "a_{\widetilde{X}}"] \arrow[d, two heads] \arrow[drr, phantom, "\square"] & & K_1 \times K_2 \times K_3 \arrow[d, two heads, "\eta"] \\
X \arrow[rr, two heads, "a_X"'] & & A_X.
\end{tikzcd}$$

where ε is a desingularization.

The birational geometry of varieties of maximal Albanese dimension is governed by cohomological support loci of the push-forward of the canonical sheaf $a_{X*}\omega_X$. It is well-known that $a_{X*}\omega_X$ is a GV-sheaf. The technical advance of this article is indeed the following decomposition theorem (see Theorem 3.4 and Theorem 3.5 for details), which implies that $a_{X*}\omega_X$ is not far from being M-regular.

Theorem 1.2. *Let $f : X \to A$ be a generically finite morphism to an abelian variety. Then, we have*

$$f_*\omega_X \simeq \bigoplus_i (p_i^*\mathscr{F}_i \otimes P_i),$$

where $p_i : A \to A_i$ are quotients of abelian varieties, $\mathscr{F}_i$ are M-regular sheaves on A_i, and P_i are torsion line bundles on A.

Remark 1.3. In the above formulation, we allow $p_i : A \to A_i$ to be trivial quotients, namely p_i could be an isomorphism or a fibration to $\operatorname{Spec}\mathbb{C}$.

This decomposition theorem can certainly be applied to prove globally generated properties for canonical or pluricanonical bundles. Another important application is a criterion for birationality of morphisms between varieties of maximal Albanese dimension, which is a main ingredient of the proof of Theorem 1.1.

The paper is organized as follows. In section 2 we introduce definitions and prove some basic results on Fourier-Mukai transform of GV sheaves. Section 3 is devoted to prove the decomposition theorem and its corollaries. In section 4 we provide several birational criterion of morphisms between varieties of maximal Albanese dimension. In section 5 we study the general structure of varieties X of general type and of maximal Albanese dimension with $\chi(X, \omega_X) = 0$. We formulate a conjectural characterization of such varieties. Finally, in section 6, we restrict ourselves to the case when A_X has only three simple factors and prove Theorem 1.1.

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2. Notation and Preliminaries

For any smooth projective variety X, we will denote by $a_X : X \to A_X$ the Albanese morphism of X and $\widehat{A}_X = \mathrm{Pic}^0(X)$ the dual of the Albanese variety. We will denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves on X. Following [PP2], for any object $\mathscr{E} \in \mathbf{D}(X)$, we write $\mathbf{R}\Delta(\mathscr{E}) := \mathbf{R}\mathscr{H}om(\mathscr{E}, \omega_X)$.

For an abelian variety A and its dual $\widehat{A}$, we denote by $\mathscr{P}_A$ the normalized Poincaré line bundle on $A \times \widehat{A}$. For $\alpha \in \widehat{A}$, we denote by P_α the line bundle that represents α. By [Mu], the following functors give equivalence between $\mathbf{D}(A)$ and $\mathbf{D}(\widehat{A})$:

$$\mathbf{R}\Phi_{\mathscr{P}_A} : \mathbf{D}(A) \to \mathbf{D}(\widehat{A}), \;\; \mathbf{R}\Phi_{\mathscr{P}_A}(\cdot) = \mathbf{R}p_{\widehat{A}*}(p_A^*(\cdot) \otimes \mathscr{P}_A),$$
$$\mathbf{R}\Psi_{\mathscr{P}_A} : \mathbf{D}(\widehat{A}) \to \mathbf{D}(A), \;\; \mathbf{R}\Psi_{\mathscr{P}_A}(\cdot) = \mathbf{R}p_{A*}(p_{\widehat{A}}^*(\cdot) \otimes \mathscr{P}_A).$$

For any coherent sheaf $\mathscr{F}$ on X and any morphism $f : X \to A$ to an abelian variety, we define the i-th cohomological locus

$$V^i(\mathscr{F}, f) := \{\alpha \in \widehat{A} \mid H^i(X, \mathscr{F} \otimes P_\alpha) \neq 0\}.$$

If $f = a_X$ is the Albanese morphism, we will simply denote by $V^i(\mathscr{F})$ the i-th cohomological locus.

For an abelian variety A and its dual $\widehat{A}$, we always use the notation $\widehat{*}$ to denote an abelian subvariety of $\widehat{A}$, and then $* = \widehat{\widehat{*}}$ is the natural quotient of A.

We recall the definition of GV-sheaves and M-regular sheaves on abelian varieties (see [PP1] and [PP2]).

Definition 2.1. Let $\mathscr{F}$ be a coherent sheaf on an abelian variety A. Then $\mathscr{F}$ is a GV-sheaf if

$$\mathrm{codim}_{\widehat{A}} \operatorname{Supp} \mathbf{R}^i\Phi_{\mathscr{P}_A}(\mathscr{F}) \geq i$$

for all $i \geq 0$; $\mathscr{F}$ is a M-regular sheaf if

$$\operatorname{codim}_{\widehat{A}} \operatorname{Supp} \mathbf{R}^i\Phi_{\mathscr{P}_A}(\mathscr{F}) > i$$

for all $i > 0$.

Remark 2.2. The following properties of M-regular sheaves and GV-sheaves are quite useful:

1) if $\mathscr{F}$ is a GV-sheaf (resp. M-regular sheaf) on A, then

$$\widehat{\mathbf{R}\Delta(\mathscr{F})} := \mathbf{R}\Phi_{\mathscr{P}_A}(\mathbf{R}\Delta(\mathscr{F}))[g]$$

is a coherent (resp. torsion-free) sheaf of $\widehat{A}$ supported on $V^0(\mathscr{F})$ ([H1, Theorem 1.2], [PP3, Proposition 2.8]);

2) if $\mathscr{F}$ is a GV-sheaf, then

$$\mathscr{E}xt^i(\widehat{\mathbf{R}\Delta(\mathscr{F})}, \mathscr{O}_{\widehat{A}}) \simeq (-1_{\widehat{A}})^*\mathbf{R}^i\Phi_{\mathscr{P}_A}(\mathscr{F})$$

(see [PP2, Remark 3.13]);

3) $\mathscr{F}$ is a GV-sheaf (resp. M-regular sheaf) if and only if $\operatorname{codim}_{\widehat{A}} V^i(\mathscr{F}) \geq i$ (resp. $> i$) for all $i \geq 1$ ([PP2, Lemma 3.6]).

The following proposition is a generalization of [BL, Proposition 14.7.9 (b)]

Proposition 2.3. *Let $f : A \to B$ be a quotient of abelian varieties. Assume that $\dim A = g$ and $\dim B = g_1$. Then*

(1) *we have the natural isomorphism of functors from $\mathbf{D}(B)$ to $\mathbf{D}(\widehat{A})$:*

$$\mathbf{R}\Phi_{\mathscr{P}_A} \circ f^* \circ [g] \simeq \widehat{f}_* \circ \mathbf{R}\Phi_{\mathscr{P}_B} \circ [g_1];$$

(2) *and the natural isomorphism of functors from $\mathbf{D}(\widehat{B})$ to $\mathbf{D}(A)$:*

$$f^* \circ \mathbf{R}\Psi_{\mathscr{P}_B} \simeq \mathbf{R}\Psi_{\mathscr{P}_A} \circ \widehat{f}_*$$

Remark 2.4. Note that the direct image $\widehat{f}_*$ needs not to be derived for a closed embedding and similarly, f^* needs not to be derived for a smooth morphism.

Proof. We note that (2) is equivalent to (1) by Fourier-Mukai equivalence. It suffices to prove (1).

For brevity, we will abuse the notation of pull-back and push-forward with its derived functors. We consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & A\times\widehat{A} \arrow[l, "p_A"'] \arrow[d, "f_{\widehat{A}}"'] \arrow[dr, "p_{\widehat{A}}"] \arrow[dr, phantom, "\circlearrowleft", near start] & \\
B & B\times\widehat{A} \arrow[l, "\pi_B"'] \arrow[r, "\pi_{\widehat{A}}"] \arrow[dr, phantom, "\square"] & \widehat{A} \\
& B\times\widehat{B} \arrow[ul, "p_B"] \arrow[u, "f_B"'] \arrow[r, "p_{\widehat{B}}"] & \widehat{B}, \arrow[u, "\widehat{f}"']
\end{tikzcd}$$

where each map is either the natural projection, dual map, or base change. We then have

$$\begin{aligned}
\mathbf{R}\Phi_{\mathscr{P}_A}(f^*\mathscr{F}) &= p_{\widehat{A}*}(p_A^* f^*\mathscr{F}\otimes\mathscr{P}_A)\\
&\simeq p_{\widehat{A}*}(f_{\widehat{A}}^*\pi_B^*\mathscr{F}\otimes\mathscr{P}_A)\\
&\simeq \pi_{\widehat{A}*}f_{\widehat{A}*}(f_{\widehat{A}}^*\pi_B^*\mathscr{F}\otimes\mathscr{P}_A)\\
&\simeq \pi_{\widehat{A}*}(\pi_B^*\mathscr{F}\otimes f_{\widehat{A}*}\mathscr{P}_A)\quad\text{(projection formula).}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\widehat{f}_*\mathbf{R}\Phi_{\mathscr{P}_B}(\mathscr{F})) &= \widehat{f}_*p_{\widehat{B}*}(p_B^*\mathscr{F}\otimes\mathscr{P}_B)\\
&\simeq \pi_{\widehat{A}*}\widehat{f}_{B*}(\widehat{f}_B^*\pi_B^*\mathscr{F}\otimes\mathscr{P}_B)\\
&\simeq \pi_{\widehat{A}*}(\pi_B^*\mathscr{F}\otimes\widehat{f}_{B*}\mathscr{P}_B)\quad\text{(projection formula).}
\end{aligned}$$

Thus if f has connected fibers, we conclude the proof by Lemma 2.5 below.

If f has disconnected fibers, we consider the Stein factorization of f:

$$A\xrightarrow{g}B'\xrightarrow{\pi}B,$$

where π is an isogeny between abelian varieties and g is a fibration.

Let $\mathscr{F}\in\mathbf{D}(B)$. Since π is an isogeny, by [BL, Proposition 14.7.9 (b)], we have

$$\mathbf{R}\Phi_{\mathscr{P}_{B'}}(\pi^*\mathscr{F})\simeq\widehat{\pi}_*\mathbf{R}\Phi_{\mathscr{P}_B}(\mathscr{F}).$$

Since g is a fibration, we also have

$$\widehat{g}_*\mathbf{R}\Phi_{\mathscr{P}_{B'}}(\pi^*\mathscr{F})[g_1]\simeq\mathbf{R}\Phi_{\mathscr{P}_A}(g^*\pi^*\mathscr{F})[g].$$

Thus we have

$$\begin{aligned}
\widehat{f}_*\mathbf{R}\Phi_{\mathscr{P}_B}(\mathscr{F})[g_1] &\simeq \widehat{g}_*\widehat{\pi}_*\mathbf{R}\Phi_{\mathscr{P}_B}(\mathscr{F})[g_1]\\
&\simeq \widehat{g}_*\mathbf{R}\Phi_{\mathscr{P}_{B'}}(\pi^*\mathscr{F})[g_1]\\
&\simeq \mathbf{R}\Phi_{\mathscr{P}_A}(g^*\pi^*\mathscr{F})[g]\\
&\simeq \mathbf{R}\Phi_{\mathscr{P}_A}(f^*\mathscr{F})[g].
\end{aligned}$$

This completes the proof. □

Lemma 2.5. *Let $f: A\to B$ be a quotient of abelian varieties with connected fibers. Keeping the notation as in Proposition 2.3, we have*

$$\mathbf{R}f_{\widehat{A}*}\mathscr{P}_A\circ[g]\simeq\widehat{f}_{B*}\mathscr{P}_B\circ[g_1]\in\mathbf{D}(B\times\widehat{A}).$$

Proof. Let K be the kernel of f.

Step 1. *We first assume that $A=B\times K$ and $f: A\to B$ is the natural projection.*

Then $\mathscr{P}_A = p_{13}^*\mathscr{P}_B \otimes p_{24}^*\mathscr{P}_K$ on $A\times\widehat{A} = B\times K\times\widehat{B}\times\widehat{K}$ and $\widehat{f}:\widehat{B}\to\widehat{B}\times\widehat{K}$ is the closed embedding $x\to(x,0_{\widehat{K}})$ for $x\in\widehat{B}$. We consider

$$\begin{tikzcd}
A\times\widehat{A}=B\times K\times\widehat{B}\times\widehat{K} \arrow[r, "p_{24}"] \arrow[d, "f_{\widehat{A}}=(p_1,p_3,p_4)"] & K\times\widehat{K} \arrow[d, "p_{\widehat{K}}"] \\
B\times\widehat{A}=B\times\widehat{B}\times\widehat{K} \arrow[r, "q_{\widehat{K}}"] \arrow[d, "p_B"] & \widehat{K} \\
B\times\widehat{B} \arrow[u, bend left, "\widehat{f}_B"] &
\end{tikzcd}$$

Therefore,

$$\begin{aligned}
f_{\widehat{A}_*}\mathscr{P}_A &= f_{\widehat{A}*}(p_{13}^*\mathscr{P}_B\otimes p_{24}^*\mathscr{P}_K)\\
&= f_{\widehat{A}*}(f_{\widehat{A}}^*p_B^*\mathscr{P}_B\otimes p_{24}^*\mathscr{P}_K)\\
&= p_B^*\mathscr{P}_B\otimes f_{\widehat{A}*}p_{24}^*(\mathscr{P}_K)\\
&= p_B^*\mathscr{P}_B\otimes q_{\widehat{K}}^*p_{\widehat{K}*}(\mathscr{P}_K)\\
&= p_B^*\mathscr{P}_B\otimes q_{\widehat{K}}^*\mathbb{C}_{0_{\widehat{K}}}[g_1-g]\\
&\simeq (\widehat{f}_{B*}\mathscr{P}_B)[g_1-g].
\end{aligned}$$

Step 2. We assume that $f:A\to B$ has connected fibers. By Poincaré's reducibility theorem, we can take $\pi_B:\widetilde{B}\to B$ to be an isogeny such that the fiber product $\widetilde{f}:\widetilde{A}:=A\times_B\widetilde{B}\to\widetilde{B}$ is isomorphic to the projection $\widetilde{B}\times K\to\widetilde{B}$.

We then have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{A}\times\widehat{\widetilde{A}} \arrow[d, "f_{\widehat{\widetilde{A}}}"] & \widetilde{A}\times\widehat{A} \arrow[l, "\pi_1"'] \arrow[r, "\tau_1"] \arrow[d, "h_A"] & A\times\widehat{A} \arrow[d, "f_{\widehat{A}}"] \\
\widetilde{B}\times\widehat{\widetilde{A}} & \widetilde{B}\times\widehat{A} \arrow[l, "\pi_2"'] \arrow[r, "\tau_2"] & B\times\widehat{A} \\
\widetilde{B}\times\widehat{\widetilde{B}} \arrow[u, hook, "\widehat{f}_{\widetilde{B}}"] & \widetilde{B}\times\widehat{B} \arrow[l, "\pi_3"'] \arrow[r, "\tau_3"] \arrow[u, hook, "h_B"'] & B\times\widehat{B}. \arrow[u, hook, "\widehat{f}_B"']
\end{tikzcd}$$

By [BL, (14.2)], one sees that $\pi_1^*\mathscr{P}_{\widetilde{A}}=\tau_1^*\mathscr{P}_A$ and $\pi_3^*\mathscr{P}_{\widetilde{B}}=\tau_3^*\mathscr{P}_B$. Together with base change and Step 1, it is easy to verify that

$$\begin{aligned}
\tau_2^*f_{\widehat{A}*}\mathscr{P}_A[g] &\simeq \pi_2^*f_{\widehat{\widetilde{A}}*}\mathscr{P}_{\widetilde{A}}[g]\\
&\simeq \pi_2^*\widehat{f}_{\widetilde{B}*}\mathscr{P}_{\widetilde{B}}[g_1]\\
&\simeq \tau_2^*\widehat{f}_{B*}\mathscr{P}_B[g_1].
\end{aligned}$$

Thus $f_{\widehat{A}*}\mathscr{P}_A[g-g_1]$ is a coherent sheaf supported on the image of $\widehat{f}_B$ and of rank 1 on each point of its support. Therefore, we have

$$f_{\widehat{A}*}\mathscr{P}_A \simeq \big(\widehat{f}_{B*}\mathscr{P}_B \otimes L\big)[g_1 - g]$$

for some $L \in \mathrm{Im}(\mathrm{Pic}^0(B) \to \mathrm{Pic}^0(B \times \widehat{A}))$. Finally, by the fact that $\mathbf{R}p_{\widehat{A}*}\mathscr{P}_A = \mathbb{C}_{0_{\widehat{A}}}[-g]$, we have $L \cong \mathscr{O}_{B\times\widehat{A}}$. □

Proposition 2.6. *Let $f : A \to B$ be a quotient of abelian varieties and let $\mathscr{F}$ be a GV-sheaf on B. Then $\mathbf{R}\widehat{\Delta(f^*\mathscr{F})} := \mathbf{R}\Phi_{\mathscr{P}_A}(\mathbf{R}\Delta(f^*\mathscr{F}))[g]$ is a coherent sheaf supported on $\widehat{f}(\widehat{B})$. Moreover, if $\mathscr{F}$ is M-regular on B, then $\mathbf{R}\widehat{\Delta(f^*\mathscr{F})}$ is a pure sheaf supported on $\widehat{f}(\widehat{B})$.*

Proof. Since $\mathscr{F}$ is GV-sheaf, $\mathbf{R}\widehat{\Delta(\mathscr{F})} := \mathbf{R}\Phi_{\mathscr{P}_B}(\mathbf{R}\Delta(\mathscr{F}))[g_1]$ is a coherent sheaf on $\widehat{B}$ by Remark 2.2.1. By Lemma 2.3, it follows that

$$\mathbf{R}\Phi_{\mathscr{P}_A}(\mathbf{R}\Delta(f^*\mathscr{F}))[g] = \mathbf{R}\widehat{f}_*(\mathbf{R}\widehat{\Delta(\mathscr{F})}) = \widehat{f}_*(\mathbf{R}\widehat{\Delta(\mathscr{F})})$$

is also a coherent sheaf supported on $\widehat{f}(\widehat{B})$. If $\mathscr{F}$ is M-regular on B, then $\mathbf{R}\widehat{\Delta(\mathscr{F})}$ is torsion-free by Remark 2.2.1. Hence $\mathbf{R}\widehat{\Delta(f^*\mathscr{F})}$ is a pure sheaf supported on $\widehat{f}(\widehat{B})$. □

Corollary 2.7. *Let $f : A \to B$ be a quotient of abelian varieties. Let $\mathscr{F}_1$ be a M-regular sheaf on B and let $\mathscr{F}_2$ be a GV-sheaf on A. Assume that $\widehat{f}(\widehat{B})$ is not contained in $V^0(\mathscr{F}_2)$, then* $\mathrm{Hom}_A(f^*\mathscr{F}_1, \mathscr{F}_2) = 0$.

Proof. We have

$$\begin{aligned}
&\mathrm{Hom}_A(f^*\mathscr{F}_1, \mathscr{F}_2) = \mathrm{Hom}_{\mathbf{D}(A)}(f^*\mathscr{F}_1, \mathscr{F}_2)\\
\simeq\ &\mathrm{Hom}_{\mathbf{D}(A)}(\mathbf{R}\Delta(\mathscr{F}_2), \mathbf{R}\Delta(f^*\mathscr{F}_1))\\
\simeq\ &\mathrm{Hom}_{\mathbf{D}(\widehat{A})}(\mathbf{R}\widehat{\Delta(\mathscr{F}_2)}, \mathbf{R}\widehat{\Delta(f^*\mathscr{F}_1)}) \text{ by [Mu, Corollary 2.5]}\\
=\ &\mathrm{Hom}_{\widehat{A}}(\mathbf{R}\widehat{\Delta(\mathscr{F}_2)}, \mathbf{R}\widehat{\Delta(f^*\mathscr{F}_1)}).
\end{aligned}$$

By Proposition 2.6 and Remark 2.2, $\mathbf{R}\widehat{\Delta(f^*\mathscr{F}_1)}$ is a pure sheaf supported on $\widehat{f}(\widehat{B})$ and $\mathbf{R}\widehat{\Delta(\mathscr{F}_2)}$ is supported on $V^0(\mathscr{F}_2)$. Since $\widehat{f}(\widehat{B})$ is not contained in $V^0(\mathscr{F}_2)$, we have

$$\mathrm{Hom}_{\widehat{A}}(\mathbf{R}\widehat{\Delta(\mathscr{F}_2)}, \mathbf{R}\widehat{\Delta(f^*\mathscr{F}_1)}) = 0.$$

We conclude the proof. □

3. A decomposition theorem for $f_*\omega_X$

Let $f : X \to A$ be a generically finite morphism onto its image and A is an abelian variety. In this section we study the sheaf $f_*\omega_X$.

By the work of Green-Lazarsfeld ([GL2]) and Simpson [S], we know that $V^i(\omega_X, f) = V^i(f_*\omega_X)$ is a union of torsion translated abelian subvarieties of $\widehat{A}$. Moreover, each irreducible component of $V^i(\omega_X, f)$ in $\widehat{A}$ is of codimension

$\geq i$ ([GL1]). In particular $f_*\omega_X$ is a GV-sheaf on A. However $f_*\omega_X$ often fails to be M-regular, namely $V^i(f_*\omega_X)$ often has an irreducible component of codimension i in $\widehat{A}$. In particular, if f is generically finite onto A, then $\mathscr{O}_A$ is a direct summand of $f_*\omega_X$.

Definition 3.1. Let $f: X \to A$ be a generically finite morphism to an abelian variety A. When f is surjective onto A, then we define $\mathscr{W}_{X/A}$ or simply $\mathscr{W}$ to be the direct summand of $f_*\omega_X$ so that $f_*\omega_X = \mathscr{O}_A \oplus \mathscr{W}_{X/A}$. When f is not surjective onto A, we define $\mathscr{W}_{X/A}$ to be $f_*\omega_X$.

Instead, we are interesting in the M-regularity of $\mathscr{W}$. This is actually the main technical difficulty to study birational geometry of varieties of maximal Albanese dimension. Hence we consider the following set which measures how far $\mathscr{W}_{X/A}$ is from being M-regular.

Definition 3.2. Let $\mathscr{F}$ be a GV-sheaf on A, of which cohomological support loci consists of torsion translated subtori. We define $S^i(\mathscr{F})$ to be the set of torsion translated subtori of $\widehat{A}$ consisting of irreducible components of $V^i(A, \mathscr{F})$ of codimension i in $\widehat{A}$. We define $S(\mathscr{F}) := \cup_{i=0}^n S^i(\mathscr{F})$ and $S(\mathscr{F})^* := \cup_{i=1}^n S^i(\mathscr{F})$. For brevity, We use S^i_X (resp. S_X, S^*_X) to denote $S^i(\mathscr{W}_{X/A})$ (resp. $S(\mathscr{W}_{X/A})$, $S^*(\mathscr{W}_{X/A})$.

Therefore, by definition, $S(\mathscr{F})^*$ is empty if and only if $\mathscr{F}$ is M-regular. Note that $V^i(\mathscr{W}_{X/A}) = V^i(f_*\omega_X)$ unless $q(X) = \dim X$ and $i = \dim X$, and hence every irreducible component of S_X has dimension > 0 by [EL, Theorem 3].

Convention. Replacing X by a finite étale covering, we may and do assume that all irreducible component of S_X pass through the origin.

For each component $\widehat{A}_k \in S^i_X$ passing through the origin, we have the dual abelian variety A_k together with a surjection $p_k : A \to A_k$. We consider a modification of the Stein factorization:

(1)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q_k"'] & A \arrow[d, "p_k"] \\ Y_k \arrow[r, "h_k"] & A_k \end{tikzcd}$$

where q_k is a fibration, Y_k is a smooth projective variety. By [CDJ, Theorem 3.1], we know that $R^i q_{k*}\omega_X = \omega_{Y_k}$ and $V^0(h_{k*}\omega_{Y_k}) = \widehat{A}_k$, where $i = \mathrm{codim}(\widehat{A}_k, \widehat{A})$.

Lemma 3.3. *For each $\widehat{A}_k \in S^i_X$ passing through the origin of $\widehat{A}$ and each $j \geq 0$, we have*

$$S^j_{Y_k} = \{\widehat{A} \in S^{i+j}_X \mid \widehat{A} \subset \widehat{A}_k\}.$$

Proof. By Kollár's results [K1, Proposition 7.6] and [K2, Theorem 3.1], we have $R^i q_{k*}\omega_X = \omega_{Y_k}$, and for $P \in \widehat{A}_k$, we have

$$\begin{aligned} H^{i+j}(X, \omega_X \otimes q_k^* h_k^* P) &\simeq \sum_{s+t=i+j} H^s(Y_k, R^t q_{k*}\omega_X \otimes h_k^* P) \\ &\simeq \sum_{s+t=i+j} H^s(A_k, h_{k*}R^t q_{k*}\omega_X \otimes P). \end{aligned}$$

Hence $S^j_{Y_k} \subseteq \{\widehat{A} \in S^{i+j}_X \mid \widehat{A} \subset \widehat{A}_k\}$. On the other hand, by Hacon's generic vanishing theorem [H1, Corollary 4.2], $h_{k*}R^t q_{k*}\omega_X$ is a GV-sheaf on A_k for each $t \geq 0$. Thus $S^j_{Y_k} \supseteq \{\widehat{A} \in S^{i+j}_X \mid \widehat{A} \subset \widehat{A}_k\}$. $\square$

Theorem 3.4. *Assume that each component of S_X passes through the origin of $\widehat{A}$. Then for each $\widehat{A}_k \in S_X$, there exists a nontrivial M-regular sheaf $\mathscr{F}_k$ on A_k, which is a direct summand of $h_{k*}\omega_{Y_k}$. Moreover we have an isomorphism*

$$\mathscr{W}_{X/A} \simeq \bigoplus_k p_k^* \mathscr{F}_k.$$

Proof. For each $\widehat{A}_k \in S^i_X$, we use the notation in the diagram (1) and define $Z_k := Y_k \times_{A_k} A$. Then $Z_k \to Y_k$ is a smooth abelian fibration and Z_k is a smooth projective variety. We recall that the dimension of a general fiber of q_k is i (see for instance the proof of [EL, Theorem 3]). Hence we have the natural morphism

$$g_k : X \to Z_k,$$

which is generically finite and surjective. Considering the natural morphisms

$$\begin{tikzcd} X \arrow[r, "g_k"] \arrow[rr, bend left, "f"] \arrow[dr, "q_k"'] & Z_k \arrow[r, "f_k"] \arrow[d, "r_k"] \arrow[dr, phantom, "\square"] & A \arrow[d, "p_k"] \\ & Y_k \arrow[r, "h_k"] & A_k, \end{tikzcd} \tag{2}$$

we have

$$\begin{aligned} f_*\omega_X &= f_{k*}g_{k*}\omega_X \\ &= f_{k*}(\omega_{Z_k} \oplus \mathscr{Q}_k) \\ &= p_k^*(h_{k*}\omega_{Y_k}) \oplus f_{k*}\mathscr{Q}_k, \end{aligned} \tag{3}$$

where the last equality holds because the right part of diagram (12) is Cartesian.

We now let $0 < d_N < d_{N-1} < \cdots < d_2 < d_1 < n$ be the positive numbers such that $S^{d_i}_X$ is not empty.

Note that Theorem 3.4 holds in dimension 1. We argue by induction on $\dim X$. Thus we suppose that Theorem 3.4 holds for varieties of dimension $< \dim X$.

Step 1. Define $\mathscr{F}_i$ on A_i for each $\widehat{A}_i \in S^*_X$.

Suppose $\widehat{A}_m \in S^{d_r}_X$, we know by Lemma 3.3 that,

$$\begin{aligned} S_{Y_m} &= S^0_{Y_m} \cup S^{d_{r-1}-d_r}_{Y_m} \cup \ldots \cup S^{d_2-d_r}_{Y_m} \cup S^{d_1-d_r}_{Y_m} \\ &= \{\widehat{A}_m\} \cup \{\widehat{A} \in \cup_{j<r} S^{d_j}_X \mid \widehat{A} \subset \widehat{A}_m\}. \end{aligned}$$

By induction, we know that there exists a nontrivial M-regular sheaf $\mathscr{F}_m$ on A_m such that $h_{m*}\omega_{Y_m}$ is a direct sum of $\mathscr{F}_m$ with sheaves pulled back from the dual of elements of $S^*_{Y_m}$. Therefore, for $P \in \widehat{A}_m$ general, we have

$$\dim H^0(Y_m, \omega_{Y_m} \otimes h^*_m P) = \dim H^0(A_m, \mathscr{F}_m \otimes P), \tag{4}$$

and moreover, by (3), $p^*_m\mathscr{F}_m$ is a direct summand of $\mathscr{W}_{X/A}$.

Step 2. Derive the decomposition.

We start with $S^{d_1}_X$. By Step 1, for each $\widehat{A}_k \in S^{d_1}_X$, there exists a coherent sheaf $\mathscr{W}_k$ on A such that we have the decomposition:

$$\mathscr{W}_{W/A} = p^*_k\mathscr{F}_k \oplus \mathscr{W}_k.$$

For distinct components $\widehat{A}_{k_1}, \widehat{A}_{k_2}$ in $S^{d_1}_X$, we now consider the decomposition of identity

$$\mathrm{Id} \;:\; p^*_{k_2}\mathscr{F}_{k_2} \hookrightarrow \mathscr{W}_{X/A} = p^*_{k_1}\mathscr{F}_{k_1} \oplus f_{k_1*}\mathscr{W}_{k_1} \twoheadrightarrow p^*_{k_2}\mathscr{F}_{k_2}. \tag{5}$$

By Corollary 2.7, we know that

$$\mathrm{Hom}_A(p^*_{k_2}\mathscr{F}_{k_2}, p^*_{k_1}\mathscr{F}_{k_1}) = 0.$$

Hence $p^*_{k_2}\mathscr{F}_{k_2}$ is a direct summand of $\mathscr{W}_{k_1}$ and thus $p^*_{k_1}\mathscr{F}_{k_1} \oplus p^*_{k_2}\mathscr{F}_{k_2}$ is a direct summand of $\mathscr{W}_{X/A}$.

Continuing in this way, we finally get a decomposition:

$$\mathscr{W}_{X/A} = \bigoplus_{\widehat{A}_k \in S^{d_1}_X} p^*_k\mathscr{F}_k \bigoplus \mathscr{W}_{d_1}.$$

Now we apply induction on d_i. Suppose that we have the decomposition

$$\mathscr{W}_{X/A} = \bigoplus_{\widehat{A}_k \in \cup_{j\leq r-1} S^{d_j}_X} p^*_k\mathscr{F}_k \bigoplus \mathscr{W}_{d_{r-1}},$$

where $r \leq N$. Then for $\widehat{A}_m \in S^{d_r}_X$, we consider

$$\mathrm{Id} \;:\; p^*_m\mathscr{F}_m \hookrightarrow \bigoplus_{\widehat{A}_k \in \cup_{j\leq r-1} S^{d_j}_X} p^*_k\mathscr{F}_k \bigoplus \mathscr{W}_{d_{r-1}} \twoheadrightarrow p^*_m\mathscr{F}_m,$$

we again conclude by (2.7) that

$$\mathrm{Hom}_A(p^*_m\mathscr{F}_m, \bigoplus_{\widehat{A}_k \in \cup_{j\leq r-1} S^{d_j}_X} p^*_k\mathscr{F}_k) = 0.$$

It follows that $p_m^*\mathscr{F}_m$ is indeed a direct summand of $\mathscr{W}_{d_{r-1}}$. As before, we then get the decomposition for each element of $S_X^{d_r}$:

$$\mathscr{W}_{X/A} = \bigoplus_{\widehat{A}_k \in \cup_{j\leq r} S_X^{d_j}} p_k^*\mathscr{F}_k \bigoplus \mathscr{W}_{d_r}.$$

By induction, we end up with the decomposition

$$\mathscr{W}_{X/A} = \bigoplus_{\widehat{A}_k \in S_X^*} p_k^*\mathscr{F}_k \bigoplus \mathscr{W}_{d_N}. \tag{6}$$

Step 3. Show that $\mathscr{W}_{d_N}$ is either M-regular on A if $V^0(f_*\omega_X) = \widehat{A}$ or trivial otherwise.

As a direct summand of $\mathscr{W}_{X/A}$, $\mathscr{W}_{d_N}$ is a GV-sheaf (possibly trivial). It suffices to show that $S^*(\mathscr{W}_{d_N}) = \emptyset$.

Assume to the contrary that $S^*(\mathscr{W}_{d_N}) \subset S_X^*$ is not empty. We then pick $\widehat{A}_m \in S^{d_r}(\mathscr{W}_{d_N})$ for some d_r.

For $P \in \widehat{A}_m$ general, we have

$$\begin{aligned}
h^{d_r}(A, f_*\omega_X \otimes p_m^*P) &= h^{d_r}(A, \mathscr{W}_{X/A} \otimes p_m^*P) \\
&= \sum_{\widehat{A}_k \in S_X^*} h^{d_r}(A, p_k^*\mathscr{F}_k \otimes p_m^*P) + h^{d_r}(A, \mathscr{W}_{d_N} \otimes p_m^*P) \\
&\geq h^{d_r}(A, p_m^*(\mathscr{F}_m \otimes P)) + h^{d_r}(A, \mathscr{W}_{d_N} \otimes p_m^*P) \\
&> h^{d_r}(A, p_m^*(\mathscr{F}_m \otimes P)) \\
&= \sum_{i+j=d_r} h^i(A_m, R^j p_{m*}p_m^*(\mathscr{F}_m \otimes P)) \\
&= h^0(A_m, R^{d_r} p_{m*}p_m^*(\mathscr{F}_m \otimes P)) \\
&= h^0(A_m, \mathscr{F}_m \otimes P),
\end{aligned}$$

where the second equality holds because of (6) and the last two equalities holds because for each $j \geq 0$, $R^j p_{m*}p_m^*\mathscr{F}_m$ is a direct sum of copies of $\mathscr{F}_m$ and hence is M-regular on A_m, and in particular $R^{d_r}p_{m*}p_m^*(\mathscr{F}_m) = \mathscr{F}_m$.

On the other hand, since $R^j p_{m*} f_*\omega_X$ is a GV-sheaf on A_m, we have

$$\begin{aligned}
h^{d_r}(A, f_*\omega_X \otimes p_m^*P) &= h^0(A_m, R^{d_r}p_{m*}f_*\omega_X \otimes P) \\
&= h^0(A_m, h_{m*}R^{d_r}q_{m*}\omega_X \otimes P) \\
&= h^0(A_m, h_{m*}\omega_{Y_m} \otimes P).
\end{aligned}$$

Combining all the (in)equalities, we get that, for $P \in \widehat{A}_m$ general,

$$h^0(A_m, h_{m*}\omega_{Y_m} \otimes P) > h^0(A_m, \mathscr{F}_m \otimes P),$$

which is a contradiction to (4). Therefore, $S^*(\mathscr{W}_{d_N}) = \emptyset$. □

Theorem 3.5. *Let $f : X \to A$ be a generically finite morphism. Then, for each $P_k + \widehat{A}_k \in S_X$, there exists a nontrivial M-regular sheaf $\mathscr{F}_k$ on A_k such that we have an isomorphism*

$$\mathscr{W}_{X/A} \simeq \bigoplus_{P_k+\widehat{A}_k\in S_X^*} P_k^{-1} \otimes p_k^*\mathscr{F}_k.$$

Proof. We take an étale cover $\pi : A' \to A$ such that, for the induced morphism $X' := X \times_A A' \to A'$, every element of $S_{X'}$ contains the origin of $\widehat{A'}$.

Note that $\mathscr{W}_{X'/A'} \simeq \pi^*\mathscr{W}_{X/A}$ and hence $\mathbf{R}\Delta\mathscr{W}_{X'/A'} \simeq \pi^*\mathbf{R}\Delta\mathscr{W}_{X/A}$. We denote respectively by $\mathscr{M}_{X'}$ and $\mathscr{M}_X$ the sheaves $\mathbf{R}\Delta\widehat{(\mathscr{W}_{X'/A'})}$ and $\mathbf{R}\Delta\widehat{(\mathscr{W}_{X/A})}$. Then, by Proposition 2.3, $\mathscr{M}_{X'} \simeq \widehat{\pi}_*\mathscr{M}_X$.

By Theorem 3.4 and Remark 2.2, we know that $\mathscr{M}_{X'}$ is a direct sum of pure sheaves supported on abelian subvarieties of $\widehat{A'}$. Since $\widehat{\pi}$ is an isogeny between abelian varieties, $\mathscr{M}_X$ is a direct summand of $\widehat{\pi}^*\widehat{\pi}_*\mathscr{M}_X = \widehat{\pi}^*\mathscr{M}_{X'}$. Hence $\mathscr{M}_X$ is also a direct sum of pure sheaves supported on torsion translates of abelian subvarieties of $\widehat{A}$:

$$\mathscr{M}_X = \bigoplus_k \tau_{P_k}^*\iota_{k*}\mathscr{M}_k,$$

where τ_{P_k} is the translation by a torsion element $P_k \in \widehat{A}$, $\iota_k \colon \widehat{A}_k \hookrightarrow \widehat{A}$ is the embedding of an abelian subvariety of $\widehat{A}$, and $\mathscr{M}_k$ is a torsion-free sheaf on $\widehat{A}_k$.

Note that, by Fourier-Mukai duality, we have the formula:

$$(-1_A)^*\mathscr{W}_{X/A} \simeq \mathbf{R}\Delta\mathbf{R}\Psi_{\mathscr{P}_A}(\mathscr{M}_X) \simeq \mathbf{R}\Delta\mathbf{R}\Psi_{\mathscr{P}_A}(\bigoplus_k \tau_{P_k}^*\iota_{k*}\mathscr{M}_k). \tag{7}$$

Since τ_{P_k} is a translation and by Proposition 2.3, we have

$$\mathbf{R}\Psi_{\mathscr{P}_A}(\tau_{P_k}^*\iota_{k*}\mathscr{M}_k) \simeq P_k^{-1} \otimes \mathbf{R}\Psi_{\mathscr{P}_A}(\iota_{k*}\mathscr{M}_k) \simeq P_k^{-1} \otimes p_k^*\mathbf{R}\Psi_{\mathscr{P}_{A_k}}\mathscr{M}_k.$$

Since p_k is a smooth morphism, by [Hu, (3.17)], we have

$$\begin{aligned} \mathbf{R}\Delta(P_k^{-1} \otimes p_k^*\mathbf{R}\Psi_{\mathscr{P}_{A_k}}\mathscr{M}_k) &\simeq \mathbf{R}\Delta(p_k^*\mathbf{R}\Psi_{\mathscr{P}_{A_k}}\mathscr{M}_k) \otimes P_k \\ &\simeq p_k^*\big(\mathbf{R}\Delta(\mathbf{R}\Psi_{\mathscr{P}_{A_k}}\mathscr{M}_k)\big) \otimes P_k. \end{aligned}$$

We define $\mathscr{F}_k := \mathbf{R}\Delta(\mathbf{R}\Psi_{\mathscr{P}_{A_k}}(\mathscr{M}_k))$. Since $p_k^*\mathscr{F}_k \otimes P_k$ is a direct summand of the coherent sheaf $\mathscr{W}_{X/A}$, it follows that $\mathscr{F}_k$ is a sheaf. Moreover, $\mathbf{R}\Phi_{\mathscr{P}_{A_k}}\mathbf{R}\Delta(\mathscr{F}_k)[\dim A_k] \simeq (-1)_{A_k}^*\mathscr{M}_k$ is a pure sheaf on A_k, hence $\mathscr{F}_k$ is M-regular. This concludes the proof of Corollary 3.5. $\square$

The following corollary is clear from the above theorem and will be used to prove a birational criterion for morphisms between varieties of maximal Albanese dimension in Section 5.

Corollary 3.6. *Let $f : X \to A$ be a generically finite morphism. Assume that $\mathscr{Q}$ is a direct summand of $f_*\omega_X$. Then we have the decomposition for the sheaf $\widehat{\mathbf{R}\Delta(\mathscr{Q})}$:*

$$\widehat{\mathbf{R}\Delta(\mathscr{Q})} \simeq \bigoplus_{P_k+\widehat{A}_k \in S(\mathscr{Q})} \mathscr{M}_k,$$

where $\mathscr{M}_k$ is a pure sheaf supported on $-P_k + \widehat{A}_k$.

Theorem 3.7. *Let $f : X \to A$ be a generically finite morphism. Then there exists an abelian Galois étale cover $\pi_A : \widetilde{A} \to A$ with the base change $\widetilde{f} : \widetilde{X} := X \times_A \widetilde{A} \to \widetilde{A}$ such that $K_{\widetilde{X}}$ is globally generated away from the exceptional locus of $\widetilde{f}$.*

Proof. By Theorem 3.5, we have $\mathscr{W}_{X/A} \simeq \bigoplus_{P_k+\widehat{A}_k \in S_X} P_k^{-1} \otimes p_k^*\mathscr{F}_k$, where $\mathscr{F}_k$ is M-regular on A_k. By [D2, Proposition 3.1], there exists an abelian Galois étale cover $\pi_A : \widetilde{A} \to A$ such that $\pi_A^*(P_k^{-1} \otimes p_k^*\mathscr{F}_k)$ is globally generated. Considering the base change:

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \widetilde{A} \arrow[d, "\pi_A"] \\ X \arrow[r, "f"'] & A, \end{tikzcd}$$

we have $\mathscr{W}_{\widetilde{X}/\widetilde{A}} = \pi_A^*\mathscr{W}_{X/A}$. Therefore, $\mathscr{W}_{\widetilde{X}/\widetilde{A}}$ is globally generated and so is $\widetilde{f}_*\omega_{\widetilde{X}}$. This implies that $K_{\widetilde{X}}$ is globally generated away from the exceptional locus of $\widetilde{f}$. $\square$

4. Criterion of birationality

We consider a surjective and generically finite morphism between smooth projective varieties $t : X \to Y$. We are interested to know when t is birational. When there exists a generically finite morphism $p : Y \to A$ over an abelian varitey A, there are several cohomological criterion about the birationality of t, see for instance [HP2, Theorem 3.1], and [CDJ, Lemma 5.4].

In this section, we consider the case when $X \xrightarrow{t} Y \xrightarrow{g} A$ are generically finite over the abelian variety A. We denote by $f = g \circ t$. We will always assume that

$$V^0(\omega_X, f) = \cup_{i=1}^N \widehat{A}_i, \tag{8}$$

where $\widehat{A}_i$ is a proper abelian subvariety of $\widehat{A}$, and in particular, we have $\chi(X, \omega_X) = 0$.

We may write

$$t_*\omega_X = \omega_Y \oplus \mathscr{Q}. \tag{9}$$

Then t is birational if and only if $\mathscr{Q}=0$. Since $f_*\omega_X$ is a GV-sheaf, so is $g_*\mathscr{Q}$. Hence $\mathscr{Q}=0$ if and only if $V^0(\mathscr{Q},g)=\emptyset$.

For any irreducible component $\widehat{A}_i\subset V^0(\omega_X,f)$, we consider the Stein factorizations

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "h_{X_i}"] & Y \arrow[r, "g"] \arrow[d, "h_{Y_i}"] & A \arrow[d, "\pi_i"] \\ X_i \arrow[r, "t_i"] & Y_i \arrow[r, "g_i"] & A_i \end{tikzcd} \tag{10}$$

and the set

$$\Sigma_b:=\{1\leq j\leq N\mid t_j \text{ is birational}\},$$

and its complementary set

$$\Sigma_{nb}:=\{1\leq j\leq N\mid t_j \text{ is not birational}\}.$$

Proposition 4.1. *Under the above assumptions, it follows that*

1) *for any $T\in S(g_*\mathscr{Q})$, $T\subseteq\cup_{j\in\Sigma_{nb}}\widehat{A}_j$, and in particular,*

$$\bigcup_{T\in S(g_*\mathscr{Q})}T\subseteq\bigcup_{j\in\Sigma_{nb}}\widehat{A}_j;$$

2) *if all t_i are birational, t is also birational.*

Proof. We first prove 1). Assume that $S(g_*\mathscr{Q})\neq\emptyset$ and take $T\in S(g_*\mathscr{Q})\subset S(f_*\omega_X)$. By Simpson's theorem [S], we can write $T=P+\widehat{K}$, where P is a torsion point and $\widehat{K}$ is a abelian subvariety of $\widehat{A}$. Assume $P+\widehat{K}\subset\widehat{A}_1$, it suffices to prove that $1\in\Sigma_{nb}$.

Since $P\in\widehat{A}_1$, we may take an étale cover $A_1'\to A_1$ such that the pull-back of P is trivial. After base change by $A_1'\to A_1$, diagram (10) now reads:

$$\begin{tikzcd} X' \arrow[r, "t'"] \arrow[d, "h_X'"] & Y' \arrow[r, "g'"] \arrow[d, "h_Y'"] & A' \arrow[d, "\pi'"] \\ X_1' \arrow[r, "t_1'"] & Y_1' \arrow[r, "g_1'"] & A_1' \end{tikzcd}$$

where all h_X', h_Y' are fibrations and t_1' is birational if and only if t_1 is birational.

We know that $P+\widehat{K}$ is an irreducible component of $V^k(\mathscr{Q},g)$. Then we consider the composition of morphisms $X'\xrightarrow{t'}Y'\xrightarrow{g'}A'\to A\to K$ and take smooth models of the Stein factorization of $X'\to K$ and $Y'\to K$. We get

$$\begin{tikzcd} X' \arrow[r, "t'"] \arrow[d, "\mu_X"] & Y' \arrow[r, "g'"] \arrow[d, "\mu_Y"] & A' \arrow[d, "\pi_K'"] \\ Z_X \arrow[r, "t_K"] & Z_Y \arrow[r, "g_K"] & K \end{tikzcd}$$

We write $t'_*\omega_{X'} = \omega_{Y'} \oplus \mathscr{Q}'$, where $\mathscr{Q}'$ is the pull-back of $\mathscr{Q}$ on Y' and moreover, since the pull back of P is trivial on A', $\pi'^*_K\widehat{K} \hookrightarrow \widehat{A}'$ is an irreducible component of $V^k(\mathscr{Q}', g')$. We know that $R^i\mu_{Y*}\mathscr{Q}'$ is a GV-sheaf on Z_Y for each $i \geq 0$ by Hacon's version of generic vanishing theorem (see [H1, Corollary 4.2]), we conclude that $V^0(R^k\mu_{Y*}\mathscr{Q}', g_K) = \widehat{K}$.

We then observe that

$$\begin{aligned} t_{K*}\omega_{Z_X} &= t_{K*}R^k\mu_{X*}\omega_{X'} \qquad \text{by [K1, Proposition 7.6]},\\ &= R^k\mu_{Y*}t'_*\omega_{X'}\\ &= \omega_{Z_Y} \oplus R^k\mu_{Y*}\mathscr{Q}'. \end{aligned}$$

Therefore, $h^0(Z_X, \omega_{Z_X}) > h^0(Z_Y, \omega_{Z_Y})$, we conclude that $\deg t_K > 1$. Notice that we have

$$\begin{tikzcd} X' \arrow[r, "t'"] \arrow[d] & Y' \arrow[d] \\ X'_1 \arrow[r, "t'_1"] \arrow[d] & Y'_1 \arrow[d] \\ Z_X \arrow[r, "t_K"] & Z_Y \end{tikzcd}$$

where all the vertical morphisms are fibrations. Hence $\deg t_K > 1$ implies that $\deg t'_1 > 1$. Therefore, t'_1 is not birational. This implies that $1 \in \Sigma_{nb}$.

For 2), we note that it suffices to prove that $V^0(\mathscr{Q}, g) = \emptyset$. Assume that $V^0(\mathscr{Q}, g)$ is not empty, then it contains an irreducible component T of codimension $k \leq \dim X$. By [PP2, Proposition 3.15], T is also an irreducible component of $V^k(\mathscr{Q}, g)$. In particular, $T \in S(g_*\mathscr{Q})$. On the other hand, all t_i are birational by assumption, thus $\Sigma_{nb} = \emptyset$. Therefore, we have by 1) that $S(g_*\mathscr{Q}) = \emptyset$, which is a contradiction. □

It will be useful to consider birationality when X dominates more than one varieties. Let's consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, "q"] & Y_1 \arrow[d, "a_1"] \\ Y_2 \arrow[r, "a_2"] & A \end{tikzcd}$$

where all the morphisms are generically finite and surjective. We then write $f_{i*}\omega_X = \omega_{Y_i} \oplus \mathscr{Q}_i$ for $i = 1, 2$.

Proposition 4.2. *Suppose that $S(a_{1*}\mathscr{Q}_1) \cap S(a_{2*}\mathscr{Q}_2) = \emptyset$, then either f_1 or f_2 is birational.*

Proof. Let's denote by r_i the degree of f_i and denote by s_i the degree of a_i. Certainly, $r_1s_1 = r_2s_2 = \deg q$.

We may assume that f_1 is not birational. Then $\mathscr{Q}_1$ is a sheaf of rank $r_1 - 1 > 0$. We note that $a_{i*}\mathscr{Q}_1$ is a direct summand of

$$q_*\omega_X = a_{i*}\omega_{Y_2} \oplus a_{i*}\mathscr{Q}_2,$$

for $i = 1, 2$. Thus $\mathbf{R}\widehat{\Delta(a_{i*}\mathscr{Q}_i)}$ is a direct summand of $\mathbf{R}\widehat{\Delta(q_*\omega_X)}$. By Corollary 3.6, we have the decomposition for $\mathbf{R}\widehat{\Delta(a_{i*}\mathscr{Q}_i)}$:

$$\mathbf{R}\widehat{\Delta(a_{i*}\mathscr{Q}_i)} \simeq \oplus_{P_k+\widehat{A}_k \in S(a_{i*}\mathscr{Q}_i)} \mathscr{F}_k, \tag{11}$$

where $\mathscr{F}_k$ is a pure sheaf supported on $-P_k + \widehat{A}_k$.

We now consider the morphisms:

$$\mathrm{Id} : a_{1*}\mathscr{Q}_1 \xrightarrow{\Psi=(\Psi_1,\Psi_2)} a_{2*}\omega_{Y_2} \oplus a_{2*}\mathscr{Q}_2 \to a_{1*}\mathscr{Q}_1$$

We notice that

$$\begin{aligned} & \mathrm{Hom}_{\mathbf{D}(A)}(a_{1*}\mathscr{Q}_1, a_{2*}\omega_{Y_2} \oplus a_{2*}\mathscr{Q}_2) \\ \simeq \ & \mathrm{Hom}_{\mathbf{D}(A)}\big(\mathbf{R}\Delta(a_{2*}\omega_{Y_2}) \oplus \mathbf{R}\Delta(a_{2*}\mathscr{Q}_2), \mathbf{R}\Delta(a_{1*}\mathscr{Q}_1)\big) \\ \simeq \ & \mathrm{Hom}_{\mathbf{D}(\widehat{A})}\big(\mathbf{R}\widehat{\Delta(a_{2*}\omega_{Y_2})} \oplus \mathbf{R}\widehat{\Delta(a_{2*}\mathscr{Q}_2)}, \mathbf{R}\widehat{\Delta(a_{1*}\mathscr{Q}_1)}\big). \end{aligned}$$

We denote by Φ_i the image of Ψ_i under the above natural transformation. By assumption $S(a_{1*}\mathscr{Q}_1) \cap S(a_{2*}\mathscr{Q}_2) = \emptyset$, thus by (11), we have $\Phi_2 = 0$ and hence $\Psi_2 = 0$. Therefore, $a_{1*}\mathscr{Q}_1$ is a direct summand $a_{2*}\omega_{Y_2}$.

We note that $h^n(A, a_{2*}\omega_{Y_2}) = 1$ but $h^n(A, a_{1*}\mathscr{Q}_1) = h^n(A, q_*\omega_X) - h^n(A, a_{1*}\omega_{Y_1}) = 0$. Hence $\mathrm{rank}\, a_{2*}\omega_{Y_2} > \mathrm{rank}\, a_{1*}\mathscr{Q}_1$.

We then compare the rank of these two sheaves: $s_2 > s_1(r_1 - 1)$. Hence $r_1 s_1 = r_2 s_2 > r_2 s_1 (r_1 - 1)$ and $r_2 < \frac{r_1}{r_1 - 1}$. Therefore $r_2 = 1$ and f_2 is birational. □

Here is an application of Proposition 4.2 and will be used in the last section.

Corollary 4.3. *Under the setting of Proposition 4.1, suppose that all t_i but possible one are birational, then t is birational.*

Proof. Assume that t is not birational, then $\Sigma_{nb} \neq \emptyset$. We may assume that $\Sigma_{nb} = \{1\}$.

We define $Z := X_1 \times_{A_1} A$. Then the induced morphism $s : X \to Z$ is generically finite and surjective.

We now consider the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, "q"] & Y \arrow[d, "a_1"] \\ Z \arrow[r, "a_2"] & A. \end{tikzcd}$$

Write $t_*\omega_X = \omega_Y \oplus \mathscr{Q}_Y$ and $s_*\omega_X = \omega_Z \oplus \mathscr{Q}_Z$. We note by Proposition 4.1 that all elements of $S(a_{1*}\mathscr{Q}_Y)$ are contained in $\widehat{A}_1$. On the other hand, by the construction of Z, any element of $S(a_{2*}\mathscr{Q}_Z)$ is not contained in $\widehat{A}_1$.

Hence $S(a_{1*}\mathcal{Q}_Y) \cap S(a_{2*}\mathcal{Q}_Z) = \emptyset$. We now apply Proposition 4.2 to conclude that either t or s is birational. However, s cannot be birational because Z is not of general type. This is the desired contradiction. $\square$

5. Primitive varieties of $\chi = 0$

We are now ready to apply the results proved in previous sections to study the structure of a smooth projective variety X of maximal Albanese dimension, of general type, and $\chi(X, \omega_X) = 0$. In this section, we will formulate a conjecture about the structure of such varieties.

Inspired by the main theorem in [CDJ], we find the following definition useful.

Definition 5.1. Let X be a smooth projective variety of maximal Albanese dimension, of general type, and $\chi(X, \omega_X) = 0$. We say that X is *primitive of* $\chi = 0$ if

> for any non-trivial fibration $h : X \to Y$ to a normal projective variety with a general fiber F, we have $\chi(F, \omega_F) > 0$;

For a primitive variety of $\chi = 0$, we know that $q(X) = \dim X = n$, the Albanese morphism $a_X : X \to A_X$ is generically finite and surjective, and A_X has at least 3 simple factors (see [CDJ, Lemma 4.6 and Corollary 3.5]). Moreover, we see from the definition that primitive varieties of $\chi = 0$ are the building blocks of varieties of maximal Albanese dimension, of general type, and of $\chi(X, \omega_X) = 0$. But more precisely, we have the following structural result.

Proposition 5.2. *Let X be a variety of general type, of maximal Albanese dimension, and $\chi(X, \omega_X) = 0$. Then either X is primitive of $\chi = 0$ or there exists an irregular fibration*[1] *$f : X \to Z$ with a general fiber F primitive of $\chi = 0$.*

Proof. If X is not primitive of $\chi = 0$, we take $f : X \to Y$ to be a fibration with a general fiber F such that $\chi(\omega_F) = 0$ and assume that $\dim F$ is minimal among all such fibrations.

By Lemma 5.3 below and the minimality of $\dim F$, we see that f is an irregular fibration and $a_X(F)$ is a translate of an abelian variety K of A_X. We claim that F is primitive of $\chi = 0$.

Otherwise, there exists a fibration of F, whose general fiber has $\chi = 0$. Considering the generically finite morphism $a_X|_F : F \to K$, we conclude again by Lemma 5.3 that there exists an abelian subvariety K' of K such that an irreducible component F' of a general fiber of $F \to K \to K/K'$ has $\chi(\omega_{F'}) = 0$. Then considering the Stein factorization $X \xrightarrow{g} Z \to A_X/K'$, F' is a general fiber of g, which is again a contradiction to the minimality of the dimension of F. $\square$

[1] Here, by definition, an irregular fibration $f : X \to Z$ is the Stein factorization of a morphism $g : X \to A$ to an abelian variety. Hence Z is a normal projective variety and there exists a finite morphism $Z \to A$.

Lemma 5.3. *Let $\alpha : X \to A$ be a generically finite morphism from a smooth projective variety of general type to an abelian variety. Assume that we have a fibration $f : X \to Y$ with a general fiber F such that $\chi(\omega_F) = 0$. Then there exists a quotient of abelian varieties $A \to B$ such that f factors birationally through the Stein facorization $g : X \to Z$ of the induced morphism $X \to A \to B$. Moreover, we have $\chi(\omega_{F'}) = 0$ for a general fiber F' of g.*

Proof. We consider the morphism $\alpha|_F : F \to A$. By [CH2, Theorem 4.2], there exists an abelian subvariety K of A such that $\alpha|_F(F)$ is fibred by K and moreover, an irreducible component F' of a general fiber $F \to A \to A/K$ has $\chi(\omega_{F'}) = 0$. We then take B to be A/K and let $X \xrightarrow{g} Z \to B$ be the Stein factorization of $X \to B$. It is easy to check that g is the irregular fibration we are looking for. □

Here we have some basic properties for primitive varieties of $\chi = 0$.

Lemma 5.4. *Let X be a primitive variety of $\chi = 0$, then*

1) *for any simple abelian sub-variety $\widehat{K} \hookrightarrow \widehat{A}_X$, there exists an irreducible component $\widehat{A}_i$ of $V^0(\omega_X, a_X)$ such that the composition of morphisms $\widehat{A}_i \to \widehat{A}_X \to \widehat{A}_X/\widehat{K}$ is an isogeny;*
2) *for any simple abelian variety $\widehat{K} \hookrightarrow \widehat{A}_X$, the induced morphism $X \to A_X \to K$ is a fibration.*

Proof. For 1), we consider the surjective morphism $g : X \to A_X \to K$. Since X is primitive, for an irreducible component F of a general fiber of g, we have $\chi(F, \omega_F) > 0$. Hence the natural morphism $V^0(\omega_X, a_X) \to \widehat{A}_X \to \widehat{A}_X/\widehat{K}$ is surjective. Therefore there exists an irreducible component $\widehat{A}_i$ of $V^0(\omega_X, a_X)$ such that the natural morphism $\widehat{A}_i \to \widehat{A}_X/\widehat{K}$ is an isogeny.

We argue by contradiction to prove 2). Assume that $X \to A$ is not a fibration, we take a modification of Stein factorization $X \to M \to K$ such that M is a smooth projective variety. Then the morphism $M \to K$ is not an étale morphism and is of degree ≥ 1. Since K is simple, we have $\chi(M, \omega_M) > 0$. On the other hand, by 1), there is an irreducible component T_i of $V^0(\omega_X)$ such that $T_i \times \widehat{K} \to \widehat{A}_X$ is an isogeny. After taking an étale cover of X, we may assume that $T_i = \widehat{A}_i$ passes through the origin of $\widehat{A}_X$. Hence $X \to M \times Y_i$ is generically finite and surjective. However, we then have $\chi(X, \omega_X) \geq \chi(M, \omega_M)\chi(Y_i, \omega_{Y_i}) > 0$, which is a contradiction. □

Corollary 5.5. *Let X be primitive of $\chi = 0$. Then,*

(1) *if A_X has m simple factors, then $V^0(\omega_X)$ has at least m irreducible components;*
(2) *if A_X has 3 simple factors, then each irreducible component $\widehat{A}_i$ of $V^0(\omega_X)$ has 2 simple factors.*

We end this section with a conjectural characterization of primitive varieties of $\chi = 0$.

Conjecture 5.6. Let X be a smooth projective variety. Then X is a primitive variety of $\chi = 0$ if and only if there exist simple abelian varieties $K_1, K_2, \ldots, K_{2k+1}$, double coverings from normal projective varieties of general type $F_i \to K_i$ with involutions σ_i, and an isogeny $K_1 \times \cdots \times K_{2k+1} \to A_X$ such that the base change $\widetilde{X}$ is birational to

$$(F_1 \times \cdots \times F_{2k+1})/\langle \sigma_1 \times \cdots \times \sigma_{2k+1} \rangle.$$

Together with Proposition 5.2, Conjecture 5.6 gives all possible structures for varieties of general type, of maximal Albanese dimension, and of $\chi(\omega_X) = 0$. We note one direction of Conjecture 5.6 is fairly standard but the other direction seems rather difficult. In next section, we will prove Conjecture 5.6 when A_X has only 3 factors.

6. Three simple factors

In this section we assume that X is of general type, of maximal Albanese dimension, with $\chi(X, \omega_X) = 0$, and A_X has only three simple factors. In particular, X is primitive of $\chi = 0$ (see [CDJ, Proposition 4.5]). We are interested in the structure (up to étale covers and birational modifications) of such varieties.

We are free to take étale covers of X and hence we always assume that each component of $V^0(\omega_X)$ passes through the origin of $\widehat{A}_X$ in this section. We then write $V^0(\omega_X) = \cup_{i=1}^N \widehat{A}_i$. We know that the complementary of $\widehat{A}_i \hookrightarrow \widehat{A}_X$ is a simple abelian variety and $N \geq 3$.

The main theorem of this section is the following generalization of [CDJ, Theorem 5.1].

Theorem 6.1. *Let X be a variety of general type, of maximal Albanese dimension, with $\chi(X, \omega_X) = 0$, and assume that A_X has only three simple factors. There exist simple abelian varieties K_1, K_2, K_3, double coverings from normal varieties $F_i \to K_i$ with associated involution τ_i, and an isogeny $\eta\colon K_1 \times K_2 \times K_3 \to A_X$, such that the base change $\widetilde{X}$ is birational to*

$$(F_1 \times F_2 \times F_3)/\langle \sigma \rangle$$

where $\sigma = \tau_1 \times \tau_2 \times \tau_3$ is the diagonal involution. That is, we have the following commutative diagram:

$$\begin{tikzcd}
 & (F_1 \times F_2 \times F_3)/\langle \sigma \rangle \arrow[dr, two heads] & \\
\widetilde{X} \arrow[ur, two heads, "\varepsilon"] \arrow[rr, two heads, "a_{\widetilde{X}}"] \arrow[d, two heads] \arrow[drr, phantom, "\square"] & & K_1 \times K_2 \times K_3 \arrow[d, two heads, "\eta"] \\
X \arrow[rr, two heads, "a_X"'] & & A_X.
\end{tikzcd}$$

where ε is a desingularization.

Let $X \xrightarrow{f_i} Y_i \xrightarrow{t_i} A_i$ be a modification of the Stein factorization of the morphism $X \xrightarrow{a_X} A_X \xrightarrow{p_i} A_i$ such that Y_i is smooth projective, and denote by $a_{X*}\omega_X = \mathscr{O}_{A_X} \oplus \mathscr{W}_{X/A_X}$ and $t_{i*}\omega_{Y_i} = \mathscr{O}_{A_i} \oplus \mathscr{F}_i$, where $\mathscr{F}_i$ is M-regular by Lemma 5.4.

Since A_X has only three simple factors and X is primitive, by Lemma 5.4 2), we know that

$$S_X = \{\widehat{A}_i \mid 1 \leq i \leq N\}.$$

By Theorem 3.4, we have the following proposition, which is our starting point.

Proposition 6.2. *We have $\mathscr{W}_{X/A_X} \simeq \bigoplus_{i=1}^N p_i^*\mathscr{F}_i$. In particular,*

$$\deg a_X - 1 = \sum_{i=1}^{N} (\deg t_i - 1).$$

6.1. **Characterization of special primitive varieties.** In this subsection, we are going to prove Theorem 6.1 for special primitive varieties of $\chi = 0$ satisfying :

(†1) $A_X = K_1 \times K_2 \times K_3$.

(†2) $V^0(\omega_X)$ contains three components $\widehat{A}_1 = \{0\} \times \widehat{K}_2 \times \widehat{K}_3$, $\widehat{A}_2 = \widehat{K}_1 \times \{0\} \times \widehat{K}_3$, and $\widehat{A}_3 = \widehat{K}_1 \times \widehat{K}_2 \times \{0\}$, and

(†3) the induced morphism $X \to Z_k := Y_i \times_{K_k} Y_j$ is birational, for $\{i, j, k\} = \{1, 2, 3\}$.

Recall that we have generically finite morphism $Y_i \to A_i$ and the induced fibrations $h_{ij}\colon Y_i \to K_j$. These fit into the following diagram

$$\begin{tikzcd}
Z_1 \arrow[d, "g_{12}"'] \arrow[dr, "g_{13}"] & Z_2 \arrow[d, "g_{23}"'] \arrow[dr, "g_{21}"] & Z_3 \arrow[d, "g_{31}"] \arrow[dll, "g_{32}"'] \\
Y_2 \arrow[d, "h_{23}"'] \arrow[dr, "h_{21}"] & Y_3 \arrow[d, "h_{31}"'] \arrow[dr, "h_{32}"] & Y_1 \arrow[d, "h_{12}"] \arrow[dll, "h_{13}"'] \\
K_3 & K_1 & K_2,
\end{tikzcd} \tag{12}$$

where Z_i is the main component of the fiber product over K_i. We note that, by [CH1, Theorem 2.3], Z_i is of general type.

We denote $a_i = \deg t_i = \deg(Y_i/A_i)$ for $i = 1, 2, 3$. Then by assumption (†3), $\deg a_X = a_i a_j$ for any $1 \leq i \neq j \leq 3$. Hence $a := a_1 = a_2 = a_3$.

Lemma 6.3. *There are smooth varieties of general type F_1, F_2, F_3 generically finite over K_1, K_2, K_3 respectively such that the fibration $h_{ij} : Y_i \to K_j$ is isotrivial with a general fiber F_k, for $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. We will show that $h_{13} : Y_1 \to K_3$ is isotrivial and the general fiber of h_{13} is birational to a general fiber of $h_{31} : Y_3 \to K_1$. The same argument works for other fibrations.

Since $X \to Z_1$ is birational, there exists a dominant map $\tau : Z_1 \dashrightarrow X \to Y_1$ fits into the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[drr, "f_1"'] & Z_1 \arrow[rr, "{(f_{12},f_{13})}"] \arrow[dr, dashed, "\tau"] & & K_2 \times K_3 , \\
& & Y_1 \arrow[ur, "{(h_{12},h_{13})}"'] &
\end{tikzcd}$$

where $f_{12} = h_{32} \circ g_{13}$ and $f_{13} = h_{23} \circ g_{12}$.

Fix a general point $t \in K_1$, and denote respectively by F_{2t} and F_{3t} the fibers of h_{31} and h_{21} over t. Then the fiber of $Z_1 \to K_1$ is isomorphic to $F_{3t} \times F_{2t}$ with the commutative diagram

$$\begin{tikzcd}
& F_{2t} \times F_{3t} \arrow[dl] \arrow[d, dashed, "\tau"] \arrow[dr] & \\
F_{2t} \arrow[ddd, "h_{32}"'] & Y_1 \arrow[d] \arrow[dddl, "h_{12}"'] \arrow[dddr, "h_{13}"] & F_{3t} \arrow[ddd, "h_{23}"] \\
& K_2 \times K_3 \arrow[ddl] \arrow[ddr] & \\
& & \\
K_2 & & K_3
\end{tikzcd} \tag{13}$$

Since h_{21} is a fibration, we know that

$$\deg(F_{3t}/K_3) = \deg(Y_2/A_2) = a.$$

Let F_2 be a general fiber of $h_{13} : Y_1 \to K_3$. We also have that $\deg(F_2/K_2) = \deg(Y_1/K_2 \times K_3) = a$. Let W be the main component of $Y_1 \times_{K_3} F_{3t}$. From the right part of diagram (13), we see that there is a induced rational dominant map $\tau' : F_{2t} \times F_{3t} \dashrightarrow W$ over Y_1. Since

$$\begin{cases} \deg(F_{2t} \times F_{3t}/Y_1) & = a^2/\deg(Y_1/K_2 \times K_3) = a, \\ \deg(W/Y_1) & = \deg(F_{3t}/K_3) = a. \end{cases}$$

It follows that τ' is birational. Thus $h_{13} : Y_1 \to K_3$ is isotrivial with a general fiber F_2 birational to F_{2t} for any general t. $\square$

Lemma 6.4. *$V^0(\omega_X, a_X)$ has exactly three irreducible components $\widehat{A}_1$, $\widehat{A}_2$, and $\widehat{A}_3$.*

Proof. Let $F_{ij} := F_i \times F_j$ and $F_{123} := F_1 \times F_2 \times F_3$. Since there are dominant generically finite rational map $F_{12} \dashrightarrow Y_3$ (resp. $F_{13} \dashrightarrow Y_2$, $F_{23} \dashrightarrow Y_1$), it is straightforward to see that there exists dominant generically finite rational map $F_{123} \dashrightarrow Z_k$ for $k = 1, 2, 3$. In particular, they induce a dominant generically finite rational map $F_{123} \dashrightarrow X$. Resolve the indeterminancy via $\nu : X' \to F_{123}$, we have a generically finite morphism ρ: $X' \to X$.

We claim that $V^1(X', a_X\circ\rho)$ consists of finite unions of translation of $\widehat{A}_1$, $\widehat{A}_2$, and $\widehat{A}_3$. Since $\chi(\omega_X)=0$,

$$V^0(\omega_X, a_X)=V^1(\omega_X, a_X)\subset V^1(\omega_{X'}, a_X\circ\rho)$$

and each component of $V^0(\omega_X, a_X)$ is a non-simple abelian subvariety passing through the origin, we conclude the Lemma.

It remains to prove the claim. Let $p=(\alpha,\beta,\gamma): F_{123}\to A_X=K_1\times K_2\times K_3$ be the given surjective morphism. Clearly, ν is birational and $p\circ\nu=a_X\circ\rho$. Hence

$$V^1(\omega_{X'}, a_X\circ\rho)=V^1(\omega_{X'}, p\circ\nu)=V^1(\omega_{F_{123}}, p).$$

Since K_1, K_2 and K_3 are simple abelian varieties, $V^1(\omega_{F_1},\alpha)$, $V^1(\omega_{F_2},\beta)$ and $V^1(\omega_{F_3},\gamma)$ are finite union of isolated points. By Künneth formula,

$$\begin{aligned} V^1(\omega_{F_{123}}, p) \quad = \quad & \big(V^1(\omega_{F_1},\alpha)\times\widehat{K}_2\times\widehat{K}_3\big)\bigcup\big(\widehat{K}_1\times V^1(\omega_{F_2},\beta)\times\widehat{K}_3\big)\\ & \bigcup\big(\widehat{K}_1\times\widehat{K}_2\times V^1(\omega_{F_3},\gamma)\big). \end{aligned}$$

This verifies the claim. □

By Lemma 6.4, $V^0(\omega_X)=\widehat{A}_1\cup\widehat{A}_2\cup\widehat{A}_3$. Thus we apply Proposition 6.2 and get $a^2-1=3(a-1)$. Hence $a=2$ and $\deg(Y_i/A_i)=\deg(F_i/K_i)=2$, for $i=1,2,3$.

Replace F_i by its Stein factorization over K_i, we may and do assume that F_i is normal and double cover over K_i. Let τ_i be the corresponding involution.

The covering $F_2\times F_3\to K_2\times K_3=A_1$ is Galois with Galois group $\mathbb{Z}_2\times\mathbb{Z}_2$. The function field $K(Y_1)$ is an intermediate field of the extension $K(F_2\times F_3)/K(A_1)$ and of degree 2 over $K(A_1)$. Together with the fact that Y_1 is of general type, it follows that Y_1 is birational to $(F_1\times F_2)/\langle\tau_1\times\tau_2\rangle$ by exhausting all intermediate fields.

It is clear to see that Y_2, Y_3 has the same structure. In fact, the similar argument also shows that X is birational to $(F_1\times F_2\times F_3)/\langle\tau_1\times\tau_2\times\tau_3\rangle$.

We thus conclude this subsection that

Proposition 6.5. *Under the hypothesis †, there exist simple abelian varieties K_1, K_2, K_3, double coverings from normal varieties $F_i\to K_i$ with associated involution τ_i, such that X is birational to*

$$(F_1\times F_2\times F_3)/\langle\sigma\rangle$$

where $\sigma=\tau_1\times\tau_2\times\tau_3$ is the diagonal involution.

6.2. **The general case.** We prove the main theorem in this subsection. To start with, it is convenient to introduce the following notions.

Definition 6.6. A primitive variety of $\chi=0$ is said to be *special* if it is birational to $(F_1\times F_2\times F_3)/\langle\tau_1\times\tau_2\times\tau_3\rangle$ as in Proposition 6.5. A primitive variety X of $\chi = 0$ is said to be *quasi-special* if there is an étale base change $\widetilde{X} \to X$ so that $\widetilde{X}$ is special.

Given a variety X primitive of $\chi = 0$, we said that X is *minimal* if X is minimal among smooth projective varieties of general type sitting between X and A_X (up to birational equivalent). More precisely, let X_0 be a variety of general type sit between X and A_X, then $X \to X_0$ is birational.

Fix a primitive variety X with $\chi = 0$. For any component $\widehat{A}_i$, we have an map $t_i \colon Y_i \to A_i$. Let $d_i := \deg(t_i)$.

For any two distinct components $\widehat{A}_i, \widehat{A}_j$, let $\widehat{K}_{ij}$ be the neutral component of $\widehat{A}_i \cap \widehat{A}_j$. We consider Z_{ij} a desingularization of an irreducible component of the main component of $(Y_i \times_{K_{ij}} Y_j) \times_{A_i \times_{K_{ij}} A_j} A_X$. Replacing X by its higher model, we may assume that there exists induced maps $\rho_{ij} \colon X \to Z_{ij}$ and $a_{ij} \colon Z_{ij} \to A_X$. It is easy to see the following properties of Z_{ij}:

1) Z_{ij} is of general type.
2) $\deg(Z_{ij}/A_X) | d_i d_j$.
3) $V^0(\omega_{Z_{ij}}, a_{ij}) \supset \widehat{A}_i, \widehat{A}_j$ and we have the natural morphisms $Z_{ij} \to Y_i \to A_i$ and $Z_{ij} \to Y_j \to A_j$. In particular, Z_{ij} is not quasi-special if $d_i \geq 3$ or $d_j \geq 3$.

Lemma 6.7. *Pick any three irreducible components, say $\widehat{A}_1$, $\widehat{A}_2$, and $\widehat{A}_3$ of $V^0(\omega_X)$. Assume that all $\rho_{12}, \rho_{13}, \rho_{23}$ are birational.*

Then after an abelian étale base change $\widetilde{X} \to X$, one may assume that $\widetilde{X}$ satisfying †1, †2, †3. Hence X is quasi-special.

Proof. By [CDJ, Proposition 4.3], we know that after an abelian étale cover $\widetilde{A}_X \to A_X$ and make the base change:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{a_{\widetilde{X}}} & \widetilde{A}_X \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow{a_X} & A_X, \end{array}$$

we may assume that $\widetilde{A}_X = K_1 \times K_2 \times K_3$ and $\widehat{\widetilde{A_1}} = \widehat{\pi}(\widehat{A}_1) = \{0\} \times \widehat{K}_2 \times \widehat{K}_3$, $\widehat{\widetilde{A_2}} = \widehat{\pi}(\widehat{A}_2) = \widehat{K}_1 \times \{0\} \times \widehat{K}_3$, and $\widehat{\widetilde{A_3}} = \widehat{\pi}(\widehat{A}_3) = \widehat{K}_1 \times \widehat{K}_2 \times \{0\}$. As before, for $i = 1, 2, 3$, we denote by $\widetilde{X} \xrightarrow{\widetilde{g}_i} \widetilde{Y}_i \xrightarrow{\widetilde{t}_i} \widetilde{A_i}$ a modification of the Stein factorization of the natural morphism $\widetilde{X} \to \widetilde{A_i}$ such that $\widetilde{Y}_i$ is smooth projective.

We claim that the induced morphism $\widetilde{X} \to \widetilde{Y}_i \times_{K_k} \widetilde{Y}_j$ is birational for $\{i, j, k\} = \{1, 2, 3\}$. To see this, note that since ρ_{ij} is birational, we have $\deg a_X | (d_i d_j)$, for any $i, j \in \{1, 2, 3\}$.

Thus $\deg a_{\widetilde{X}} = \deg a_X | (d_i d_j)$. We also know that t_i and $\widetilde{t}_i$ are respectively the Albanese morphisms of Y_i and $\widetilde{Y}_i$. Thus $\deg \widetilde{t}_i \geq d_i$. Thus $\deg \big((\widetilde{Y}_i \times_{K_k}$

$\widetilde{Y}_j)/\widetilde{A}_X\big) \geq d_i d_j$ for $\{i,j,k\}=\{1,2,3\}$. Hence the induced morphism $\widetilde{X} \to \widetilde{Y}_i \times_{K_k} \widetilde{Y}_j$ is birational. □

Proposition 6.8. *We have the following criterion for quasi-special primitive varieties with* $\chi = 0$*:*

1. *if* X *is minimal, then* X *is quasi-special.*
2. *if* $V^0(\omega_X, a_X)$ *has three components, then* X *is quasi-special.*
3. *if* $\deg(X/A_X) = 4$*, then* X *is quasi-special.*

Proof. (1). Suppose that X is minimal. We pick three components $\widehat{A}_1$, $\widehat{A}_2$, and $\widehat{A}_3$. For $(i,j) = (1,2),(1,3)$ and $(2,3)$, Z_{ij} of general type sitting between X and A_X and hence is birational to X by minimality of X. Therefore X is quasi-special by Lemma 6.7.

(2). For each $(i,j) = (1,2),(1,3)$ and $(2,3)$, we have the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\rho_{ij}"] \arrow[d] & Z_{ij} \arrow[r] \arrow[d] & A_X \arrow[d] \\ Y_t \arrow[r, equal] & Y_t \arrow[r] & A_t, \end{tikzcd}$$

where $t = i$ or j.

Since $V^0(\omega_X)$ consists of three components $\widehat{A}_1$, $\widehat{A}_2$, and $\widehat{A}_3$, by Corollary 4.3, ρ_{ij} is birational. Thus by Lemma 6.7, X is quasi-special

(3). Again let X_0 be a minimal primitive variety dominated by X. Since X_0 is quasi-special, $\deg(X_0/A_X) = 4$. It follows immediately that $\deg(X/X_0)$ $= 1$ and hence X is quasi-special. □

Proposition 6.9. *It* X *is primitive of* $\chi = 0$ *such that* A_X *has three simple factors, then* X *is quasi-special.*

Proof. Suppose on the contrary that X is not quasi-special. Since minimal primitive varieties are quasi-special, there exists X' sitting between X and A_X such that X' is not quasi-special but any other general type variety dominated by X' is quasi-special. Replace X by X', we may and do assume that X is not quasi-special, but any variety of general type dominated by X and not birational to X is quasi-special.

For $X \xrightarrow{f_i} Y_i \xrightarrow{t_i} A_i$, we denote by d_i the degree of t_i, for each $A_i \in S_X$. We may assume that $d_1 \geq \cdots \geq d_N \geq 2$. Since X is not quasi-special, by Proposition 6.8, we have $N \geq 4$. We will deduce a contradiction by the following steps.

Step 1. Either $d_1 > d_2 = \cdots d_N = 2$, or $d_1 = \cdots = d_N = 2$.

First of all, we see that the set $T = \{d_i \mid 1 \leq i \leq N\}$ contains at most two different numbers. Otherwise, we may assume that $d_i > d_j > d_k$. Then Z_{ik}, Z_{ik}, Z_{jk} are of general type. They are not all birational to X by Lemma 6.7, since X is not quasi-special. We may assume that Z_{jk} is not birational to X and hence Z_{jk} is quasi-special. Then $d_j = d_k = 2$, which is a contradiction.

If T contains two different numbers, we may assume that $d_1 = \cdots = d_s > d_{s+1} = \cdots = d_N \geq 2$. If $i \leq s$, then Z_{ij} can not be quasi-special and thus birational to X. If $s \geq 2$, we consider Z_{12}. Take a component $\widehat{A}_i$ of $V^0(\omega_{Z_{12}})$ for some $i \geq 3$. Then X is birational to Z_{12}, Z_{1i} and Z_{2i} and hence X is quasi-special by Lemma 6.7, which contradicts the assumption. Hence $s = 1$. Moreover, since Z_{1i} is birational to X for any $i \geq 2$, Z_{2j} should be quasi-special for any $2 \leq j \leq N$, we then have $d_2 = \cdots = d_N = 2$.

We next consider that $d := d_1 = \cdots = d_N$. If $d \geq 3$, then Z_{ij} is not quasi-special and hence is birational to X for any $1 \leq i < j \leq N$. But by Lemma 6.7, X is quasi-special, which is a contradiction. Thus $d_1 = \cdots = d_N = 2$, and each Z_{ij} is quasi-special.

Step 2. In both cases, Z_{23} is quasi-special and we may assume that $S_{Z_{23}} = \{\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\}$.

Since $\deg(Z_{23}/A_X) = 4$, by lemma 6.8, Z_{23} is always quasi-special. Moreover, since $N \geq 4$, in the second case $d_1 = \cdots = d_N = 2$, we may assume that $S_{Z_{23}} = \{\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\}$ and thus $\widehat{A}_4 \notin S_{Z_{23}}$.

Assume that $d_1 > d_2 = \cdots = d_N = 2$, then each Z_{1i} is not quasi-special and hence is birational to X, for any $2 \leq i \leq N$. Thus $\deg(X/A_X) = 2d_1$. On the other hand, Z_{23} is quasi-special. We claim that $S_{Z_{23}} = \{\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\}$. Otherwise, the composition of morphisms $Z_{23} \to A_X \to A_1$ is a fibration. Thus the main component of $Z_{23} \times_{A_1} Y_1$ is irreducible and is of degree $4d_1$ over A_X, which is absurd. Thus we again have $\widehat{A}_4 \notin S_{Z_{23}}$.

Step 3. Deduce a contradiction.

By step 2, in both cases, we may assume that $S_{Z_{23}} = \{\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\}$. Then since Z_{23} is quasi-special, after an étale cover of Z_{23}, we may assume that Z_{23} is birational to $(F_1 \times F_2 \times F_3)/\sigma$ as in Proposition 6.5.

Consider $\widehat{K}_{14}$, $\widehat{K}_{24}$, and $\widehat{K}_{34}$, we may assume that $\widehat{K}_{14}$ is different from $\widehat{K}_1 \times 0 \times 0$, $0 \times \widehat{K}_2 \times 0$, and $0 \times 0 \times \widehat{K}_3$ as an ablelian subvariety of $\widehat{A}_X = \widehat{K}_1 \times \widehat{K}_2 \times \widehat{K}_3$.

The corresponding fiber product of $(F_2 \times F_3)/\langle \tau_2 \times \tau_3 \rangle$ and Y_4 over K_{14} is quasi-special. To summarize, we have the following commutative diagram of morphisms

$$\begin{tikzcd}
F_2 \times F_3 \arrow[r, "\varphi"] \arrow[drr, "f"'] & (F_2 \times F_3)/\langle \tau_2 \times \tau_3 \rangle \arrow[r] \arrow[dr, "g"] & A_1 \arrow[d] \\
 & & K_{14},
\end{tikzcd}$$

where φ is the quotient morphism and hence is étale in codimension 1. Hence f is also isotrivial. For $t \in K_{24}$ general, the corresponding fiber F_t of f is birational to a fixed variety F. We denote by p and q the natural projections of $F_1 \times F_3$ to F_1 and F_3 and denote by $p_t : F_t \to F_1$ and $q_t : F_t \to F_3$ the restrictions of p and q on F_t. Both p_t and q_t are dominate surjective morphisms between varieties of general type. By [GP, Corollary 2.2], we know that almost all p_t are birational equivalent to each other and so is q_t. However, then for $t \in K_{24}$ general, the image of $(p_t, q_t) : F_t \to F_1 \times F_3$ is contained in a fixed proper Zariski subset of $F_1 \times F_3$. This is absurd. $\square$

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