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Reconstruction of cracks in an inhomogeneous anisotropic elastic medium

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Abstract

In this paper we give in two and three dimensions a reconstruction formula for determining cracks buried in an inhomogeneous anisotropic elastic body by making elastic displacement and traction measurements at the boundary. The information is encoded in the local Neumann-to-Dirichlet map. With the help of the Runge property, the local Neumann-to-Dirichlet map is connected to the so-called indicator function. This function can be expressed as an energy integral involving some special solutions, called reflected solutions. The heart of our method lies in analyzing the blow-up behavior at the crack of the indicator function, which is by no means an easy task for the inhomogeneous anisotropic elasticity system. To overcome the difficulties, we construct suitable approximations of the reflected solutions that capture their singularities. The indicator function is then analyzed by the Plancherel formula.

Résumé

Dans cet article, nous donnons, en deux et trois dimensions, une formule qui permet de déterminer des fissures localisées à l'intérieur d'un objet élastique, anisotrope et inhomogène, à partir des mesures du champ des déplacements et des contraintes imposés sur le bord de cet objet. L'information est contenue dans l'opérateur de Neumann–Dirichlet local. En utilisant la propriété de Runge, nous constatons que l'opérateur de Neumann–Dirichlet local est relié à ce qu'on appelle la fonction indicatrice. Cette fonction peut être exprimée comme une intégrale d'énergie faisant intervenir des solutions particulières, dites solutions réfléchies. Le coeur de notre méthode consiste à

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analyser l'explosion près de la fissure de la fonction indicatrice. Ceci n'est pas simple à réaliser pour le système d'élasticité anisotrope inhomogène. Afin de surmonter les difficultés rencontrées dans cette direction, nous construisons des approximations appropiées des solutions réfléchies qui mettent en évidence leurs singularités. La fonction indicatrice est ainsi analysée par la formule de Plancherel.

Keywords: Cracks; Anisotropic; Dirichlet-to-Neumann map

1. Introduction and statement of the results

In this paper we give a reconstruction formula in two and three dimensions for determining cracks embedded in an inhomogeneous anisotropic elastic body by making traction and displacement measurements on an open subset of the boundary of the medium. This information is encoded in the local Neumann-to-Dirichlet map. We describe below more precisely the problem and our main result.

Let $\mathcal{B}$ be an anisotropic elastic body and the reference configuration of $\mathcal{B}$ be Ω, a bounded connected domain in $\mathbb{R}^n$, $n = 2, 3$, with C^1 boundary Γ. Denote by $C(x) = (C_{ijkl}(x)) \in C^1(\overline{\Omega})$ the elastic tensor. Here and below, all Latin indices are set to be from 1 to n ($n = 2$ or 3). We assume that the elastic tensor C satisfies the full symmetry properties:

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad \forall i, j, k, l, \tag{1.1}$$

and the strong convexity condition, i.e., there exists a constant $\delta > 0$ such that

$$C(x)E \cdot E \geqslant \delta |E|^2 \tag{1.2}$$

for any symmetric matrix E and all $x \in \Omega$. Here we have used the conventions

$$(CG)_{ij} = \sum_{kl} C_{ijkl} g_{kl} \quad \text{and} \quad G \cdot H = \sum_{ij} g_{ij} h_{ij},$$

where $G = (g_{ij})$ and $H = (h_{ij})$ are real matrices. Let $u = {}^t(u_1, \ldots, u_n)$ be the displacement vector, then the equation of equilibrium, when there are no exterior forces acting on the domain, is given by [7]

$$\mathcal{L}_C u = \nabla \cdot \sigma_C(u) = 0 \quad \text{in } \Omega, \tag{1.3}$$

where $(\nabla \cdot G)_i = \sum_j \partial_j g_{ij}$ for any matrix function $G = (g_{ij})$ and $\sigma_C(u) = C\nabla u$. It is evident that $\sigma_C(u) = C\varepsilon(u)$ with $\varepsilon(u) = \mathrm{Sym}\nabla u = \frac{1}{2}({}^t\nabla u + \nabla u)$ if C satisfies (1.1). Here the superscript t denotes the transpose of vectors or matrices.

Throughout the paper we use the following notations and assumptions. Let X be an open submanifold of a manifold Y. If F is a space of distributions in Y, we set:

$$\overline{F}(X) := \{ f|_X \colon f \in F \}, \qquad \dot{F}(\overline{X}) := \{ f \in F \colon \mathrm{supp}(f) \subset \overline{X} \},$$

where $f|_X$ is the restriction of f to X. These notations will be used for some Sobolev spaces defined in X and when X, Y and the boundary ∂X have sufficient regularities. Assume that $S \subset \Omega$ is a C^2 closed Jordan curve ($n=2$) or closed connected surface ($n=3$) and $\Sigma \subset S$ is an open curve or surface. When $n=3$ we suppose that the boundary $\partial\Sigma$ of Σ is C^2. Here Σ will be considered as a crack. We can have several number of cracks. For this case our theory also works without any essential change. Let Ω_- be the open subset of Ω with boundary S and $\Omega_+ := \Omega \setminus \overline{\Omega}_-$. The trace operator to Γ is denoted by γ and those from $\Omega_\pm$ to S is denoted by $\gamma_\pm$. The directions of the unit normal ν to Γ and S are directed into $\mathbb{R}^n \setminus \overline{\Omega}$ and Ω_+, respectively.

In our problem, we take $X = \Sigma$, $k \in \mathbb{R}$ (with $|k| \leqslant 1$), and define the Sobolev spaces $\overline{H}^k(\Sigma)$ and $\dot{H}^k(\overline{\Sigma})$, which are subspaces of $H^k(S)$. Also, we denote $\overline{H}^k(\Sigma)^*$ the dual space of $\overline{H}^k(\Sigma)$. For $1/2 < s \leqslant 1$, we define $H^s(\Omega \setminus \Sigma)$ by:

$$H^s(\Omega \setminus \Sigma) := \left\{u \in \mathfrak{D}'(\Omega):\ u_\pm := u|_{\Omega_\pm} \in \overline{H}^s(\Omega_\pm);\ [u] := \gamma_+ u_+ - \gamma_- u_- = 0 \text{ on } S \setminus \overline{\Sigma}\right\}$$

with the norm $\|u\|_{H^s(\Omega\setminus\Sigma)} := \|u_+\|_{\overline{H}^s(\Omega_+)} + \|u_-\|_{\overline{H}^s(\Omega_-)}$.

To study different types of boundary measurements, we divide Γ into two parts:

$$\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N, \qquad \Gamma_D \cap \Gamma_N = \emptyset,$$

where $\Gamma_D, \Gamma_N \subset \Gamma$ are open subsets with C^1 boundaries $\partial\Gamma_D$, $\partial\Gamma_N$. One of Γ_D and Γ_N will be considered as the place where we perform the measurements. Note that we do not exclude the case $\Gamma_D = \emptyset$ or $\Gamma_N = \emptyset$.

We will give several mixed type boundary conditions. For example, fixing Dirichlet data on one part of the boundary, we measure the corresponding Dirichlet data on the other part of the boundary for given Neumann data on the same part of the boundary. By changing the fixed data, given data and measured data we obtain another type of traction and displacement measurements at the boundary. More precisely we consider two types of boundary value problems as direct problems.

Type 1. For any $g \in H_\#^{-1/2}(\Gamma) := \{g \in H^{-1/2}(\Gamma):\ \int_\Gamma g \cdot (o + Wx)\,\mathrm{d}s = 0\}$, where o is a constant n-vector and W is a skew-symmetric $n \times n$ matrix, find a solution $u \in H_\#^1(\Omega \setminus \Sigma) := \{u \in H^1(\Omega \setminus \Sigma):\ \int_\Gamma u\,\mathrm{d}s = 0,\ \mathrm{Skew}\int_\Gamma \nabla u\,\mathrm{d}s = 0\}$ to

$$\begin{cases} \mathcal{L}_C u = 0 & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(u)\nu = 0 & \text{on } \Sigma, \\ \sigma_C(u)\nu = g & \text{on } \Gamma, \end{cases} \tag{1.4}$$

where $\mathrm{d}s$ is the line or surface element and $\mathrm{Skew}\int_\Gamma \nabla u\,\mathrm{d}s$ is the skew symmetric part of $\int_\Gamma \nabla u\,\mathrm{d}s$. Here $o + Wx$ is usually called an *infinitesimal rigid displacement* (see [19]).

Type 2. Assume $\Gamma_D \neq \emptyset$. For any given $f \in \overline{H}^{1/2}(\Gamma_D)$ and $g \in \overline{H}^{-1/2}(\Gamma_N)$, find a solution $u \in H^1(\Omega \setminus \Sigma)$ to

$$\begin{cases} \mathcal{L}_C u = 0 & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(u)\nu = 0 & \text{on } \Sigma, \\ u = f & \text{on } \Gamma_D, \\ \sigma_C(u)\nu = g & \text{on } \Gamma_N. \end{cases} \tag{1.5}$$

In Section 2 (see Theorem 2.1) we prove that the corresponding boundary problems (1.4) and (1.5) are well posed. The local Neumann-to-Dirichlet map is defined by:

Definition 1.1. (i) For Type 1 direct problem, we define $\Lambda_\Sigma : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ by:

$$\Lambda_\Sigma g = u|_\Gamma,$$

where $u \in H^1_\#(\Omega \setminus \Sigma)$ is the solution to (1.4) with $g \in H^{-1/2}(\Gamma)$.

(ii) For Type 2 direct problem, we define $\Lambda_\Sigma : \overline{H}^{-1/2}(\Gamma_N) \to \overline{H}^{1/2}(\Gamma_N)$ by:

$$\Lambda_\Sigma g = u|_{\Gamma_N},$$

where $u \in H^1(\Omega \setminus \Sigma)$ is the solution to (1.5) with $g \in \overline{H}^{-1/2}(\Gamma_N)$.

(iii) For both types of direct problems, we denote Λ_Σ by $\Lambda_\emptyset$ if $\Sigma = \emptyset$.

In this paper we are concerned with reconstructing Σ from Λ_Σ. In fact, we present a reconstruction formula along the lines of the probe method [11]. This method has similarities with the point source method [18]. The probe method relies on the indicator function defined by:

$$I(t,r) := \lim_{j\to\infty} \left\langle g_j, \overline{(\Lambda_\Sigma - \Lambda_\emptyset) g_j} \right\rangle, \tag{1.6}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $\overline{H}^{-1/2}(\Gamma_N)$ and $\dot{H}^{1/2}(\overline{\Gamma}_N)$. By make an appropriate choice of the Neumann data g_j to search for the location of the crack. In order to construct this data we impose the technical assumption that the elasticity system possesses the *Runge property with constraints*.

Assumption 1.1. Suppose that U is an open subset of Ω with C^1 boundary such that $\overline{U} \subset \Omega$ and $\Omega \setminus \overline{U}$ is connected. Let X be the set of all functions $u|_U$ satisfying $u \in H^1$ in an open neighborhood of $\overline{U}$ and therein $\mathcal{L}_C u = 0$; let Y denote the set of all functions $v|_U$ satisfying $v \in H^1(\Omega)$ and $\mathcal{L}_C v = 0$ in Ω with $\mathrm{supp}(v) \subset \Gamma_0$, where Γ_0 is any fixed open subset of Γ. Then Y is dense in X with respect to the H^1 topology.

It is well known that the Runge property with constraints is an easy consequence of the unique continuation property. When the elastic medium is homogeneous or analytic, the unique continuation property is obvious. Recently, the first and third authors proved the unique continuation property for a generic class of two-dimensional inhomogeneous anisotropic elasticity systems [17]. The unique continuation property for three-dimensional inhomogeneous anisotropic elasticity systems is still an open problem. We would like to emphasize that our method is valid in greater generality than the unique continuation property; we only need the Runge property. Therefore, in the two-dimensional case, our method works for any inhomogeneous anisotropic elasticity system satisfying some generic conditions (see [17] for the precise conditions).

Now we state the main result of the paper.

Main Theorem. *Let Assumption* 1.1 *hold. Then there is a reconstruction formula for identifying Σ from Λ_Σ.*

We will summarize the reconstruction formula of our method at the end of the last section.

In the paper we only consider a single crack. Nevertheless, the same method works without any change for multiple cracks. A similar problem is considered in [13] of determination of crack embedded in an inhomogeneous isotropic conductive medium. There are several difficulties in generalizing the approach to the case of anisotropic elastic medium. The analysis of the blow-up behavior of the indicator function at the crack is considerably more complicated. The indicator function, although is defined in terms of boundary measurements, can be expressed as an integral which contains a special solution for the cracked domain (see (3.10)). This special solution is called *reflected solution*. The construction of the reflected solution and the analysis of the behavior of the indicator function become extremely complicated in the case of an inhomogeneous anisotropic medium. We construct the reflected solution by suitable change of local coordinates near the crack, freezing the elastic coefficients, and the Fourier transform method which is based on a factorization of the elasticity system (see Section 4). Then we analyze the behavior of the indicator function by the Plancherel formula and the form of the reflected solution. In Section 3 we outline the proof and the steps in the reconstruction method. In Section 4 we give the details of the proofs.

In addition to the recent paper [13], there are several related results on crack determination in different contexts. We mention Bryan and Vogelius [6], Kress [16], Ben Abda et al. [1–3], Brühl et al. [5]. Ben Abda et al. assumed the non-vanishing of the stress intensity factor for a surface breaking crack in a two-dimensional medium and the non-vanishing of the displacement gap across a two-dimensional crack in a plane and used the reciprocity gap principle to reconstruct the crack. Brühl et al. used Kirsch's linear sampling method (see [15]). Others' results reduce the problems to some optimization problems and use a Newton type algorithm to solve these.

2. The direct problem

In the following theorem we prove the well posedness for the Type 1 and Type 2 mixed boundary value problem stated in the introduction including the case $\Gamma_D = \emptyset$.

Theorem 2.1. *For any given $p \in \overline{H}^{-1/2}(\Sigma)$, $f \in \overline{H}^{1/2}(\Gamma_D)$ and $g \in \overline{H}^{-1/2}(\Gamma_N)$, there exists a unique solution $u \in H^1(\Omega \setminus \Sigma)$ to*

$$\begin{cases} \mathcal{L}_C u = 0 & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(u)\nu = p & \text{on } \Sigma, \\ u = f & \text{on } \Gamma_D, \\ \sigma_C(u)\nu = g & \text{on } \Gamma_N. \end{cases} \tag{2.1}$$

Moreover it satisfies the estimate:

$$\|u\|_{H^1(\Omega\setminus\Sigma)} \leqslant c\big(\|p\|_{\overline{H}^{-1/2}(\Sigma)} + \|f\|_{\overline{H}^{1/2}(\Gamma_D)} + \|g\|_{\overline{H}^{-1/2}(\Gamma_N)}\big), \tag{2.2}$$

where, hereafter, c denotes a general positive constant. In the case where $\Gamma_D = \emptyset$, we assume $g \in H_{\#}^{-1/2}(\Gamma)$ and take $u \in H_{\#}^1(\Omega \setminus \Sigma)$.

Proof. The same theorem for the conductivity equation was proved in [13]. Here we modify their arguments to handle the elasticity system. We first consider the case $\Gamma_D \neq \emptyset$. By virtue of the definition $\overline{H}^{1/2}(\Gamma_D)$, there exists an extension $\tilde{f} \in H^{1/2}(\Gamma)$ of f such that $\|\tilde{f}\|_{H^{1/2}(\Gamma)} \leqslant \|f\|_{\overline{H}^{1/2}(\Gamma_D)}$. Let $u_0 \in H^1(\Omega)$ be the solution to

$$\begin{cases} \mathcal{L}_C u_0 = 0 & \text{in } \Omega, \\ u_0 = \tilde{f} & \text{on } \Gamma, \end{cases}$$

then we can easily show that u_0 satisfies:

$$\|u_0\|_{H^1(\Omega)} \leqslant c\|\tilde{f}\|_{H^{1/2}(\Gamma)} \quad \text{and} \quad \big\|\sigma_C(u_0)\nu\big\|_{\overline{H}^{-1/2}(\Gamma_N)} \leqslant c\|\tilde{f}\|_{H^{1/2}(\Gamma)}.$$

Now let $\chi \in C^\infty(\overline{\Omega})$ satisfy $\operatorname{supp}(\chi) \cap \Sigma = \emptyset$ and $\chi = 1$ near Γ. Define $u_1 := u - \chi u_0$. We obtain from (2.1) that u_1 solves:

$$\begin{cases} \mathcal{L}_C u_1 = F & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(u_1)\nu = p & \text{on } \Sigma, \\ u_1 = 0 & \text{on } \Gamma_D, \\ \sigma_C(u_1)\nu = h & \text{on } \Gamma_N, \end{cases} \tag{2.3}$$

where

$$F = -\nabla \cdot \big(C(u_0 \otimes \nabla\chi)\big) - (C\nabla u_0)\nabla\chi \in H^1(\Omega \setminus \Sigma)^* \quad \text{and}$$
$$h = g - \chi\sigma_C(u_0)\nu \in \overline{H}^{-1/2}(\Gamma_N).$$

Here the tensor product of two vectors a and b is defined as $(a \otimes b)_{ij} = a_i b_j$. We now formulate (2.3) in a variational form, namely, finding u_1 solving

$$\int_{\Omega\setminus\Sigma} C\varepsilon(u_1)\cdot\varepsilon(v)\,\mathrm{d}x = -\int_{\Omega\setminus\Sigma} F\cdot v\,\mathrm{d}x + \int_{\Gamma_N} h\cdot v\,\mathrm{d}s - \int_{\Sigma} p\cdot[v]\,\mathrm{d}s \tag{2.4}$$

for any $v \in \mathcal{V} := \{v \in H^1(\Omega\setminus\Sigma)\colon\ v = 0 \text{ on } \Gamma_D\}$. Note that $[v] = \gamma_+ v_+ - \gamma_- v_- \in \dot{H}^{1/2}(\overline{\Sigma})$ and $\gamma v \in \dot{H}^{1/2}(\overline{\Gamma}_N)$. Also, we can see that $\overline{H}^{-1/2}(\Gamma_N) = \dot{H}^{1/2}(\overline{\Gamma}_N)^*$. Therefore, to prove (2.4) has a unique solution in $\mathcal{V}$ with the estimate:

$$\|u_1\|_{H^1(\Omega\setminus\Sigma)} \leqslant c\big(\|F\|_{H^1(\Omega\setminus\Sigma)^*} + \|h\|_{\overline{H}^{1/2}(\Gamma_N)} + \|p\|_{\overline{H}^{-1/2}(\Sigma)}\big), \tag{2.5}$$

by the Lax–Milgram theorem, it suffices to establish the coercive estimate:

$$a(v,v) := \int\limits_{\Omega\setminus\Sigma} C\varepsilon(v)\cdot\varepsilon(v)\,\mathrm{d}x \geqslant c\|v\|^2_{H^1(\Omega\setminus\Sigma)} \quad \forall v\in\mathcal{V}. \tag{2.6}$$

It is obvious that (2.5) implies (2.2). In view of the strong convexity condition (1.2), to prove (2.6), we only need to show that $\|v\|^2_{H^1(\Omega\setminus\Sigma)}$ and $\alpha(v) := \int_{\Omega\setminus\Sigma}|\varepsilon(v)|^2\,\mathrm{d}x$ are equivalent norms in $\mathcal{V}$. It is easy to see that $\alpha(v)=0$ if and only if v is an infinitesimal rigid displacement, i.e., $v = o + Wx$. Since $v=0$ on Γ_D, $v\equiv 0$ in $\Omega\setminus\Sigma$. In other words, $\alpha(\cdot)$ defines a norm on $\mathcal{V}$. To prove that $\|\cdot\|^2_{H^1(\Omega\setminus\Sigma)}$ and $\alpha(\cdot)$ are equivalent, it is enough to show:

$$\alpha(v) \geqslant c\|v\|^2_{L^2(\Omega\setminus\Sigma)} \quad \forall v\in\mathcal{V} \tag{2.7}$$

due to Korn's inequality

$$\int\limits_{\Omega\setminus\Sigma} \left|\varepsilon(v)\right|^2\mathrm{d}x + \int\limits_{\Omega\setminus\Sigma}|v|^2\,\mathrm{d}x \geqslant c\|v\|^2_{H^1(\Omega\setminus\Sigma)}.$$

The estimate (2.7) can be proved by standard contradiction arguments as in [8, Theorem 3.3]. So we omit the details here.

When $\Gamma_D=\emptyset$, the existence and uniqueness of solution u to (2.1) can be shown using a similar variational formulation. Note that in this case we take $\mathcal{V} = H^1_{\#}(\Omega\setminus\Sigma)$. □

3. Proof of Main Theorem

Here we will only prove the theorem for Type 2 problem. The same proof works for Type 1 problem. As mentioned above, we will design our reconstruction formula based on the probe method.

To begin, let $r := \{r(t)\in\overline{\Omega}\colon\ 0\leqslant t\leqslant 1\}$ be a non-selfintersecting continuous curve joining $r(0), r(1)\in\Gamma$ with $r(t)\in\Omega$ for $0<t<1$. This curve r is called a needle. Define:

$$T(r,\Sigma) := \sup\bigl\{t\colon\ 0<t<1,\ r(s)\notin\overline{\Sigma}\text{ for }0<s<t\bigr\}.$$

Physically, $T(r,\Sigma)$ can be interpreted as the first hitting time of the needle r to Σ. It is clear that if $T(r,\Sigma)=1$ then the needle r does not touch the crack Σ. For any given needle r, we would like to find a characterization of $T(r,\Sigma)$. To do this, as indicated in the introduction, we define the indicator function $I(t,r)$ by:

$$I(t,r) := \lim_{j\to\infty}\bigl\langle g_j, \overline{(\Lambda_\Sigma-\Lambda_\emptyset)g_j}\bigr\rangle, \tag{3.1}$$

where here $\langle\cdot\,,\cdot\rangle$ is the pairing between $\overline{H}^{-1/2}(\Gamma_N)$ and $\dot{H}^{1/2}(\overline{\Gamma}_N)$. The Neumann data g_j requires further explanations. Let $v', v''_j \in H^1(\Omega)$ ($j \in \mathbb{N}$) be defined as follows. v' is the solution to

$$\begin{cases} \mathcal{L}_C v' = 0 & \text{in } \Omega, \\ v' = f & \text{on } \Gamma_D, \\ \sigma_C(v')\nu = 0 & \text{on } \Gamma_N \end{cases} \tag{3.2}$$

and v''_j satisfy

$$\begin{cases} \mathcal{L}_C v''_j = 0 & \text{in } \Omega, \\ \operatorname{supp}(v''_j) \subset \Gamma_0, \\ v''_j \to G(\cdot\,, r(t)) & (j \to \infty) \text{ in } H^1_{\text{loc}}(\Omega \setminus r_t), \end{cases} \tag{3.3}$$

where Γ_0 is a fixed open subset of Γ_N and

$$r_t := \{r(s)\colon\ 0 < s \leqslant t\}.$$

Here the distribution $G(\cdot\,, x^0)$ in $x^0 \in \Omega$ satisfies:

$$\mathcal{L}_C G(\cdot\,, x^0) + \delta(x - x^0)b = 0$$

and

$$\bigl(G(\cdot\,, x^0) - E(\cdot\,, x^0)b\bigr)_{x^0 \in \Omega} \quad \text{is bounded in } H^1(\Omega), \tag{3.4}$$

where $0 \neq b \in \mathbb{C}$ and the distribution $E(x, x^0)$ in $x^0 \in \mathbb{R}^n$ satisfies:

$$\mathcal{L}_{C(x^0)} E(x, x^0) + \delta(x - x^0) I_n = 0. \tag{3.5}$$

Note that $C(x^0)$ is a homogeneous elastic tensor with $C(x) = C(x^0)$ for all $x \in \Omega$ and I_n the identity matrix. The existence v''_j is guaranteed by the Runge property with constraints (Assumption 1.1). The proof for the existence of $G(x, x^0)$ can be found in [12]. To deal with the inverse problem here, we will give an explicit construction of E in the next section. Now we define:

$$v_j = v' + v''_j \quad \text{and} \quad g_j = \sigma_C(v_j)\nu|_{\Gamma_N}.$$

To relate the indicator function $I(t, r)$ to $T(r, \Sigma)$, we define the quantity:

$$t(r, \Sigma) := \sup\Bigl\{0 < t < 1\colon\ \sup_{0<s<t} \bigl|I(s, r)\bigr| < \infty\Bigr\}.$$

Our aim now is to show that

$$t(r, \Sigma) = T(r, \Sigma) \quad \text{if } r \cap \Sigma \neq \emptyset, \tag{3.6}$$

namely, the indicator function $I(t,r)$ will become unbounded once the tip of the needle touches the crack. Proving (3.6) requires some delicate analysis. It is the main technical part of the proof.

First of all, we would like to rewrite the indicator function $I(t,r)$ which involves the so-called *reflected solution* defined as follows. Let $u_j \in H^1(\Omega \setminus \Sigma)$ be the solution of

$$\begin{cases} \mathcal{L}_C u_j = 0 & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(u_j)\nu = 0 & \text{on } \Sigma, \\ u_j = f & \text{on } \Gamma_D, \\ \sigma_C(u_j)\nu = g_j & \text{on } \Gamma_N \end{cases}$$

and $w_j = u_j - v_j \in H^1(\Omega \setminus \Sigma)$; then we can show:

Lemma 3.1 (reflected solution). *If $r_t \cap \overline{\Sigma} = \emptyset$, then $w_j \to w'$ in $H^1(\Omega \setminus \Sigma)$ and $w' \in H^1(\Omega \setminus \Sigma)$ satisfies*:

$$\begin{cases} \mathcal{L}_C w' = 0 & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(w')\nu = -\sigma_C\big(v' + G\big(\cdot, r(t)\big)\big)\nu & \text{on } \Sigma, \\ w' = 0 & \text{on } \Gamma_D, \\ \sigma_C(w')\nu = 0 & \text{on } \Gamma_N. \end{cases} \tag{3.7}$$

Proof. In view of the definitions of v_j and u_j, we obtain that

$$\begin{cases} \mathcal{L}_C w_j = 0 & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(w_j)\nu = -\sigma_C(v_j)\nu & \text{on } \Sigma, \\ w_j = 0 & \text{on } \Gamma_D, \\ \sigma_C(w_j)\nu = 0 & \text{on } \Gamma_N. \end{cases} \tag{3.8}$$

Applying Theorem 2.1 to (3.8) yields:

$$\|w_j - w_k\|_{H^1(\Omega\setminus\Sigma)} \leqslant c\big\|\sigma_C(v_j - v_k)\nu\big\|_{\overline{H}^{-1/2}(\Sigma)} = c\big\|\sigma_C\big(v_j'' - v_k''\big)\nu\big\|_{\overline{H}^{-1/2}(\Sigma)}. \tag{3.9}$$

Now let D be a bounded domain with C^1 boundary such that $\overline{\Sigma} \subset D \subset \overline{D} \subset \Omega \setminus r_t$. We can see that $v_j'' - v_k'' \in H^1(D)$ and $\mathcal{L}_C(v_j'' - v_k'') = 0$ in D. So by the trace theorem (Lemma A.2 in Appendix A), we have that

$$\big\|\sigma_C\big(v_j'' - v_k''\big)\nu\big\|_{\overline{H}^{-1/2}(\Sigma)} \leqslant c\big\|v_j'' - v_k''\big\|_{H^1(D)}.$$

Thus, this lemma is proved using (3.3) and (3.9). □

With the reflected solution w, we can give another form of the indicator $I(t,r)$.

Lemma 3.2. *Assume $r_t \cap \overline{\Sigma} = \emptyset$. Then we have*:

$$I(t,r) = \int_{\Omega\setminus\Sigma} \sigma_C(w') \cdot \varepsilon(\overline{w'})\,\mathrm{d}x + \int_{\Gamma_D} f\sigma_C(\overline{w'})\nu\,\mathrm{d}s. \tag{3.10}$$

Proof. In view of Lemma 3.1 and the definition of $I(t,r)$, it suffices to show that

$$\langle g_j, \overline{(\Lambda_\Sigma - \Lambda_\emptyset)g_j}\rangle = \int_{\Omega\setminus\Sigma} \sigma_C(w_j)\cdot\varepsilon(\overline{w_j})\,\mathrm{d}x + \int_{\Gamma_D} f\sigma_C(\overline{w_j})\nu\,\mathrm{d}s. \tag{3.11}$$

The derivation of (3.11) is based on Green's formula (A.1) in Lemma A.3. By means of (A.1) or usual Green's formula, we have that

$$\int_{\Omega\setminus\Sigma} \sigma_C(v_j)\cdot\varepsilon(\overline{v_j})\,\mathrm{d}x = \int_{\Gamma} \overline{v_j}\sigma_C(v_j)\nu\,\mathrm{d}s = \int_{\Gamma_D} \overline{f}\sigma_C(v_j)\nu\,\mathrm{d}s + \langle g_j, \overline{\Lambda_\emptyset g_j}\rangle. \tag{3.12}$$

Similarly, it follows from (A.1) that

$$\int_{\Omega\setminus\Sigma} \sigma_C(v_j)\cdot\varepsilon(\overline{u_j})\,\mathrm{d}x = \int_{\Gamma_D} \overline{f}\sigma_C(v_j)\nu\,\mathrm{d}s + \langle g_j, \overline{\Lambda_\Sigma g_j}\rangle - \int_{\Sigma_\pm} \overline{u_j}\sigma_C(v_j)\nu\,\mathrm{d}s, \tag{3.13}$$

where

$$\begin{aligned}\int_{\Sigma_\pm} \overline{u_j}\sigma_C(v_j)\nu\,\mathrm{d}s &:= \int_{\Sigma_+} \gamma_+\big(\overline{u_j}\sigma_C(v_j)\nu\big)\,\mathrm{d}s - \int_{\Sigma_-} \gamma_-\big(\overline{u_j}\sigma_C(v_j)\nu\big)\,\mathrm{d}s \\ &= \int_{\Sigma} \big(\gamma_+\overline{u_j}\big[\sigma_C(v_j)\nu\big] + [\overline{u_j}]\gamma_-\sigma_C(v_j)\nu\big)\,\mathrm{d}s \\ &= \int_{\Sigma} [\overline{u_j}]\gamma_-\sigma_C(v_j)\nu\,\mathrm{d}s.\end{aligned}$$

Combining (3.12) and (3.13) yields:

$$\langle g_j, \overline{(\Lambda_\Sigma - \Lambda_\emptyset)g_j}\rangle = \int_{\Omega\setminus\Sigma} \sigma_C(v_j)\cdot\varepsilon(\overline{w_j})\,\mathrm{d}x + \int_{\Sigma_\pm} \overline{u_j}\sigma_C(v_j)\nu\,\mathrm{d}s. \tag{3.14}$$

On the other hand, using Green's formula (A.1) again, we can compute:

$$\begin{aligned}\int_{\Omega\setminus\Sigma} \sigma_C(v_j)\cdot\varepsilon(\overline{w_j})\,\mathrm{d}x &= \int_{\Omega\setminus\Sigma} \varepsilon(v_j)\cdot\sigma_C(\overline{w_j})\,\mathrm{d}x \\ &= \int_{\Gamma_D} v_j\sigma_C(\overline{w_j})\nu\,\mathrm{d}s + \int_{\Gamma_N} v_j\sigma_C(\overline{w_j})\nu\,\mathrm{d}s - \int_{\Sigma_\pm} v_j\sigma_C(\overline{w_j})\nu\,\mathrm{d}s \\ &= \int_{\Gamma_D} f\sigma_C(\overline{w_j})\nu\,\mathrm{d}s - \int_{\Sigma} [v_j]\gamma_-\sigma_C(\overline{w_j})\nu\,\mathrm{d}s\end{aligned}$$

$$= \int\limits_{\Gamma_D} f\sigma_C(\overline{w_j})\nu \, \mathrm{d}s. \tag{3.15}$$

Finally, in view of Green's formula, we obtain that

$$\int\limits_{\Sigma_\pm} \overline{u_j}\sigma_C(v_j)\nu \, \mathrm{d}s = -\int\limits_{\Sigma_\pm} \overline{w_j}\sigma_C(w_j)\nu \, \mathrm{d}s = \int\limits_{\Omega\setminus\Sigma} \sigma_C(w_j)\cdot\varepsilon(\overline{w_j}) \, \mathrm{d}x. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) immediately yields (3.11). □

By virtue of Lemma 3.2, we can show a distinct feature of $I(t,r)$ when $r \cap \Sigma \neq \emptyset$.

Theorem 3.1. *If $r(T(r,\Sigma)) \in \Sigma$, then $|I(t,r)| \to \infty$ as $t \to T(r,\Sigma)$.*

By Theorem 3.1, we can prove (3.6) in the same way as [14]. The proof of Theorem 3.1 relies on the analysis of behavior of the reflected solution $w' = w'(x, r(t))$. We will give its proof in the next section. We end this section by giving the reconstruction algorithm of our method.

Reconstruction Algorithm.

Step 1. Given a needle $r = \{r(t)\colon\ 0 \leqslant t \leqslant 1\}$ and consider the domain $\Omega \setminus r_t$.

Step 2. Solve (3.2) for v' and find a sequence of functions v''_j satisfying (3.3).

Step 3. Compute $g_j = \sigma_C(v' + v''_j)\nu|_{\Gamma_N}$ and evaluate the indicator function

$$I(t,r) := \lim_j \langle g_j, \overline{(\Lambda_\Sigma - \Lambda_\emptyset)g_j} \rangle.$$

Step 4. Increase t and search for t where $|I(r,t)|$ becomes very large. Denote this t by $t_a(r,\Sigma)$.

Step 5. Choose many needles r and repeat all previous steps. Draw some surface Σ_a which is close enough to the points $t_a(r,\Sigma)$ for these r. Σ_a is gives an approximation of Σ.

4. Blow-up behavior of the indicator function

This section is devoted to the proof of Theorem 3.1. The analysis here is different from that in [13]. There the authors worked in the space coordinates. Here we will use the inverse Fourier transform which is more flexible and can be applied to several other equations.

The main step in proving Theorem 3.1 is to analyze the behavior of the reflected solution $w'(x, r(t))$ near $r(t)$ which is sufficiently close to the crack. It turns out the behavior of $|I(t,r)|$ as $t \to T(r,\Sigma)$ is determined by the local property of $w'(x, r(t))$ near $r(t)$. To proceed the analysis, let $x^0 = r(t) \in \Omega \setminus \overline{\Sigma}$ and $a = x(T(r, \Sigma))$. Assume that x^0 is sufficiently close to a, denoted by $x^0 \sim a$. Let $y = (y_1, \dots, y_n) = (y_1(x, x^0), \dots, y_n(x, x^0))$ be the boundary normal coordinates near a such that

$$y(a) = 0, \quad \frac{\partial y(x, x^0)}{\partial x}\bigg|_{x=x^0} = I_n, \quad \text{and} \quad \Omega_- = \{y_1 < 0\} \quad \text{near } a.$$

Let $J(x) = \frac{\partial y(x,x^0)}{\partial x} = (q_{ij}(x))$ and $x = x(y(x, x^0))$. Denote:

$$\widetilde{C}_{i\tilde{j}k\tilde{l}} = \left|J(x)\right|^{-1} C_{ijkl} q_{\tilde{j}j} q_{l\tilde{l}}, \qquad \widetilde{C} = (\widetilde{C}_{i\tilde{j}k\tilde{l}}) \quad \text{for } 1 \leqslant i, \tilde{j}, k, \tilde{l} \leqslant n,$$

and

$$\tilde{u}(y) = u\big(x\big(y, x^0\big)\big), \qquad y^0 = y\big(x^0, x^0\big) \quad \text{with } y_1^0 > 0.$$

Then we can see that

(i) $\widetilde{C}(y) \in C^1$ near $y = 0$;
(ii) $|J|^{-1}\mathcal{L}_C u = \mathcal{L}_{\widetilde{C}}\tilde{u}$ near $y = 0$;
(iii) $\nu|_\Sigma = e_1 = (1, 0, \dots, 0)$ near $y = 0$;
(iv) $\delta(x(y, x^0) - x^0) = \delta(y - y^0)$.

In view of this choice of the coordinates $\{y_1, \dots, y_n\}$, we have that

$$\widetilde{C}_{i\tilde{j}k\tilde{l}}\big(y^0\big) = C_{i\tilde{j}k\tilde{l}}\big(x^0\big) \quad \text{and} \quad \widetilde{C}_{i\tilde{j}k\tilde{l}}(y) = \widetilde{C}_{k\tilde{l}i\tilde{j}}(y) \quad \forall i, \tilde{j}, k, \tilde{l}. \tag{4.1}$$

We now adopt a definition introduced in [13] to simplify some expressions in our arguments below.

Definition 4.1. Let X be a function space defined in an open subset of $\mathbb{R}^n$ and $\{\Phi(\cdot, x^0)\}$, $\{\Psi(\cdot, x^0)\}$ be family of distributions defined in this open set depending on $x^0 \sim a$. We denote $\Phi(\cdot, x^0) \sim \Psi(\cdot, x^0)$ in X if $\{\Phi(\cdot, x^0) - \Psi(\cdot, x^0)\colon x^0 \sim a\}$ is bounded in X.

We use this definition even for distributions defined in terms of the boundary normal coordinates $y = y(x, x^0)$. In this case $x^0 \sim a$ changes to $y^0 \sim 0$.

Let $V \subset \mathbb{R}^n$ be a small open neighborhood of $y = 0$ with C^1 boundary. Define $V_\pm := V \cap \mathbb{R}^n_\pm$. Assume that $\beta_\pm$, β_0 are open subsets of the boundary $\partial V_\pm$ of $V_\pm$ such that

$$\partial V_\pm = \bar{\beta}_\pm \cup \bar{\beta}_0, \quad \beta_\pm \cap \beta_0 = \emptyset, \; \beta_\pm \subset \mathbb{R}^n_\pm, \text{ and } \beta_0 \subset \{y_1 = 0\}.$$

Now let $E(y, y^0)$ satisfy:

$$\mathcal{L}_{\widetilde{C}(y^0)} E\big(y, y^0\big) + \delta\big(y - y^0\big) I_n = 0.$$

It turns out the distribution $E(y, y^0)$ plays an important role in the analysis of the indicator function $I(t, r)$. Here, we would like to construct a particular $E(y, y^0)$ which will meet our needs. In what follows, to simply the notation, the homogeneous tensor $\widetilde{C}(y^0)$ is denoted by $\widetilde{C}$. Also, we denote $y' = (y_2, \ldots, y_n)$ and $y^{0\prime} = (y_2^0, \ldots, y_n^0)$.

Lemma 4.1. *Assume that* $y^0 \in \mathbb{R}^n$ *and* $E_\pm(y, y^0) \in C^\infty(\{\pm(y_1 - y_1^0) \geqslant 0\}, \mathfrak{D}'(\mathbb{R}^{n-1}_{y'}))$ *satisfy*:

$$\begin{cases} \mathcal{L}_{\widetilde{C}} E_\pm = 0 \quad \text{in } \pm\big(y_1 - y_1^0\big) > 0, \\ E_+|_{y_1 = y_1^0 + 0} - E_-|_{y_1 = y_1^0 - 0} = 0, \\ \sigma_{\widetilde{C}}(E_+)e_1|_{y_1 = y_1^0 + 0} - \sigma_{\widetilde{C}}(E_-)e_1|_{y_1 = y_1^0 - 0} = -\delta\big(y' - y^{0\prime}\big) I_n. \end{cases} \tag{4.2}$$

Let E *be defined by*:

$$E\big(y, y^0\big) = \begin{cases} E_+\big(y, y^0\big) & \text{in } y_1 - y_1^0 \geqslant 0, \\ E_-\big(y, y^0\big) & \text{in } y_1 - y_1^0 \leqslant 0; \end{cases}$$

then

$$\mathcal{L}_{\widetilde{C}} E + \delta\big(y - y^0\big) I_n = 0 \quad \text{in } \mathbb{R}^n.$$

Proof. Without loss of generality, we take $y^0 = 0$. For all $\varphi \in \mathfrak{D}(\mathbb{R}^n)$, we have:

$$\langle E, \mathcal{L}_{\widetilde{C}} \varphi \rangle = \int_{\mathbb{R}^n_+} E_+ \mathcal{L}_{\widetilde{C}} \varphi \, \mathrm{d}y + \int_{\mathbb{R}^n_-} E_- \mathcal{L}_{\widetilde{C}} \varphi \, \mathrm{d}xy.$$

For further computations, we introduce some notations:

$$Q(\xi') = \left(\sum_{j,l=2}^n \widetilde{C}_{ijkl} \xi_j \xi_l; \; \begin{matrix} i \downarrow 1, \ldots, n \\ k \to 1, \ldots, n \end{matrix} \right), \quad \xi' = (\xi_2, \ldots, \xi_n),$$

$$R(\xi') = \left(\sum_{j=2}^n \widetilde{C}_{ijk1} \xi_j; \; \begin{matrix} i \downarrow 1, \ldots, n \\ k \to 1, \ldots, n \end{matrix} \right), \quad A(\xi') = R(\xi') + {}^t R(\xi'),$$

$$T = \left(\widetilde{C}_{i1k1}; \; \begin{matrix} i \downarrow 1, \ldots, n \\ k \to 1, \ldots, n \end{matrix} \right).$$

Then we can see that

$$\mathcal{L}_{\widetilde{C}} = T \partial_1^2 + A(\partial') \partial_1 + Q(\partial')$$

which immediately gives:

$$
\begin{aligned}
\langle E, \mathcal{L}_{\widetilde{C}}\varphi\rangle &= \int_{\mathbb{R}^n} E\big(T\partial_1^2 + A\partial_1\big)\varphi \,\mathrm{d}y + \int_{\mathbb{R}^n} EQ\varphi \,\mathrm{d}y \\
&= -\int_{\mathbb{R}^{n-1}} TE_+\partial_1\varphi \,\mathrm{d}y'|_{y_1=+0} + \int_{\mathbb{R}^{n-1}} TE_-\partial_1\varphi \,\mathrm{d}y'|_{y_1=-0} - \int_{\mathbb{R}^n_+} T\partial_1 E\partial_1\varphi \,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^n_-} T\partial_1 E\partial_1\varphi \,\mathrm{d}y - \int_{\mathbb{R}^{n-1}} E_+A\varphi \,\mathrm{d}y'|_{y_1=+0} + \int_{\mathbb{R}^{n-1}} E_-A\varphi \,\mathrm{d}y'|_{y_1=-0} \\
&\quad - \int_{\mathbb{R}^n_+} \partial_1 EA\varphi \,\mathrm{d}y - \int_{\mathbb{R}^n_-} \partial_1 EA\varphi \,\mathrm{d}y + \int_{\mathbb{R}^n_+} EQ\varphi \,\mathrm{d}y + \int_{\mathbb{R}^n_-} EQ\varphi \,\mathrm{d}y \\
&= \int_{\mathbb{R}^{n-1}} T\partial_1 E\varphi \,\mathrm{d}y'|_{y_1=+0} - \int_{\mathbb{R}^{n-1}} T\partial_1 E\varphi \,\mathrm{d}y'|_{y_1=-0} + \int_{\mathbb{R}^n_+} T\partial_1^2 E\varphi \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n_-} T\partial_1^2 E\varphi \,\mathrm{d}y + \int_{\mathbb{R}^n_+} A\partial_1 E\varphi \,\mathrm{d}y + \int_{\mathbb{R}^n_-} A\partial_1 E\varphi \,\mathrm{d}y + \int_{\mathbb{R}^n_+} QE\varphi \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n_-} QE\varphi \,\mathrm{d}y \\
&= \int_{\mathbb{R}^{n-1}} (T\partial_1 E|_{y_1=+0} - T\partial_1 E|_{y_1=-0})\varphi(0, y') \,\mathrm{d}y' + \int_{\mathbb{R}^n_+} (\mathcal{L}_{\widetilde{C}} E_+)\varphi \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n_-} (\mathcal{L}_{\widetilde{C}} E_-)\varphi \,\mathrm{d}y \\
&= \int_{\mathbb{R}^{n-1}} (T\partial_1 E|_{y_1=+0} - T\partial_1 E|_{y_1=-0})\varphi(0, y') \,\mathrm{d}y' \\
&= -\varphi(0),
\end{aligned}
$$

where the last equality follows from the facts that

$$
\sigma_{\widetilde{C}}(E_\pm) = \big(T\partial_1 + {}^tR(\partial')\big)E_\pm,
$$
$$
{}^tR(\partial')E_+|_{y_1=+0} = {}^tR(\partial')E_-|_{y_1=-0},
$$

and

$$
\sigma_{\widetilde{C}}(E_+)e_1|_{y_1=y_1^0+0} - \sigma_{\widetilde{C}}(E_-)e_1|_{y_1=y_1^0-0} = -\delta\big(y' - y^{0\prime}\big)I_n.
$$

Therefore, we obtain that

$$\mathcal{L}_{\widetilde{C}} E + \delta\big(y - y^0\big) I_n = 0. \qquad \square$$

Now let $M(\xi) := T\xi_1^2 + A(\xi')\xi_1 + Q(\xi')$, i.e., the symbol of $\mathcal{L}_{\widetilde{C}}$, then M admits the factorization:

$$M(\xi) = \big(\xi_1 - B_\pm(\xi')^*\big) T \big(\xi_1 - B_\pm(\xi')\big),$$

where

$$B_\pm(\xi') = \left(\oint_{\ell_\pm} \zeta M(\zeta, \xi')^{-1}\, \mathrm{d}\zeta\right)\left(\oint_{\ell_\pm} M(\zeta, \xi')^{-1}\, \mathrm{d}\zeta\right)^{-1}$$

and $\ell_+ \subset \mathbb{C}_+$ (or $\ell_- \subset \mathbb{C}_-$) is a C^1 Jordan closed curves enclosing all roots of $\det M = 0$ in ξ_1 with positive (or negative) imaginary parts (see [10]). As defined in [4],

$$Z_\pm(\xi') := \mp \mathrm{i}\big(T B_\pm(\xi') + {}^t R(\xi')\big)$$

is called the surface impedance tensor for the half space $\mathbb{R}^n_\pm$. Also, it is known that $Z_\pm(\xi')$ are positive Hermitian matrices. It is not difficult to show that

Lemma 4.2. (i) $\overline{B}_+(\xi') = B_-(\xi')$;
(ii) $\overline{Z}_+(\xi') = Z_-(\xi')$.

Taking advantage of the surface impedance tensor, we can give an explicit representation of $E_\pm$ in terms of the oscillatory integral.

Lemma 4.3. *Let $E_\pm(y, y^0)$ satisfy* (4.2)*, then $E_\pm(y, y^0)$ can be written as*

$$E_\pm\big(y, y^0\big) = \text{Os-}\int \mathrm{e}^{\mathrm{i}(y' - y^{0\prime})\cdot\xi'} \mathrm{e}^{\mathrm{i}(y_1 - y_1^0) B_\pm(\xi')} \left(\frac{1}{2}\right) (\operatorname{Re} Z_+)^{-1}\, \text{đ}\xi' \qquad \big(\pm\big(y_1 - y_1^0\big) > 0\big), \tag{4.3}$$

where $\text{đ}\xi' = (2\pi)^{-(n-1)}\, \mathrm{d}\xi'$.

We use the notation Os—to refer to an oscillatory integral.

Remark. Note that $\operatorname{Re} Z_+$ is invertible since Z_+ is positive Hermitian.

Proof. For simplicity, we let $y^{0\prime} = 0$. In view of the factorization of M, it is clear that $E_\pm$ defined in (4.3) satisfy:

$$\mathcal{L}_{\widetilde{C}} E_\pm = 0 \quad \text{in } \pm\big(y_1 - y_1^0\big) > 0.$$

So we only need to check jump conditions. We first note that

$$E_+|_{y_1=y_1^0+0} = E_-|_{y_1=y_1^0-0}.$$

Next we observe that

$$\begin{aligned}
\sigma_{\widetilde{C}}(E_+)e_1|_{y_1=y_1^0+0} &= \text{Os-}\int e^{iy'\cdot\xi'} i\big(TB_+(\xi') + {}^tR(\xi')\big)\left(\frac{1}{2}\right)\big(\operatorname{Re} Z_+(\xi')\big)^{-1}\, đ\xi' \\
&= \text{Os-}\int e^{iy'\cdot\xi'}\big(-Z_+(\xi')\big)\left(\frac{1}{2}\right)\big(\operatorname{Re} Z_+(\xi')\big)^{-1}\, đ\xi'
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\widetilde{C}}(E_-)e_1|_{y_1=y_1^0-0} &= \text{Os-}\int e^{iy'\cdot\xi'} i\big(TB_-(\xi') + {}^tR(\xi')\big)\left(\frac{1}{2}\right)\big(\operatorname{Re} Z_+(\xi')\big)^{-1}\, đ\xi' \\
&= \text{Os-}\int e^{iy'\cdot\xi'}\big(Z_-(\xi')\big)\left(\frac{1}{2}\right)\big(\operatorname{Re} Z_+(\xi')\big)^{-1}\, đ\xi'.
\end{aligned}$$

Thus, from (ii) of Lemma 4.2 we have that

$$\begin{aligned}
&\sigma_{\widetilde{C}}(E_+)e_1|_{y_1=y_1^0+0} - \sigma_{\widetilde{C}}(E_-)e_1|_{y_1=y_1^0-0} \\
&\quad= \text{Os-}\int e^{iy'\cdot\xi'}(-1)\big(Z_+(\xi') + Z_-(\xi')\big)\left(\frac{1}{2}\right)\big(\operatorname{Re} Z_+(\xi')\big)^{-1}\, đ\xi' \\
&\quad= \text{Os-}\int e^{iy'\cdot\xi'}(-1)\big(2\operatorname{Re} Z_+(\xi')\big)\left(\frac{1}{2}\right)\big(\operatorname{Re} Z_+(\xi')\big)^{-1}\, đ\xi' \\
&\quad= -\delta(y'). \qquad \square
\end{aligned}$$

For a constant vector $b \in \mathbb{R}^n \setminus \{0\}$, we define $\widetilde{w}_\pm^0$ satisfying:

$$\begin{cases}
\mathcal{L}_{\widetilde{C}}\widetilde{w}_\pm^0 = 0 & \text{in } \mathbb{R}^n_\pm, \\
\sigma_{\widetilde{C}}(\widetilde{w}_\pm^0)e_1 = -\sigma_{\widetilde{C}}\big(E_-(y, y^0)b\big)e_1 & \text{on } \{y_1 = 0\}.
\end{cases}$$

Then we can show the:

Lemma 4.4. $\widetilde{w}_\pm^0$ *can be given by*:

$$\begin{aligned}
\widetilde{w}_\pm^0(y, y^0) = \pm\text{Os-}\int & e^{i(y'-y^{0\prime})\cdot\eta'} e^{iy_1B_\pm(\eta')} e^{-iy_1^0B_-(\eta')} \\
&\times \left(\frac{1}{2}\right) Z_\pm(\eta')^{-1} Z_-(\eta')\big(\operatorname{Re} Z_+(\eta')\big)^{-1} b\, đ\eta'.
\end{aligned}$$

Proof. It suffices to verify the boundary condition. Note that

$$\sigma_{\widetilde{C}}\big(\widetilde{w}^0_\pm\big)e_1|_{y_1=0} = \text{Os-}\int e^{i(y'-y^{0\prime})\cdot\eta'} e^{-iy_1^0 B_-(\eta')}(\pm)\left(\frac{1}{2}\right)(\mp)Z_\pm(\eta')Z_\pm(\eta')^{-1} \times Z_-(\eta')\big(\operatorname{Re} Z_+(\eta')\big)^{-1} b\, đ\eta'$$
$$= \text{Os-}\int e^{i(y'-y^{0\prime})\cdot\eta'} e^{-iy_1^0 B_-(\eta')}\left(-\frac{1}{2}\right)Z_-(\eta')\big(\operatorname{Re} Z_+(\eta')\big)^{-1} b\, đ\eta'.$$

On the other hand, we have:

$$\sigma_{\widetilde{C}}\big(E_-\big(y, y^0\big)b\big)e_1|_{y_1=0} = \text{Os-}\int e^{i(y'-y^{0\prime})\cdot\eta'} e^{-iy_1^0 B_-(\eta')}\left(\frac{1}{2}\right)Z_-(\eta')\big(\operatorname{Re} Z_+(\eta')\big)^{-1} b\, đ\eta'.$$

So this lemma is proved. □

To avoid confusion, we would like to point out that $E_\pm$ is defined for $\pm(y_1 - y_1^0) > 0$ while $\widetilde{w}^0_\pm$ is defined for $\pm y_1 > 0$. Let $\widetilde{W}_\pm$ solve:

$$\begin{cases} \mathcal{L}_{\widetilde{C}(y)}\widetilde{W}_\pm = -\mathcal{L}_{\{\widetilde{C}(y)-\widetilde{C}\}}\big(\widetilde{w}^0_\pm + E\big(y, y^0\big)b\big) & \text{in } V_\pm, \\ \sigma_{\widetilde{C}(y)}(\widetilde{W}_\pm)e_1 = -\sigma_{\widetilde{C}(y)}\big(\widetilde{w}^0_\pm + E\big(y, y^0\big)b\big)e_1 & \text{on } \beta_0, \\ \widetilde{W}_\pm = 0 & \text{on } \beta_\pm. \end{cases} \tag{4.4}$$

We would like to show that $\widetilde{W}_\pm \sim 0$ in $\overline{H}^1(V_\pm)$. To verify this, it is enough to prove:

$$\sigma_{\{\widetilde{C}(y)-\widetilde{C}\}}\big(\widetilde{w}^0_\pm + E\big(y, y^0\big)b\big) \sim 0 \quad \text{in } L^2(V_\pm)$$

and

$$\sigma_{\widetilde{C}(y)}\big(\widetilde{w}^0_\pm + E\big(y, y^0\big)b\big)e_1 \sim 0 \quad \text{in } \overline{H}^{-1/2}(\beta_0).$$

We first observe that

$$\widetilde{w}^0_-\big(y, y^0\big) + E\big(y, y^0\big)b = \widetilde{w}^0_-\big(y, y^0\big) + E_-\big(y, y^0\big)b \sim 0 \quad \text{in } \overline{H}^1(V_-),$$

which implies $\widetilde{W}_- \sim 0$ in $\overline{H}^1(V_-)$. So we only need to check that

$$\sigma_{\{\widetilde{C}(y)-\widetilde{C}\}}\big(\widetilde{w}^0_+ + E\big(y, y^0\big)b\big) \sim 0 \quad \text{in } L^2(V_+)$$

and

$$\sigma_{\widetilde{C}(y)}\big(\widetilde{w}^0_+ + E\big(y, y^0\big)b\big)e_1 \sim 0 \quad \text{in } L^2(\beta_0).$$

Lemma 4.5. (i) $\sigma_{\{\widetilde{C}(y)-\widetilde{C}\}}(\widetilde{w}_+^0) \sim 0$ *in* $L^2(V_+)$;
(ii) $\sigma_{\{\widetilde{C}(y)-\widetilde{C}\}}(E(y,y^0)b) \sim 0$ *in* $L^2(V_+)$;
(iii) $\sigma_{\widetilde{C}(y)}(\widetilde{w}_+^0 + E(y,y^0)b)e_1 \sim 0$ *in* $\overline{H}^{-1/2}(\beta_0)$.

Proof. (i) It suffices to prove that $(y_j - y_j^0)\varepsilon(\widetilde{w}_+^0) \sim 0$ in $L^2(V_+)$ for $1 \leqslant j \leqslant n$. Without loss of generality, we choose $y^{0\prime} = 0$. Note that $B_\pm(\eta')$ and $Z_\pm(\eta')$ are homogeneous of degree one in η'. We will estimate $\varepsilon(\widetilde{w}_+^0)$ by the Plancherel formula and the way to estimate them is essentially the same as estimating

$$A\bigl(y, y^0\bigr) = \text{Os-}\int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}a(\eta')\,\text{đ}\eta', \tag{4.5}$$

where $a(\eta') \in C^\infty(\mathbb{R}^{n-1} \setminus \{0\})$ is homogeneous of degree zero in η' and α is a positive constant. Notice that α is related to $\inf\{\operatorname{Im}\zeta\colon\ \zeta \in \operatorname{Spec}(B_+(\eta')|\eta'|^{-1})\}$ $(= -\sup\{\operatorname{Im}\zeta\colon$ $\zeta \in \operatorname{Spec}(B_-(\eta')|\eta'|^{-1})\})$. Let $\delta > 0$ such that $0 < y_1 < \delta$ for all $y = (y_1, \ldots, y_n) \in V_+$ and $\phi(\eta') \in C^\infty(\mathbb{R}^{n-1})$ so that $0 \leqslant \phi(\eta') \leqslant 1$ with

$$\phi(\eta') = \begin{cases} 1 & \text{if } |\eta'| \leqslant 1, \\ 0 & \text{if } |\eta'| \geqslant 2. \end{cases}$$

Let $A(y, y^0)$ be decomposed into $A = A_1 + A_2$ with

$$\begin{aligned} A_1\bigl(y, y^0\bigr) &:= \text{Os-}\int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\phi(\eta')a(\eta')\,\text{đ}\eta' \\ &= \int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\phi(\eta')a(\eta')\,\text{đ}\eta' \end{aligned} \tag{4.6}$$

and

$$A_2\bigl(y, y^0\bigr) := \text{Os-}\int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\bigl(1 - \phi(\eta')\bigr)a(\eta')\,\text{đ}\eta'. \tag{4.7}$$

It is clear that

$$\begin{aligned} \int\limits_{V_+} \bigl|\bigl(y_j - y_j^0\bigr)A_1\bigl(y, y^0\bigr)\bigr|^2\,\mathrm{d}y &\leqslant c\int\limits_0^\delta \int\limits_{\mathbb{R}^{n-1}} \mathrm{e}^{-2\alpha|\eta'|y_1}\mathrm{e}^{-2\alpha|\eta'|y_1^0}\phi^2(\eta')\bigl|a(\eta')\bigr|^2\,\mathrm{d}\eta'\,\mathrm{d}y_1 \\ &< \infty \quad \bigl(\text{uniformly in } y_1^0 \sim 0\bigr), \end{aligned} \tag{4.8}$$

for $1 \leqslant j \leqslant n$.

Next we observe that

$$\bigl(\bigl(y_1 - y_1^0\bigr)\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\bigr)^2 \leqslant 2\bigl(y_1^2 + \bigl(y_1^0\bigr)^2\bigr)\mathrm{e}^{-2\alpha|\eta'|y_1}\mathrm{e}^{-2\alpha|\eta'|y_1^0}$$

and

$$\int_0^\infty y_1^2 \mathrm{e}^{-2\alpha|\eta'|y_1}\,\mathrm{d}y_1 = 2\bigl(2\alpha|\eta'|\bigr)^{-3}.$$

Thus, we can find that

$$\begin{aligned}
&\int_{V_+} \bigl|\bigl(y_1 - y_1^0\bigr)A_2\bigl(y, y^0\bigr)\bigr|^2\,\mathrm{d}y \\
&\quad\leqslant \int_{\mathbb{R}^{n-1}} \int_0^\infty \bigl(y_1 - y_1^0\bigr)^2 \mathrm{e}^{-2\alpha|\eta'|y_1}\mathrm{e}^{-2\alpha|\eta'|y_1^0}\bigl(1-\varphi(\eta')\bigr)^2\bigl|a(\eta')\bigr|^2\,\mathrm{d}y_1\,\mathrm{d}\eta' \\
&\quad\leqslant c \int_{|\eta'|\geqslant 1} \bigl(|\eta'|^{-3} + \bigl(y_1^0\bigr)^2|\eta'|^{-1}\bigr)\mathrm{e}^{-2\alpha|\eta'|y_1^0}\,\mathrm{d}\eta'.
\end{aligned}\tag{4.9}$$

We now check the right-hand side of (4.9) term by term. It is obvious that

$$\int_{|\eta'|\geqslant 1} |\eta'|^{-3}\mathrm{e}^{-2\alpha|\eta'|y_1^0}\,\mathrm{d}\eta' \leqslant c\int_1^\infty s^{-3}s^{n-2}\,\mathrm{d}s < \infty, \tag{4.10}$$

since $n = 2, 3$. Furthermore, we note that

$$\bigl(y_1^0\bigr)^2 \int_{|\eta'|\geqslant 1} |\eta'|^{-1}\mathrm{e}^{-2\alpha|\eta'|y_1^0}\,\mathrm{d}\eta' \to 0 \quad \text{as } y_1^0 \to 0. \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) yields:

$$\int_{V_+} \bigl|\bigl(y_1 - y_1^0\bigr)A_2\bigl(y, y^0\bigr)\bigr|^2\,\mathrm{d}y < \infty \quad \text{uniformly in } y_1^0 \sim 0. \tag{4.12}$$

Finally, for $j \neq 1$ we have:

$$\begin{aligned}
y_j A_2\bigl(y, y^0\bigr) &= \text{Os-}\int y_j \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\bigl(1-\phi(\eta')\bigr)a(\eta')\,\mathrm{đ}\eta' \\
&= \text{Os-i}\int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\partial_{\eta_j}\bigl\{\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\bigl(1-\phi(\eta')\bigr)a(\eta')\bigr\}\,\mathrm{đ}\eta'.
\end{aligned}$$

It is readily seen that

$$\begin{aligned}&\bigl|\partial_{\eta_j}\bigl\{\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\bigl(1-\phi(\eta')\bigr)a(\eta')\bigr\}\bigr|\\&\quad\leqslant c\bigl(y_1+y_1^0+|\eta'|^{-1}\bigr)\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\bigl(1-\phi(\eta')\bigr)+c\mathrm{e}^{-\alpha|\eta'|y_1}\mathrm{e}^{-\alpha|\eta'|y_1^0}\bigl|\partial_{\eta_j}(1-\varphi)\bigr|.\end{aligned}$$

Therefore, carrying out the same arguments as above, we can show that

$$\int_{V_+}\bigl|y_jA_2\bigl(y,y^0\bigr)\bigr|^2\,\mathrm{d}y<\infty\quad\text{uniformly in } y_1^0\sim 0. \tag{4.13}$$

The estimates (4.8), (4.12) and (4.13) immediately lead to the result (i).

(ii) Using the same idea, we can see that the way to estimate $\varepsilon(E(y,y^0)b)$ is essentially the same as estimating

$$B_1\bigl(y,y^0\bigr):=\text{Os-}\int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha(y_1-y_1^0)|\eta'|}a(\eta')\,\text{đ}\eta'\quad\text{for }\bigl(y_1-y_1^0\bigr)>0$$

and

$$B_2\bigl(y,y^0\bigr):=\text{Os-}\int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha(y_1^0-y_1)|\eta'|}a(\eta')\,\text{đ}\eta'\quad\text{for }\bigl(y_1-y_1^0\bigr)<0,$$

where α and $a(\eta')$ are defined as in (4.5). Now it is easy to compute

$$\int_{y_1^0}^{\infty}\bigl(y_1-y_1^0\bigr)^2\mathrm{e}^{-2\alpha(y_1-y_1^0)|\eta'|}\,\mathrm{d}y_1=\int_0^{\infty}\tilde{y}_1^2\mathrm{e}^{-2\alpha\tilde{y}_1|\eta'|}\,\mathrm{d}\tilde{y}_1=2\bigl(2\alpha|\eta'|\bigr)^{-3} \tag{4.14}$$

and

$$\begin{aligned}&\int_0^{y_1^0}\bigl(y_1^0-y_1\bigr)^2\mathrm{e}^{-2\alpha(y_1^0-y_1)|\eta'|}\,\mathrm{d}y_1=\int_0^{y_1^0}\tilde{y}_1^2\mathrm{e}^{-2\alpha\tilde{y}_1|\eta'|}\,\mathrm{d}\tilde{y}_1\leqslant\int_0^{\infty}\tilde{y}_1^2\mathrm{e}^{-2\alpha\tilde{y}_1|\eta'|}\,\mathrm{d}\tilde{y}_1\\&\quad=2\bigl(2\alpha|\eta'|\bigr)^{-3}.\end{aligned} \tag{4.15}$$

By virtue of (4.14) and (4.15) and using the same arguments in (i), we can prove that $(y_1-y_1^0)\varepsilon(E(y,y^0)b)$ belongs to $L^2(V_+)$ uniformly in $y_1^0\sim 0$. Additionally, the same estimates can be obtained for $y_j\varepsilon(E(y,y^0)b)$ $(j\neq 1)$ by the similar derivations.

(iii) Observe that $\sigma_{\widetilde{C}(y)}(\widetilde{w}_+^0+E(y,y^0)b)e_1=\sigma_{\{\widetilde{C}(y)-\widetilde{C}\}}(\widetilde{w}_+^0+E(y,y^0)b)e_1$ on $\{y_1=0\}$ since

$$\sigma_{\widetilde{C}}\bigl(\widetilde{w}_+^0+E\bigl(y,y^0\bigr)b\bigr)e_1=\sigma_{\widetilde{C}}\bigl(\widetilde{w}_+^0+E_-\bigl(y,y^0\bigr)b\bigr)e_1=0\quad\text{on }\{y_1=0\}.$$

Argued as above, to derive the estimate, it suffices to estimate:

$$y_j\tilde{A}\bigl(y',y^0\bigr)=\text{Os-}\int y_j\mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha|\eta'|y_1^0}a(\eta')\,\text{đ}\eta'\quad\text{for } j=2,3,$$

where $a(\eta')$ and α are defined similarly. Here we also set $y^{0\prime} = 0$. Decompose $y_j\tilde{A} = y_j\tilde{A}_1 + y_j\tilde{A}_2$ as in (4.6) and (4.7). Then $y_j\tilde{A}_1$ clearly satisfies the estimate, i.e., $y_j\tilde{A}_1 \sim 0$ in $\overline{H}^{-1/2}(\beta_0)$. For $y_j\tilde{A}_2$ we observe that

$$y_jA_2\big(y, y^0\big) = \text{Os-}\int y_j \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\mathrm{e}^{-\alpha|\eta'|y_1^0}\big(1-\phi(\eta')\big)a(\eta')\,\text{đ}\eta'$$
$$= \text{Os-i}\int \mathrm{e}^{\mathrm{i}y'\cdot\eta'}\partial_{\eta_j}\big\{\mathrm{e}^{-\alpha|\eta'|y_1^0}\big(1-\phi(\eta')\big)a(\eta')\big\}\,\text{đ}\eta'$$

and

$$\big|\partial_{\eta_j}\big\{\mathrm{e}^{-\alpha|\eta'|y_1^0}\big(1-\phi(\eta')\big)a(\eta')\big\}\big|$$
$$\leqslant c\big(y_1^0 + |\eta'|^{-1}\big)\mathrm{e}^{-\alpha|\eta'|y_1^0}\big(1-\phi(\eta')\big) + c\mathrm{e}^{-\alpha|\eta'|y_1^0}\big|\partial_{\eta_j}(1-\varphi)\big|.$$

Obviously, we have that $\mathrm{e}^{-\alpha|\eta'|y_1^0}|\partial_{\eta_j}(1-\varphi)| \in L^2(\mathbb{R}^{n-1})$ uniformly in $y_1^0 \sim 0$. By virtue of the inequality

$$\big(y_1^0 + |\eta'|^{-1}\big)^2 \leqslant 2\big(y_1^0\big)^2 + 2|\eta'|^{-2}$$

and the estimate

$$\big(y_1^0\big)^2\int_{|\eta'|\geqslant 1}\mathrm{e}^{-2\alpha|\eta'|y_1^0}\,\mathrm{d}\eta' < \infty \quad \text{uniformly in } y_1^0 \sim 0,$$

to get the estimate (iii), it remains to compute:

$$\int_{|\eta'|\geqslant 1}\big(1+|\eta'|^2\big)^{-1/2}|\eta'|^{-2}\mathrm{e}^{-2\alpha|\eta'|y_1^0}\,\mathrm{d}\eta'$$
$$\leqslant \int_{|\eta'|\geqslant 1}\big(1+|\eta'|^2\big)^{-1/2}|\eta'|^{-2}\,\mathrm{d}\eta' \leqslant c\int_1^\infty s^{-3}s^{n-2}\,\mathrm{d}s < \infty \quad \text{uniformly in } y_1^0 \sim 0. \qquad \square$$

The results in Lemma 4.5 justify the estimate $\widetilde{W}_\pm \sim 0$ in $H^1(V_\pm)$, where $\widetilde{W}_\pm$ are defined in (4.4). Now we set:

$$\widetilde{w}_\pm = \big(\widetilde{w}_\pm^0 + \widetilde{W}_\pm\big) - (\widetilde{E}b - Eb). \tag{4.16}$$

By a straightforward computation, we can easily find that

$$\begin{cases} \mathcal{L}_{\widetilde{C}(y)}\widetilde{w}_\pm = 0 & \text{in } V_\pm, \\ \sigma_{\widetilde{C}(y)}(\widetilde{w}_\pm)e_1 = -\sigma_{\widetilde{C}(y)}\big(\widetilde{E}\big(y, y^0\big)b\big)e_1 & \text{on } \beta_0. \end{cases}$$

Now let $0 \leqslant \chi \in C_0^\infty(\Omega)$ satisfy $\chi = 1$ in a sufficiently small neighborhood of a and be supported near a such that $\operatorname{supp}(\chi) \cap S \subset \Sigma$. Recall that S is the closed hypersurface where the crack lies. We then define $w \in H^1(\Omega \setminus \Sigma)$ by:

$$w = \chi w_0' \quad \text{with } w_0' = \begin{cases} \widetilde{w}_+\big(y\big(x, x^0\big), y^0\big) & \text{in } \Omega_+ \cap \operatorname{supp}(\chi), \\ \widetilde{w}_-\big(y\big(x, x^0\big), y^0\big) & \text{in } \Omega_- \cap \operatorname{supp}(\chi). \end{cases} \tag{4.17}$$

It follows from (3.7) that $\widetilde{w} := w' - w$ satisfies:

$$\begin{cases} \mathcal{L}_C \widetilde{w} = -\nabla \cdot \big(C\big(w_0' \otimes \nabla\chi\big)\big) - \big(C\nabla w_0'\big)\nabla\chi & \text{in } \Omega \setminus \overline{\Sigma}, \\ \sigma_C(\widetilde{w})\nu = -\sigma_C(v' + G - Eb)\nu - B(x, \nabla\chi, \nu)w_0' & \\ \qquad\qquad + (\chi - 1)\sigma_C(Eb)\nu & \text{on } \Sigma, \\ \widetilde{w} = 0 & \text{on } \Gamma_D, \\ \sigma_C(\widetilde{w})\nu = 0 & \text{on } \Gamma_N, \end{cases} \tag{4.18}$$

where $B(x, \nabla\chi, \nu) = (\sum_{j,l} C_{ijkl}\partial_l\chi\nu_j)$. Note that $\nabla\chi$ and $B(x, \nabla\chi, \nu)$ are supported away from a. Therefore, by taking into account of (3.4), we obtain that $\widetilde{w} \sim 0$ in $H^1(\Omega \setminus \Sigma)$. In other words, the behavior of the reflected solution w' as x_0 approaching to a is determined by that of w.

Now we are ready to prove Theorem 3.1. To begin, we note that the second term $\int_{\Gamma_D} f\sigma_C(\overline{w'})\nu\,\mathrm{d}s$ stays bounded as $t \to T(r, \Sigma)$. Thus, we only need to deal with the first term of $I(t, r)$. By the strong convexity condition (1.2), we know that the "inverse" of C exists, which is called the compliance tensor, satisfies the same condition. Therefore, to get the blow-up behavior of $I(r, t)$, it suffices to consider the integral

$$\int\limits_{\Omega\setminus\Sigma} \big|\sigma_C(w')\big|^2\,\mathrm{d}x. \tag{4.19}$$

Based on the previous analysis, we can replace the reflected solution w' in (4.19) by the localized function w defined in (4.17) near $a = T(r, \Sigma)$. Working in the local coordinates $y = (y_1, \ldots, y_n)$ near a as described before, we aim to show that

$$\int\limits_{\mathbb{R}^n_\pm} \varrho\big|\sigma_{\widetilde{C}}(\widetilde{w}_\pm)\big|^2\,\mathrm{d}y \to \infty \quad \text{as } y_1^0 \to 0, \tag{4.20}$$

where $\varrho(y) = \chi(x(y, x^0))$ which is supported near $y = 0$. In view of the definition of $\widetilde{w}_\pm$ (see (4.16)), proving (4.20) is equivalent to showing that

$$\int\limits_{\mathbb{R}^n_\pm} \varrho\big|\sigma_{\widetilde{C}}\big(\widetilde{w}_\pm^0\big)\big|^2\,\mathrm{d}y \to \infty \quad \text{as } y_1^0 \to 0. \tag{4.21}$$

To this end, we first prove the following estimate:

Lemma 4.6.

$$\int_{\mathbb{R}^n_\pm \cap V_\delta} \left|\sigma_{\widetilde{C}}\left(\widetilde{w}^0_\pm\right)\right|^2 \mathrm{d}y \to \infty \quad \textit{as } y_1^0 \to 0,$$

where $V_\delta := \{y \in \mathbb{R}^n\colon |y_1| \leqslant \delta\}$ *such that* $\operatorname{supp}(\varrho) \cap V_\delta \subsetneq \operatorname{supp}(\varrho)$.

Proof. Clearly,

$$\int_{\mathbb{R}^n_\pm \cap V_\delta} \left|\sigma_{\widetilde{C}}\left(\widetilde{w}^0_\pm\right)\right|^2 \mathrm{d}y \geqslant \int_{\mathbb{R}^n_\pm \cap V_\delta} \left|\sigma_{\widetilde{C}}\left(\widetilde{w}^0_\pm\right)e_1\right|^2 \mathrm{d}y.$$

In the proof of Lemma 4.4, we note that

$$\begin{aligned}\sigma_{\widetilde{C}}\left(\widetilde{w}^0_\pm\right)e_1 &= \text{Os-}\int \mathrm{e}^{\mathrm{i}(y'-y^{0\prime})\cdot\eta'}\mathrm{e}^{\mathrm{i}y_1 B_\pm(\eta')}\mathrm{e}^{-\mathrm{i}y_1^0 B_-(\eta')}(\pm)\left(\frac{1}{2}\right)(\mp)Z_\pm(\eta')Z_\pm(\eta')^{-1}\\ &\quad\times Z_-(\eta')\left(\operatorname{Re} Z_+(\eta')\right)^{-1} b\, \text{đ}\eta'\\ &= \text{Os-}\int \mathrm{e}^{\mathrm{i}(y'-y^{0\prime})\cdot\eta'}\mathrm{e}^{\mathrm{i}y_1 B_\pm(\eta')}\mathrm{e}^{-\mathrm{i}y_1^0 B_-(\eta')}\left(-\frac{1}{2}\right)Z_-(\eta')\left(\operatorname{Re} Z_+(\eta')\right)^{-1} b\, \text{đ}\eta'.\end{aligned}$$

Since $Z_-(\eta')$ and $\operatorname{Re} Z_+(\eta')$ are homogeneous of degree one in η' and nonsingular for all $\eta' \neq 0$, we can deduce that

$$\left|Z_-(\eta')\left(\operatorname{Re} Z_+(\eta')\right)^{-1} b\right| \geqslant c|b| \quad \text{with } c > 0,\ \forall \eta' \neq 0.$$

On the other hand, let

$$\beta := \sup\left\{\operatorname{Im}\zeta\colon\ \zeta \in \operatorname{Spec}\left(B_+(\eta')|\eta'|^{-1}\right)\right\}$$

$(= -\inf\{\operatorname{Im}\zeta\colon\ \zeta \in \operatorname{Spec}(B_-(\eta')|\eta'|^{-1})\})$ then we can get that

$$\int_{\mathbb{R}^n_\pm \cap V_\delta} \left|\sigma_{\widetilde{C}}\left(\widetilde{w}^0_\pm\right)e_1\right|^2 \mathrm{d}y \geqslant \pm c \int_{\mathbb{R}^{n-1}} \int_0^{\pm\delta} \mathrm{e}^{\mp 2\beta y_1|\eta'|}\mathrm{e}^{-2\beta y_1^0|\eta'|}\,\mathrm{d}y_1\,\mathrm{d}\eta'.$$

It is enough compute:

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \int_0^{\delta} \mathrm{e}^{-2\beta y_1|\eta'|}\mathrm{e}^{-2\beta y_1^0|\eta'|}\,\mathrm{d}y_1\,\mathrm{d}\eta' &= \int_{\mathbb{R}^{n-1}} \left(1-\mathrm{e}^{-2\beta\delta|\eta'|}\right)\left(2\beta|\eta'|\right)^{-1}\mathrm{e}^{-2\beta y_1^0|\eta'|}\,\mathrm{d}\eta'\\ &\geqslant c\int_0^\infty \left(1-\mathrm{e}^{-2\beta\delta\rho}\right)\mathrm{e}^{-2\beta y_1^0\rho}\rho^{n-3}\,\mathrm{d}\rho.\end{aligned}$$

Thus, when $n=3$, we have:

$$\int_0^\infty \big(1-\mathrm{e}^{-2\beta\delta\rho}\big)\mathrm{e}^{-2\beta y_1^0\rho}\,\mathrm{d}\rho = \frac{1}{2\beta y_1^0} - \frac{1}{2\beta(y_1^0+\delta)} \to \infty \quad \text{as } y_1^0 \to 0.$$

On the other hand, for $n=2$, we see that

$$\begin{aligned}
&\int_0^\infty \rho^{-1}\big(1-\mathrm{e}^{-2\beta\delta\rho}\big)\mathrm{e}^{-2\beta y_1^0\rho}\,\mathrm{d}\rho \\
&\quad = \int_0^\varepsilon \rho^{-1}\big(1-\mathrm{e}^{-2\beta\delta\rho}\big)\mathrm{e}^{-2\beta y_1^0\rho}\,\mathrm{d}\rho + \int_\varepsilon^\infty \rho^{-1}\big(1-\mathrm{e}^{-2\beta\delta\rho}\big)\mathrm{e}^{-2\beta y_1^0\rho}\,\mathrm{d}\rho \\
&\quad = c' + \int_\varepsilon^\infty \rho^{-1}\big(1-\mathrm{e}^{-2\beta\delta\rho}\big)\mathrm{e}^{-2\beta y_1^0\rho}\,\mathrm{d}\rho \\
&\quad = c' + \int_{2\beta y_1^0\varepsilon}^\infty \rho^{-1}\big(1-\mathrm{e}^{-(\delta/y_1^0)\rho}\big)\mathrm{e}^{-\rho}\,\mathrm{d}\rho \to \infty \quad \text{as } y_1^0 \to 0.
\end{aligned}$$

Consequently, we conclude that

$$\int_{\mathbb{R}^{n-1}} \int_0^\delta \mathrm{e}^{-2\beta y_1|\eta'|}\mathrm{e}^{-2\beta y_1^0|\eta'|}\,\mathrm{d}y_1\,\mathrm{d}\eta' \to \infty \quad \text{as } y_1^0 \to 0$$

and the proof of the lemma is complete. □

By virtue of Lemma 4.6, we can arrive at the conclusion (4.21) if we can show that the integral $\int_{\mathbb{R}^n_\pm \cap V_\delta}(1-\varrho)|\sigma_{\widetilde{C}}(\widetilde{w}^0_\pm)|^2\,\mathrm{d}y$ stays bounded as $y_1^0 \to 0$.

Lemma 4.7.

$$\int_{\mathbb{R}^n_\pm \cap V_\delta} (1-\varrho)\big|\sigma_{\widetilde{C}}\big(\widetilde{w}^0_\pm\big)\big|^2\,\mathrm{d}y < \infty \quad \textit{uniformly in } y_1^0 \sim 0.$$

Proof. Notice that

$$\int_{\mathbb{R}^n_\pm \cap V_\delta} (1-\varrho)\big|\sigma_{\widetilde{C}}\big(\widetilde{w}^0_\pm\big)\big|^2\,\mathrm{d}y \leqslant \int_{\{|y|>\tilde{\varepsilon},\ \pm y_1>0\}\cap V_\delta} \big|\sigma_{\widetilde{C}}\big(\widetilde{w}^0_\pm\big)\big|^2\,\mathrm{d}y$$

for some small positive number $\tilde{\varepsilon}$. Let $D_{\pm,2}, \ldots, D_{\pm,n}$ be open domains in $\mathbb{R}^n$ defined by:

$$D_{\pm,j} = \{y \in \mathbb{R}^n\colon\ \pm y_1 > 0,\ |y_j| > \tilde{\varepsilon}/2\}, \quad j = 2, \ldots, n.$$

It is clear that $\{|y| > \tilde{\varepsilon}, \pm y_1 > 0\} \cap V_\delta \subset \bigcup_{j=2,\ldots,n}(D_{\pm,j} \cap V_\delta)$. Now we observe that

$$\int\limits_{D_{\pm,j}\cap V_\delta} \left|\sigma_{\widetilde{C}}\big(\widetilde{w}^0_\pm\big)\right|^2 \mathrm{d}y \leqslant (2/\tilde{\varepsilon}) \int\limits_{\mathbb{R}^n_\pm\cap V_\delta} \left|y_j \sigma_{\widetilde{C}}\big(\widetilde{w}^0_\pm\big)\right|^2 \mathrm{d}y.$$

Repeating the arguments in the proof of Lemma 4.4 (see the part (i)), we can derive that for $2 \leqslant j \leqslant n$

$$\int\limits_{\mathbb{R}^n_\pm\cap V_\delta} \left|y_j \sigma_{\widetilde{C}}\big(\widetilde{w}^0_\pm\big)\right|^2 \mathrm{d}y < \infty$$

uniformly in $y^0_1 \sim 0$. So the lemma is proved and the proof of Theorem 3.1 is now complete. □

Appendix A

In this appendix, we state a trace theorem and Green's formula in an elastic medium with a crack. These results can be proved along the same lines of Eller's paper [9] where he consider the Laplace equation. Let us define:

$$C^\infty(\widetilde{\Omega}) = \{u\colon\ u_+ \in C^\infty(\overline{\Omega}_+),\ u_- \in C^\infty(\overline{\Omega}_-),\ \gamma_+\partial^\alpha u = \gamma_-\partial^\alpha u \text{ on } S \setminus \Sigma\ \forall\alpha\}$$

and

$$E(\mathcal{L}, \Omega \setminus \Sigma) = \Big\{u \in H^1(\Omega \setminus \Sigma)\colon \exists h \in L^2(\Omega) \text{ such that } \int\limits_\Omega \sigma_C(u)\cdot\varepsilon(\varphi)\,\mathrm{d}x = -\int\limits_\Omega h\cdot\varphi\,\mathrm{d}x,\ \forall\varphi \in \mathfrak{D}(\Omega \setminus \overline{\Sigma})\Big\}.$$

Lemma A.1 [9, Lemma 2.8]. *The space $C^\infty(\widetilde{\Omega})$ is dense in $E(\mathcal{L}, \Omega \setminus \Sigma)$.*

Lemma A.2 [9, Lemma 2.9]. *The mapping*

$$u \to \{\gamma\sigma_C(u)\nu,\ [\sigma_C(u)\nu],\ \gamma_-\sigma_C(u)\nu\}$$

which is defined on $C^\infty(\widetilde{\Omega})$ has a unique extension to an operator from $E(\mathcal{L}, \Omega \setminus \Sigma)$ into $H^{-1/2}(\partial\Omega) \times H^{-1/2}(\Sigma) \times (\dot{H}^{1/2}(\overline{\Sigma}))^*$.

Note that Eller used the notation $\widetilde{H}^{1/2}(\Sigma)$ in his paper [9], which is nothing but $\dot{H}^{1/2}(\overline{\Sigma})$. Also, it is easily seen that $(\dot{H}^{1/2}(\overline{\Sigma}))^* = \overline{H}^{-1/2}(\Sigma)$.

Lemma A.3 [9, Lemma 2.10]. *Let $u \in E(\mathcal{L}, \Omega \setminus \Sigma)$ and $v \in H^1(\Omega \setminus \Sigma)$, then*:

$$\begin{aligned}
&\int_{\Omega\setminus\Sigma} v\mathcal{L}_C u \,\mathrm{d}x + \int_{\Omega\setminus\Sigma} \sigma_C(u)\cdot\varepsilon(v)\,\mathrm{d}x \\
&\quad = \langle \gamma\sigma_C(u)\nu, \gamma v\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} - \langle [\sigma_C(u)\nu], \gamma_+ v\rangle_{H^{-1/2}(\Sigma), H^{1/2}(\Sigma)} \\
&\qquad - \langle \gamma_-\sigma_C(u)\nu, [v]\rangle_{\overline{H}^{-1/2}(\Sigma), \dot{H}^{1/2}(\overline{\Sigma})}.
\end{aligned} \tag{A.1}$$

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