r.v., Z-transform, Laplace transform

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Outline

- 2 or more r.v.
- z transform
- Laplace transform

PDF for 2 R.V.

$$F_{XY}(x,y) = P[X \le x, Y \le y];$$

$$f_{XY}(x,y) = \frac{d^2 F_{XY}(x,y)}{dxdy}$$

- Marginal Density Function

$$P[x_1 < X \le x_2, y_1 < Y \le y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(xy)dydx$$

$$\text{Independent}: F_{XY}(xy) = F_X(x)F_Y(y) = \text{Prob}[X \le x]\text{Prob}[Y \le y]$$

$$F_{X|Y}(x \mid y) = \frac{F_{XY}(x,y)}{F_Y(y)} = \text{Prob}[X \le x \mid Y \le y]$$

Function of Random Variable

- Func. of r.v.

$$Y = g(X)$$
$$F_Y(y) = P[Y \le y] = P[\{w : g(X(w)) \le y\}]$$

- One important r.v. is where X_i are independent

$$Y = \sum_{i=1}^{n} X_i$$

$Y = X_1 + X_2$

- PDF

$$F_Y(y) = P[Y \le y] = P[X_1 + X_2 \le y]$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y-x_2} f_{X_1X_2}(x_1x_2)dx_1dx_2$$
$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{y-x_2} f_{X_1}(x_1)dx_1] f_{X_2}(x_2)dx_2$$
$$= \int_{-\infty}^{\infty} F_{X_1}(y-x_2) f_{X_2}(x_2)dx_2$$

- pdf

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(y-x_2) f_{X_2}(x_2)dx_2$$

- Convolution

$$f_Y(y) = f_{X_1}(y) \otimes f_{X_2}(y)$$

Convolution Ex.

- f(n)
  - 2/3 as n = 1
  - 1/3 as n = 2
- g(n)
  - 1/2 as n = 1
  - 1/2 as n = 2

- h(n) = f(n)⊗g(n) = ?

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Ex. (cont.)

- h(0) = f(0)g(0) = 0
- h(1) = f(1)g(0) + f(0)g(1) = 0
- h(2) = f(2)g(0) + f(1)g(1) + f(0)g(2) = 1/3
- h(3) = f(3)g(0)+f(2)g(1)+f(1)g(2)+f(0)g(3) = ½
- h(4) = f(4)g(0)+f(3)g(1)+f(2)g(2)+f(1)g(3)+f(0)g(4) = 1/6

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z transform

- Consider a function of discrete time fn s.t.
  - fn ≥ 0 for n = 0, 1, 2, …
  - fn = 0 for n = -1, -2, …
  -

$$f_n \Leftrightarrow F(z) \text{ where } F(z) = \sum_{n=0}^{\infty} f_n z^n$$

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Examples

- Ex1:

$$f_n = A\alpha^n \text{ for } n = 0,1,2,\ldots$$

$$\therefore F(z) = \sum_{n=0}^{\infty} A\alpha^n z^n = A\sum_{n=0}^{\infty}(\alpha Z)^n$$

$$= A\frac{1}{1-\alpha z}$$

- Ex2: Convolution property $f_n \otimes g_n = \sum_{k=0}^{n} f_{n-k} g_k$

Convolution Property

$$f_n \otimes g_n \Leftrightarrow \sum_{n=0}^{\infty}[\sum_{k=0}^{n} f_{n-k} g_k] z^n$$

$$= \sum_{n=0}^{\infty}[\sum_{k=0}^{n} f_{n-k} g_k] z^{n-k} z^k$$

$$= \sum_{k=0}^{\infty}[\sum_{n=k}^{\infty} g_k z^k] f_{n-k} z^{n-k}$$

$$= \sum_{k=0}^{\infty} g_k z^k \sum_{n=k}^{\infty} f_{n-k} z^{n-k}$$

$$= \sum_{k=0}^{\infty} g_k z^k \sum_{m=0}^{\infty} f_m z^m$$

$$= G(z)F(z)$$

Properties of z transform

1. $f_n \Leftrightarrow F(z) = \sum_{n=0}^{\infty} f_n z^n$
2. $af_n + bg_n \Leftrightarrow aF(z) + bG(z)$
3. $a^n f_n \Leftrightarrow F(az)$
4. $f_{n+1} \Leftrightarrow \sum_{n=0}^{\infty} f_{n+1} z^n = \frac{1}{z}\sum_{n=0}^{\infty} f_{n+1} z^{n+1} = \frac{1}{z}[F(z) - f_0]$
5. $f_{n-1} \Leftrightarrow \sum_{n=0}^{\infty} f_{n-1} z^n = z\sum_{n=0}^{\infty} f_{n-1} z^{n-1} = zF(z)$
6. $nf_n \Leftrightarrow \sum_{n=0}^{\infty} nf_n z^n = \sum_{n=0}^{\infty} f_n \frac{dz^n}{dz} z = z\frac{d}{dz}F(z)$
7. $f_n \otimes g_n \Leftrightarrow F(z)G(z)$

Properties (cont.)

8. $f_n - f_{n-1} \Leftrightarrow F(z) - zF(z) = (1-z)F(z)$
9. $\sum_{k=0}^{n} f_k, n = 0,1,2,\ldots \Leftrightarrow \sum_{n=0}^{\infty}\sum_{k=0}^{n} f_k z^n = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} f_k z^k z^{n-k} = \frac{F(z)}{1-z}$
10. $\frac{\partial}{\partial a} f_n$ (n is a parameter of f_n) $\Leftrightarrow \frac{\partial}{\partial a}F(z)$
11. $F(1) = \sum_{n=0}^{\infty} f_n$
12. $F(0) = f_0$

z Transform pair

$1. U_n = \begin{cases} 1 & n=0 \\ 0 & otherwise \end{cases} \Leftrightarrow F(z)=1$

$2. U_{n-k} \Leftrightarrow z^k$

$3. \delta_n = 1 \ for \ n=0,1,... \Leftrightarrow \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

$4. \delta_{n-k} \Leftrightarrow \frac{z^k}{1-z}$

$5. A\alpha^n \Leftrightarrow \sum_{n=0}^{\infty} A\alpha^n z^n = \frac{A}{1-\alpha z}$

$6. n\alpha^n \Leftrightarrow \frac{\alpha z}{(1-\alpha z)^2}$

$7. n \Leftrightarrow \frac{z}{(1-z)^2}$

$8. \frac{1}{n!} \Leftrightarrow F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$

Read and derive Table I.1 and I.2

z transform : difference equation

- Ex:

$$6g_n - 5g_{n-1} + g_{n-2} = 6(\frac{1}{5})^n \quad n = 2,3,4,...$$

$$g_0 = 0; g_1 = \frac{6}{5}$$

z-transform and moment

$$G(z) = \sum_{n=0}^{\infty} f_n z^n$$

$$\frac{dG(z)}{dz} = \sum_{n=0}^{\infty} n f_n z^{n-1} \Rightarrow \frac{dG(z)}{dz}\Big|_{z=1} = \sum_{n=0}^{\infty} n f_n = \overline{X}$$

$$\frac{d^2G(z)}{dz^2} = \sum_{n=0}^{\infty} n(n-1) f_n z^{n-2} \Rightarrow$$

$$\frac{d^2G(z)}{dz^2}\Big|_{z=1} = \sum_{n=0}^{\infty} (n^2 - n) f_n = \overline{X^2} - \overline{X}$$

Laplace transform

- Def:

$$1. F^*(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt;$$

$$2. f(t) \Leftrightarrow F^*(s)$$

- Ex 1.

$$f(t) = \begin{cases} Ae^{-at} & t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

- Ex 2.

$$\delta(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Convolution

- f(t) and g(t) take on non-zero values for $t \geq 0$

$$f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(t-x)g(x)dx$$

Properties

1. $af(t) + bg(t) \Leftrightarrow aF^*(s) + bG^*(s)$
2. $f(\frac{t}{a}) \Leftrightarrow aF^*(as)$
3. $f(t-a) \Leftrightarrow e^{-as}F^*(s)$
4. $e^{-at}f(t) \Leftrightarrow F^*(s+a)$
5. $tf(t) \Leftrightarrow -\frac{d}{ds}F^*(s)$
6. $t^n f(t) \Leftrightarrow (-1)^n \frac{d^n F^*(s)}{ds^n}$
7. $\frac{f(t)}{t} \Leftrightarrow \int_{s1=s}^{\infty} F^*(s1)ds1$

Properties (cont.)

8. $\frac{df(t)}{dt} \Leftrightarrow ?(sF^*(s) - f(0^-))$
9. $\frac{d^n f(t)}{dt^n} \Leftrightarrow s^n F^*(s) - s^{n-1}f(0^-) - \ldots - f^{n-1}(0^-)$
10. $\int_{-\infty}^{t} f(t)dt \Leftrightarrow \frac{F^*(s)}{s}$
11. $\int_{-\infty}^{t} \ldots \int_{-\infty}^{t} f(t)(dt)^n \Leftrightarrow \frac{F^*(s)}{s^n}$

Differential eq.

- Find f(t)

$$\frac{d^2 f(t)}{dt^2} - \frac{6f(t)}{dt} + 9f(t) = 2t;$$

$$f(0^-) = \frac{df(0^-)}{dt} = 0;$$

Random Sum

Given $Y = X_1 + X_2 + \ldots + X_{\tilde{N}}$, find $\overline{Y}$

X_i are i.i.d. $\tilde{N}$ is a discrete non-negative r.v.

$$Y^*(s) = \sum_{n=0}^{\infty} Y^*(s \mid \tilde{N} = n) \mathrm{P}[\tilde{N} = n]$$

$$Y^*(s \mid \tilde{N} = n) = [X^*(s)]^n$$

$$Y^*(s) = \sum_{n=0}^{\infty} [X^*(s)]^n P[\tilde{N} = n]$$

Note that $N(z) = \sum_{n=0}^{\infty} P[\tilde{N} = n] z^n$

Relate z with $[X^*(s)]^n$

$Y^*(s) = N[X^*(s)]$

Ex.

$Y = X_1 + X_2 + \ldots + X_{\tilde{N}}$

X_i are i.i.d. exponentally distributed,

$\tilde{N}$ is geometrically distributed.

(a) Find $E[Y]$.

(b) what is $\mathrm{var}[Y]$.