Natural Frequencies and Stability of a Flexible Spinning Disk-Stationary Load System With Rigid-Body Tilting

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The effects of rigid-body tilting on the natural frequencies and stability of the head-disk interface of a flexible spinning disk are studied both by analysis and numerical computation. Another problem of a stationary flexible disk with rigid body tilting subject to a rotating load system is also studied. In the conventional case without rigid-body tilting, these two problems are mathematically identical except for the additional membrane stress terms arising from the centrifugal force. When the rigid-body tilting is included, however, these two problems differ not only by the centrifugal effect, but also by additional terms which arise from the gyrodynamic effect. After properly defining the inner product between two state vectors, the set of equations of motion is proven to fall into the category of gyrosopic systems. Then the derivatives of the eigenvalues of the coupled system are obtained to provide a better understanding of the numerical results.

Introduction

Interest in the dynamics of a spinning disk in contact with a stationary load system arises from its applications in such fields as computer disk drives and circular saws. Conventionally, the circular disk is free at the outer radius and clamped at the inner radius by a central clamp, usually assumed to be rigid. The stationary load system, which may contain mass, damping, and stiffness elements, both linear and rotary, represents the head suspension in the computer disk drive, or the saw guide in the circular saw, respectively. Various techniques have been used to predict the natural frequencies and the stability properties of such a coupled system, for example, the classical eigenfunction expansion (Iwan and Moeller, 1976; Shen and Mote, 1991) and the finite element method (Ono et al., 1991). In all these analyses the disk, together with the rigid central clamp, rotate about a fixed axis of symmetry of the disk.

In a recent experiment with hard disk drives, Jeong and Bogy (1991) reported that the disk may undergo a small rigid-body displacement as well as deformation when the read-write system is loaded onto the spinning disk. It has also been known in the wood cutting industry that allowing the central clamping collar to slide transversely along the axis of rotation can improve the stability of the cutting process. In order to understand the effects of the rigid-body motion on the natural frequencies and stability of the spinning flexible disk, it is necessary to generalize the conventional model to include the rigid-body motion of the clamp or collar.

The dynamic response of the head-disk interface of a spinning flexible disk with axial spindle translation has been thoroughly studied (Mote, 1977; Price, 1987; Chen and Bogy, 1992c). However, the effects of rigid-body tilting on the natural frequencies and stability of the system are still not well understood. Arnold and Maunder (1961) derived the equations of motion and described the gyrodynamic behavior of an elastically supported spinning rigid disk. Stahl and Iwan (1973a) investigated the dynamic response of an elastically supported stationary rigid disk subject to a rotating load system. It was shown that the response behavior of the rigid disk is directly analogous to that of a stationary elastic disk without rigid-body tilting and subject to a circumferentially moving load system. Yang (1988) formulated the equations of motion for a general system with coupled rigid and flexible body motion, and applied it to the stability analysis of a spinning flexible disk with rigid body translation and tilting. However, his numerical results showed the combined effects of rigid-body translation and tilting, from which it is difficult to obtain physical insight into the gyrodynamic effects of rigid-body tilting alone.

In the present paper we consider a spinning flexible disk clamped at the inner radius and subject to a stationary load system. The clamp is considered as rigid and allowed to tilt about a fixed point in space, but resisted by a torsional spring

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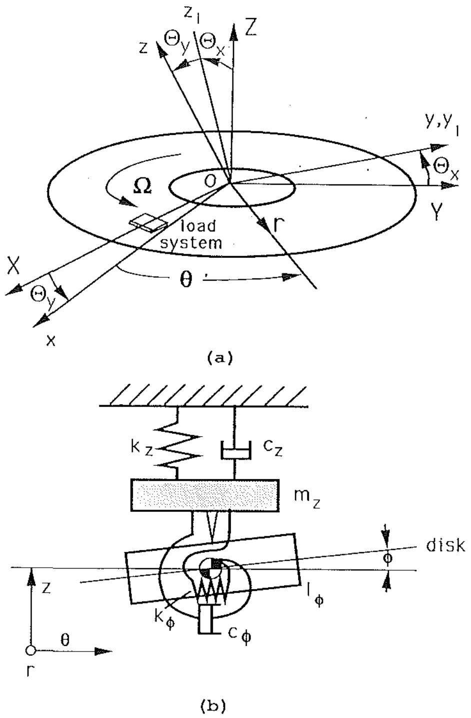

Fig. 1 (a) Global picture of a spinning disk with rigid-body tilting and the stationary load system; (b) parameters in the load system

and a torsional damper. In the special case when the stiffness of the clamp spring and the value of the clamp damping are zero, we show by numerical simulation, as well as by an analytical method, that the natural frequencies of all the one-nodal diameter modes (as seen by an observer moving with the tilting clamp) increase, as could be expected due to the inertial effect. Another problem of a stationary flexible disk with rigid-body tilting and subject to a rotating load system is also studied. It is found that these two types of problems differ not only by the stress terms arising from the centrifugal force, but also by additional terms arising from the gyrodynamic effects. After properly defining the inner product between two state vectors, we show that the set of equations fall into the category of gyroscopic systems. Then the derivatives of the eigenvalues of the system with respect to various load parameters in the load system are derived to explain several phenomena discovered in the numerical simulation when the load parameters are introduced.

Equations of Motion and Numerical Results

Figure 1 shows a flexible spinning disk with rigid body tilting. The spinning disk is clamped at the inner radius $r = a$ by a rigid clamp, and is free at the outer radius $r = b$. The rigid clamp is allowed to tilt, but it is supported by a torsional spring k_Θ and a torsional damper c_Θ. The center of the clamp is fixed at a point O in space. The motion of a spinning flexible disk can be described in two coordinate frames, as shown in Fig. 1. The frame o-xyz, together with the rigid clamp, tilts relative to the inertial frame O-XYZ. The orientation of O-xyz with respect to O-XYZ is obtained by rotation Θ_x about OX to O-$x_1y_1z_1$ and then rotation Θ_y about Oy_1 to O-xyz. The elastic transverse displacement w of the disk is measured with respect to the local noninertial O-xyz frame. The disk rotates about the Oz-axis and is in contact with a stationary load system containing transverse inertia m_z, spring k_z, dashpot c_z, and the analogous pitching elements I_ϕ, k_ϕ, and c_ϕ. By using D'Alembert's principle and by assuming that tilting angles Θ_x and Θ_y are infinitesimal, the equation of motion of the disk in terms of w and with respect to the local coordinate system (r, θ) can be written as

$$\rho h(w_{,tt} + 2\Omega w_{,t\theta} + \Omega^2 w_{,\theta\theta}) + D\nabla^4 w - \frac{h}{r}(\sigma_r r w_{,r})_{,r} - \frac{h\sigma_\theta}{r^2} w_{,\theta\theta} + r\rho h(\ddot{\Theta}_x \sin\theta - \ddot{\Theta}_y \cos\theta) + 2\Omega r\rho h(\dot{\Theta}_x \cos\theta + \dot{\Theta}_y \sin\theta) = \frac{1}{r}\delta(r-\xi)\delta(\theta)F_z - \frac{1}{r^2}\delta(r-\xi)[M_\phi \delta(\theta)]_{,\theta} \quad (1)$$

where

$$F_z = -m_z(w_{,tt} - \xi\ddot{\Theta}_y) - c_z(w_{,t} - \xi\dot{\Theta}_y) - k_z(w - \xi\Theta_y)$$

$$M_\phi = -\frac{1}{r}[I_\phi(w_{,tt\theta} + \xi\ddot{\Theta}_x) + c_\phi(w_{,t\theta} + \xi\dot{\Theta}_x) + k_\phi(w + \xi\Theta_x)]$$

and

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

The standard subscript notation for partial derivatives is used here, and the superposed dot means material time derivative. The parameters Ω, ρ, h, E, ν, and D are the rotation speed, density, thickness, Young's modulus, Poisson's ratio, and flexural rigidity of the disk. $\delta(:)$ is the Dirac delta function. The coupling position between the load system and disk is assumed to be $r = \xi$ and $\theta = 0$. The generalized plane stress components σ_r and σ_θ are due to the centrifugal effect.

We next consider a particle on the flexible spinning disk with coordinates $(r\cos\theta, r\sin\theta, w)$ with respect to the local O-xyz frame. The coordinates of the position vector $\mathbf{p}$ with respect to the inertial frame O-XYZ can be calculated by the following transformation

$$\mathbf{p} = \begin{Bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{Bmatrix} = \begin{bmatrix} \cos\Theta_y & 0 & \sin\Theta_y \\ 0 & 1 & 0 \\ -\sin\Theta_y & 0 & \cos\Theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Theta_x & -\sin\Theta_x \\ 0 & \sin\Theta_x & \cos\Theta_x \end{bmatrix} \times \begin{Bmatrix} r\cos\theta \\ r\sin\theta \\ w \end{Bmatrix} = \mathbf{Q}\begin{Bmatrix} r\cos\theta \\ r\sin\theta \\ w \end{Bmatrix}$$

where $\mathbf{Q}$ is the product matrix of the two rotation matrices. The absolute velocity of this particle is

$$\dot{\mathbf{p}} = \dot{\mathbf{Q}}\begin{Bmatrix} r\cos\theta \\ r\sin\theta \\ w \end{Bmatrix} + \mathbf{Q}\begin{Bmatrix} -r\Omega\sin\theta \\ r\Omega\cos\theta \\ \dot{w} \end{Bmatrix}.$$

The total angular momentum $\mathbf{H}$ of the disk, with respect to the origin O, is

$$\mathbf{H} = \rho h \int_0^{2\pi}\int_a^b \mathbf{p} \times \dot{\mathbf{p}}\, r dr d\theta.$$

The material derivative of $\mathbf{H}$ is

$$\dot{\mathbf{H}} = \rho h \int_0^{2\pi}\int_a^b \mathbf{p} \times \ddot{\mathbf{p}}\, r dr d\theta.$$

By considering the balance between $\dot{\mathbf{H}}$ and the resultant moment of the forces exerted by the stationary load system, the clamp spring k_Θ, and the clamp damper c_Θ, and by linearizing the equations in Θ_x, Θ_y, and w, one can obtain two additional equations of motion,

$$(I + I_c)(\ddot{\Theta}_x + 2\Omega\dot{\Theta}_y) + \int_0^{2\pi}\int_a^b \rho h \sin\theta (w_{,tt} + 2\Omega w_{,t\theta}) r^2 dr d\theta = M_\phi \delta(r-\xi)\delta(\theta) - k_\Theta \Theta_x - c_\Theta \dot{\Theta}_x \quad (2)$$

$$(I+I_c)(\ddot{\Theta}_y-2\Omega\dot{\Theta}_x)-\int_0^{2\pi}\int_a^b \rho h\cos\theta(w_{,tt}+2\Omega w_{,t\theta})r^2drd\theta$$
$$=\xi F_z\delta(r-\xi)\delta(\theta)-k_\Theta\Theta_y-c_\Theta\dot{\Theta}_y, \quad (3)$$

where $I = \pi\rho h(b^2-a^4)/4$ is the moment of inertia of the circular plate and I_c is the moment of inertia of the rigid clamp. These additional equations of motion account for the fact that there are two additional degrees-of-freedom, i.e., Θ_x and Θ_y, of the present system. For the conventional freely spinning disk with $k_\Theta = \infty$, $c_\Theta = 0$, $F_z = 0$, and $M_\phi = 0$, the eigenvalues $\lambda_{mn} = \lambda^0_{mn}$ are all purely imaginary, i.e., $\lambda^0_{mn} = i\omega^0_{mn}$, where ω^0_{mn} is real. The corresponding eigenfunctions are, in general, complex and assume the form

$$w^0_{mn}=R^0_{mn}(r)e^{\pm i\, n\theta}, \quad n=1,2,3,\ldots \quad (4)$$

in which R^0_{mn} is a real-valued function of r. The eigenfunction corresponding to $\bar{\lambda}^0_{mn}$ is $\bar{w}^0_{mn}$, where the overbar means complex conjugate. If we consider only the positive ω^0_{mn}, then w^0_{mn} in Eq. (4) with $+in\theta$ is a backward travelling wave with n nodal diameters and m nodal circles, which is also denoted by $(m, n)_b$. Similarly, w^0_{mn} with $-in\theta$ is a forward travelling wave $(m, n)_f$. The critical speed Ω_c for the mode (m, n) is defined as the rotation speed at which ω^0_{mn} of the backward travelling wave $(m, n)_b$ is zero. For Ω greater than Ω_c, this mode is a forward travelling wave, and is called a "reflected wave," denoted by $(m, n)_r$.

In the case of a freely spinning disk with the clamp allowed to tilt freely, i.e., $k_\Theta = 0$ and $c_\Theta = 0$, one can multiply Eqs. (2) and (3) by $\sin\theta$ and $\cos\theta$, respectively, and substitute the results into Eq. (1) to obtain

$$\rho h(w_{,tt}+2\Omega w_{,t\theta}+\Omega^2 w_{,\theta\theta})+D\nabla^4 w-\frac{h}{r}(\sigma_r r w_{,r})_{,r}-\frac{h\sigma_\theta}{r^2}w_{,\theta\theta}$$
$$-\frac{r\rho^2h^2}{(I+I_c)}\left[\cos\theta\int_0^{2\pi}\int_a^b r^2\cos\theta(w_{,tt}+2\Omega w_{,t\theta})drd\theta\right.$$
$$\left.+\sin\theta\int_0^{2\pi}\int_a^b r^2\sin\theta(w_{,tt}+2\Omega w_{,t\theta})drd\theta\right]=0. \quad (5)$$

The only difference between Eq. (5) and that of a conventional freely spinning disk with a nontilting clamp is the additional integral terms at the end of Eq. (5), which account for the inertial coupling between the elastic modes and the rigid-body tilting. It has been shown by Chen and Bogy (1992a,b) that the eigenfunctions w^0_{mn} are orthogonal and form a complete system of functions in r and θ in the domain $0 \le \theta \le 2\pi$, $a \le r \le b$. We therefore can express the solution w of Eq. (5) as the following infinite series:

$$w=\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}c_{mn}w^0_{mn}(r,\theta). \quad (6)$$

From Eqs. (5) and (6) it is obvious that due to the integration only the one-nodal diameter modes will be affected by the inertial force associated with the rigid-body tilting. Using a technique presented in Chen and Bogy (1992a), the natural frequency change for mode $(m, 1)$ due to the rigid-body tilting can be estimated as

$$\Delta\lambda_{m1}=\frac{i\pi\rho^2h^2\omega^0_{m1}(\omega^0_{m1}\pm 2\Omega)\left[\int_a^b R^0_{m1}(r)rdr\right]^2}{2(I+I_c)(\omega^0_{m1}\pm\Omega)\int_a^b[R^0_{m1}(r)]^2rdr}\begin{cases}+\text{ for backward wave}\\-\text{ for forward wave.}\end{cases} \quad (7)$$

It is emphasized here that the displacement w in Eqs. (1)–(6) is measured relative to the noninertial tilting frame. So, the $(m, 1)$ mode of a spinning disk with a tilting clamp in Eq. (7) is a mode with m nodal circles and one nodal diameter as seen by an observer moving with the tilting frame. To an observer in the inertial frame, however, this mode appears to be an $(m+1, 1)$ mode. Therefore, $\Delta\lambda_{m1}$ in Eq. (7) can be reinterpretated by using the mode label system with respect to the inertial frame as the change of natural frequency from an $(m, 1)$ mode of the spinning disk without rigid body tilting to an $(m+1, 1)$ mode of the spinning disk with rigid-body tilting. The advantage of using the model label with respect to the inertial frame is that as the clamp torsional spring k_Θ varies from zero to infinity, the natural frequency of the $(m, 1)$ mode with rigid-body tilting converges to that of the $(m, 1)$ mode without rigid-body tilting. Although the mode label with respect to the tilting frame is more convenient for analysis and understanding in the context of our system of equations, the mode label with respect to the inertial frame gives a more standard notation. Therefore we will adopt the latter mode label with respect to the inertial frame in the following discussions and all the figures, unless stated otherwise explicitly. However, the switch between these two slightly different notations can be easily made.

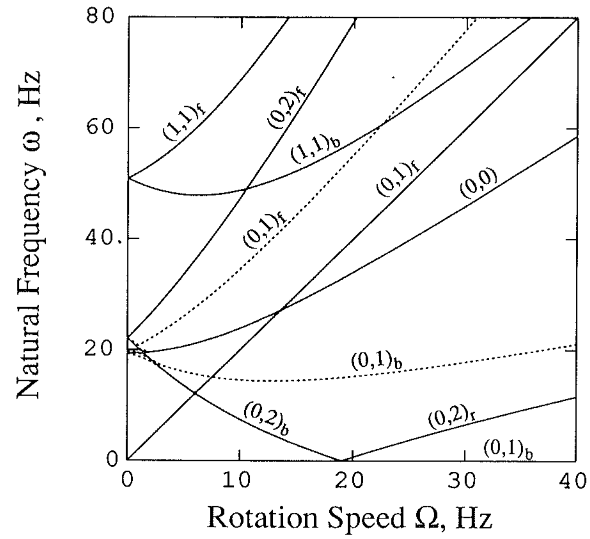

Fig. 2 Natural frequency versus rotation speed for a freely spinning flexible disk

It is obvious that $\Delta\lambda_{m1}$ in Eq. (7) is nonnegative for a backward travelling wave. However, for a forward travelling wave the term $\omega^0_{m1} - 2\Omega$ casts some uncertainties on whether $\Delta\lambda_{m1}$ is positive or negative. Figure 2 shows the relation between the natural frequencies of the spinning disk and the rotation speed. This result is obtained by a finite element computation which is similar to the one presented in Ono et al. (1991). The material properties of the disk used in the calculation are $\rho = 1.3 \times 10^3$kg/m^3, $I_c = 0$, $E = 4.9 \times 10^9$N/m^2, $\nu = 0.3$, $h = 0.078$ mm, $a = 30.0$ mm, $b = 65.0$ mm, and $\xi/b = 0.9$. For simplicity, only the first few modes with less than three nodal diameters are presented here. The dashed lines and the solid lines are the results for the spinning disk without and with rigid-body tilting, respectively. It is noted that for the system with rigid-body tilting, the mode label system with respect to the inertial frame has been adopted. The $(0, 1)_f$ mode (the diagonal line) and the $(0, 1)_b$ mode (the horizontal line with zero natural frequency) for the system with rigid-body tilting are the rigid displacement modes, i.e., $w = 0$. This can be easily checked by showing that the mode shapes $w = 0$, $\Theta_x = 1$, and $\Theta_y = \pm i$ with natural frequencies $\omega = 2\Omega$ and 0 satisfy

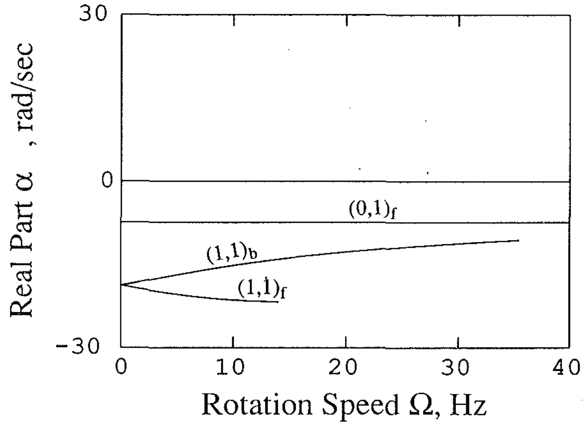

Fig. 3 Effects of clamp damping $c_\Theta = 10^{-5}$ Nms/rad

Eqs. (1)–(3). Because the ratio of the vertical and horizontal scales in Fig. 2 is 2, the slope of the diagonal line is actually 2. Now, in order for the term $\omega_{m1}^0 - 2\Omega$ in Eq. (7) to be negative, the dashed curve corresponding to the forward travelling wave would have to intersect the diagonal line. As we can see from Fig. 2, the dashed curve starts with unit slope at $\Omega = 0$, and increases its slope to more than 2 due to the effect of the centrifugal force. Consequently, the dashed line does not meet the diagonal line and the term $\omega_{m1}^0 - 2\Omega$ is always positive. This explains why the natural frequencies of both the forward and backward travelling waves (as seen by an observer moving with the tilting frame) increase for fixed Ω when the clamp is allowed to tilt freely. The modeshape of each one-nodal diameter mode (except for the rigid displacement modes $(0, 1)_f$ and $(0, 1)_b$) involves rigid body tiltings Θ_x and Θ_y with amplitudes

$$\Theta_{ym1} = \frac{\pi\rho h}{I}\int_a^b R_{m1}(r) r^2 dr = \pm i\Theta_{xm1}\begin{cases} +\text{ for backward wave} \\ -\text{ for forward wave}\end{cases}$$

where $R_{m1}(r)$ corresponds to the new modeshape of the disk with rigid-body tilting.

We now consider the effects of changing the value of clamp damping c_Θ from 0 to 10^{-5} Nms/rad on the eigenvalues of the freely spinning disk with rigid-body tilting. It is found that the natural frequency-rotation speed diagram is not affected by the change of c_Θ and remains the same as the solid lines in Fig. 2. However, the real parts corresponding to the one-nodal diameter modes become negative for all the rotation speeds, as shown in Fig. 3. The real parts of all other modes are not affected by the change of c_Θ.

It is interesting to investigate the role that the centrifugal force plays in the dynamic behavior of this system. Figure 4 is the calculation result similar to that plotted in Fig. 2, except that the membrane stresses σ_r and σ_θ in Eq. (1) are set to zero. Now the slope of the dashed frequency locus corresponding to the forward travelling wave with one nodal diameter is exactly one and it meets the diagonal line at rotation speed Ω_A, at which $\omega_{m1}^0 - 2\Omega_A$ is equal to zero. To an observer moving with the tilting frame, as can be predicted by Eq. (7), the natural frequency of the forward travelling wave increases (from dashed line $(0, 1)_f$ to solid line $(1, 1)_f$ in Fig. 4) due to rigid body tilting when $\Omega < \Omega_A$, and decreases when $\Omega > \Omega_A$. The natural frequency of the backward travelling wave increases for all Ω except at the critical speed, at which the natural frequency remains zero. As Ω exceeds Ω_A, the forward travelling wave decreases and eventually merges with the reflected wave at point B, and instability is induced. Of course no physical phenomenon corresponding to Fig. 4 will occur because the centrifugal force always exists when the disk is spinning. However, this discussion shows that the centrifugal force plays a very important role in stabilizing the present system with rigid body

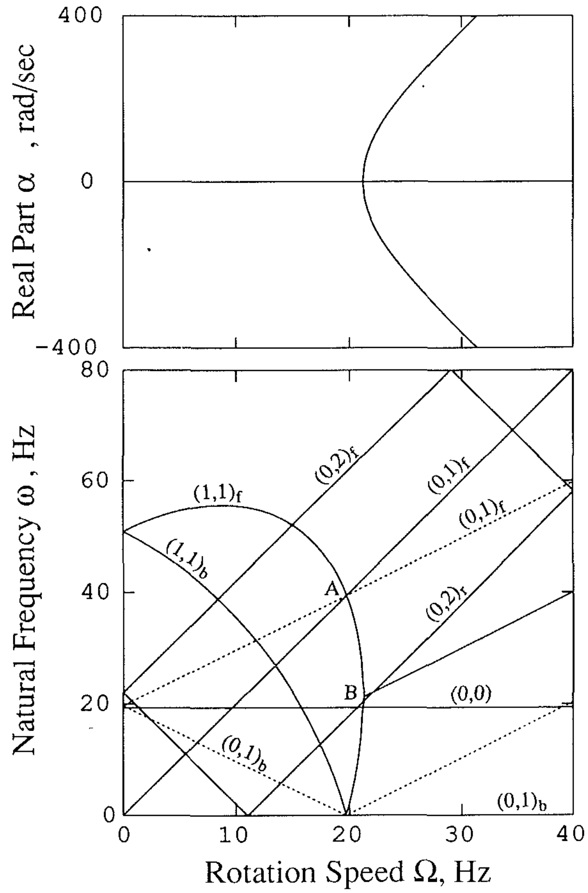

Fig. 4 The same as in Fig. 2 except that the membrane stresses are neglected in the calculation

tilting. In the conventional case without rigid-body tilting, on the other hand, neglecting the centrifugal force in the calculation will not induce any instability (Chen and Bogy, 1992a).

Stationary Disk With Rotating Load System

It is interesting to compare the results obtained in the previous section to the case of a stationary flexible disk with rigid-body tilting and subject to a rotating load system. To derive the equations of motion for this system, we first neglect F_z and M_ϕ and set Ω to zero in Eqs. (1), (2), and (3) to get the equations of motion of a stationary disk with respect to the $O\text{-}XYZ$ frame. Suppose the $O\text{-}xyz$ frame is rotating with the load system about the OZ-axis with rotation speed $-\Omega$. The terms $(\ddot{w})_{XYZ}$, $(\ddot{\Theta}_x)_{XYZ}$, and $(\ddot{\Theta}_y)_{XYZ}$ observed in the $O\text{-}XYZ$ frame are related to the corresponding terms observed in the $O\text{-}xyz$ frame through the equations

$$(\ddot{w})_{XYZ} = (w_{,tt})_{xyz} + 2\Omega(w_{,t\theta})_{xyz} + \Omega^2(w_{,\theta\theta})_{xyz} \tag{8}$$

$$(\ddot{\Theta}_x)_{XYZ} = (\ddot{\Theta}_x)_{xyz} + 2\Omega(\dot{\Theta}_y)_{xyz} - \Omega^2(\Theta_x)_{xyz} \tag{9}$$

$$(\ddot{\Theta}_y)_{XYZ} = (\ddot{\Theta}_y)_{xyz} - 2\Omega(\dot{\Theta}_x)_{xyz} - \Omega^2(\Theta_y)_{xyz}. \tag{10}$$

Equation (8) shows the relation between the Lagrangian and Eulerian descriptions of the function w. Equations (9) and (10) are the component forms of the well-known transformation equation of a single vector observed in two different frames (Meriam, 1980),

$$(\ddot{\mathbf{\Theta}})_{XYZ} = (\ddot{\mathbf{\Theta}})_{xyz} + 2\mathbf{\Omega}\times(\dot{\mathbf{\Theta}})_{xyz} + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{\Theta}$$

where vector $\mathbf{\Theta}$ is in the XY-plane with components $(\Theta_x, \Theta_y, 0)$, and vector $\mathbf{\Omega}$ is in the OZ-direction with amplitude Ω. As a consequence, the equations of motion for a stationary flexible disk with a rotating load system are, (relative to the rotating coordinate system)

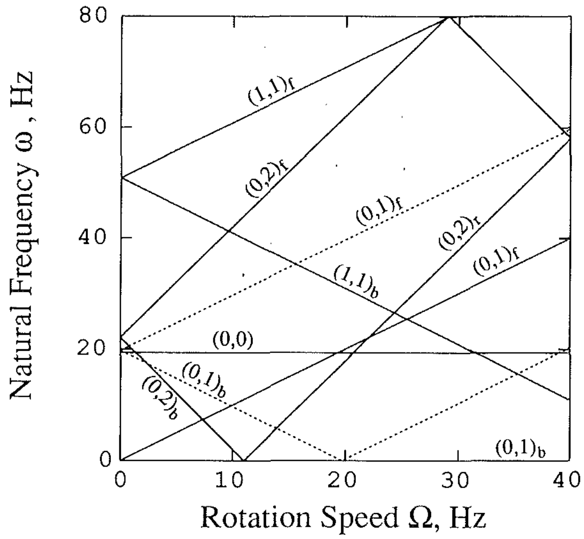

Fig. 5 Natural frequency versus rotation speed for a stationary flexible disk with rotating observer

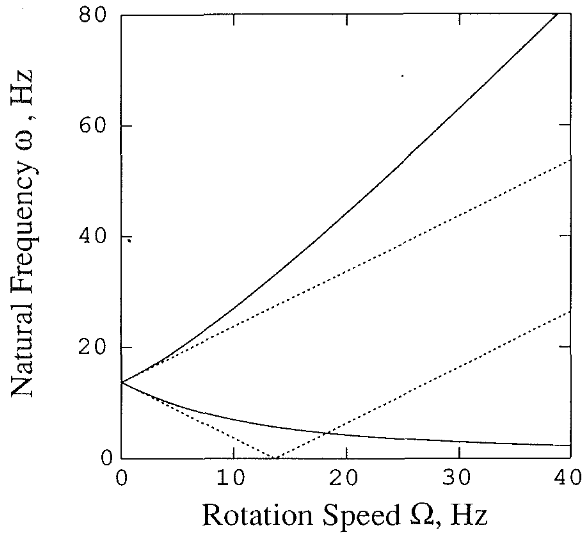

Fig. 6 Natural frequency versus rotation speed for an elastically supported rigid disk; k_Θ = 0.01 Nms/rad

$$\rho h(w_{,tt}+2\Omega w_{,t\theta}+\Omega^2 w_{,\theta\theta})+D\nabla^4 w+\rho hr[\ddot{\Theta}_x+2\Omega\dot{\Theta}_y-\Omega^2\Theta_x)\sin\theta$$
$$-(\ddot{\Theta}_y-2\Omega\dot{\Theta}_x-\Omega^2\Theta_y)\cos\theta]=\frac{1}{r}\,\delta(r-\xi)\delta(\theta)F_z$$
$$-\frac{1}{r^2}\,\delta(r-\xi)[M_\phi\delta(\theta)]_{,\theta} \quad (11)$$

$$(I+I_c)(\ddot{\Theta}_x+2\Omega\dot{\Theta}_y-\Omega^2\Theta_x)+\int_0^{2\pi}\int_a^b \rho h\sin\theta(w_{,tt}+2\Omega w_{,t\theta}$$
$$+\Omega^2 w_{,\theta\theta})r^2 dr d\theta=M_\phi\delta(r-\xi)\delta(\theta)-k_\Theta\Theta_x-c_\Theta\dot{\Theta}_x \quad (12)$$

$$(I+I_c)(\ddot{\Theta}_y-2\Omega\dot{\Theta}_x-\Omega^2\Theta_y)-\int_0^{2\pi}\int_a^b \rho h\cos\theta(w_{,tt}+2\Omega w_{,t\theta}$$
$$+\Omega^2 w_{,\theta\theta})r^2 dr d\theta=\xi F_z\delta(r-\xi)\delta(\theta)-k_\Theta\Theta_y-c_\Theta\dot{\Theta}_y. \quad (13)$$

After setting $k_\Theta = 0$, $c_\Theta = 0$, $F_z = 0$, $M_\phi = 0$, and substituting Eqs. (12) and (13) into Eq. (11), we obtain for the freely tilting disk in the absence of the load system

$$\rho h(w_{,tt}+2\Omega w_{,t\theta}+\Omega^2 w_{,\theta\theta})+D\nabla^4 w$$
$$-\frac{r\rho^2 h^2}{(I+I_c)}\left[\cos\theta\int_0^{2\pi}\int_a^b r^2\cos\theta(w_{,tt}+2\Omega w_{,t\theta}+\Omega^2 w_{,\theta\theta})dr d\theta\right.$$
$$\left.+\sin\theta\int_0^{2\pi}\int_a^b r^2\sin\theta(w_{,tt}+2\Omega w_{,t\theta}+\Omega^2 w_{,\theta\theta})dr d\theta\right]=0 \quad (14)$$

It is important to note that neglecting the membrane stresses σ_r and σ_θ in Eq. (5) does not result in Eq. (14) exactly, which contains the additional $w_{,\theta\theta}$ terms in the integrals. The natural frequency change for mode $(m, 1)$ (with respect to the tilting frame) due to rigid-body tilting can now be estimated as

$$\Delta\lambda_{m1}=\frac{i\pi\rho^2 h^2(\omega^0_{m1}\pm\Omega)\left[\int_a^b R^0_{m1}(r)rdr\right]^2}{2(I+I_c)\int_a^b [R^0_{m1}(r)]^2 rdr}\begin{cases}+\text{ for backward wave}\\ -\text{ for forward and reflected waves.}\end{cases} \quad (15)$$

It is noted that $R^0_{m1}(r)$ in Eq. (15) is independent of rotation speed Ω because no centrifugal force is involved. Furthermore, the term $\omega^0_{m1}\pm\Omega$ is a constant and is equal to the natural frequency when $\Omega = 0$ (Chen and Bogy, 1992a). Consequently, $\Delta\lambda_{m1}$ is a constant and independent of the rotation speed. Figure 5 shows the relation between the natural frequency of the stationary flexible disk as observed from a rotating frame and the rotation speed of the frame. The dashed lines and the solid lines are the results for the cases without and with rigid-body tilting, respectively. The dashed lines in Fig. 5 are the same as those in Fig. 4, as explained in (Chen and Bogy, 1992a). The solid lines for the (1, 1) modes are parallel to the dashed lines for the (0, 1) modes with a constant shifting upward. No instability is induced due to the addition of the rigid-body tilting.

A Special Case: Elastically Supported Rigid Disk

The equations of motion in the inertial frame of a spinning rigid disk with tilting supported by a torsional spring k_Θ are obtained from Eqs. (2) and (3) by setting $w = 0$,

$$I(\ddot{\Theta}_x+2\Omega\dot{\Theta}_y)+k_\Theta\Theta_x+c_\Theta\dot{\Theta}_x=M_\phi\delta(r-\xi)\delta(\theta) \quad (16)$$
$$I(\ddot{\Theta}_y-2\Omega\dot{\Theta}_x)+k_\Theta\Theta_y+c_\Theta\dot{\Theta}_y=\xi F_z\delta(r-\xi)\delta(\theta). \quad (17)$$

On the other hand, the equations of motion in the rotating frame for a stationary rigid-disk subjected to a rotating load system are derived from Eqs. (12) and (13),

$$I(\ddot{\Theta}_x+2\Omega\dot{\Theta}_y-\Omega^2)+k_\Theta\Theta_x+c_\Theta\dot{\Theta}_x=M_\phi\delta(r-\xi)\delta(\theta) \quad (18)$$
$$I(\ddot{\Theta}_y-2\Omega\dot{\Theta}_x-\Omega^2)+k_\Theta\Theta_y+c_\Theta\dot{\Theta}_y=\xi F_z\delta(r-\xi)\delta(\theta). \quad (19)$$

It is noted that Stahl and Iwan (1973a,b) used different Euler angles in writing the equations of motion for a stationary disk with rotating load system. As a result their equations are more complicated than Eqs. (18) and (19), although they are essentially the same. We find here that the problem of a spinning rigid disk with a stationary load system is not exactly equivalent to the problem of a stationary rigid disk with rotating load system when rigid-body tilting is included. The differences between Eqs. (16), (17) and Eqs. (18), (19) are the terms containing Ω^2. In the case without rigid-body tilting, the equations of these two problems are exactly the same. The dashed straight lines in Fig. 6 are the natural frequencies of the stationary rigid disk with rotating observer. The stiffness of the clamp spring k_Θ is 0.01 Nm/rad and the value of the clamp damping c_Θ is 0. The slopes of the forward and backward waves are 1 and -1, respectively. There is a critical speed, beyond which the backward wave becomes a reflected wave. The solid lines are the natural frequencies of a spinning rigid disk observed

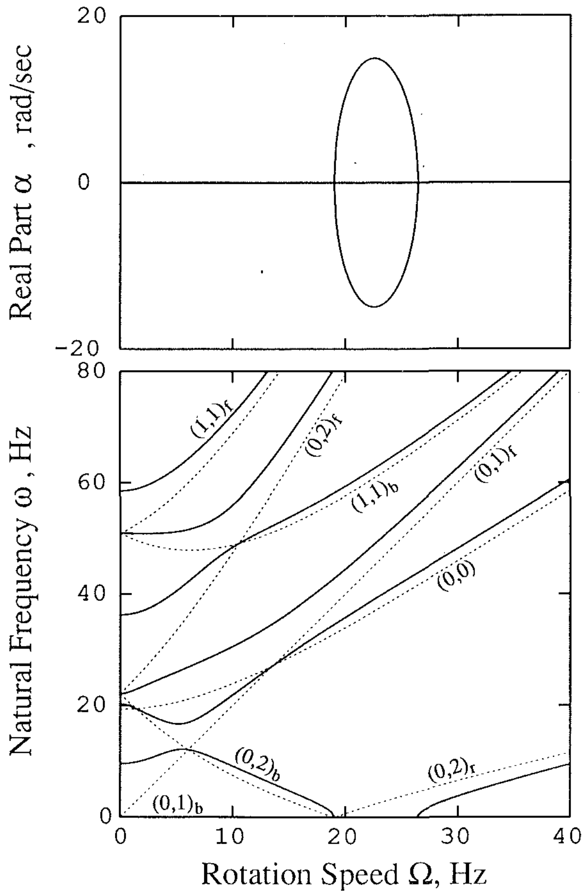

Fig. 7 Effects of transverse stiffness k_z = 5 N/m on the natural frequencies and stability of a flexible disk

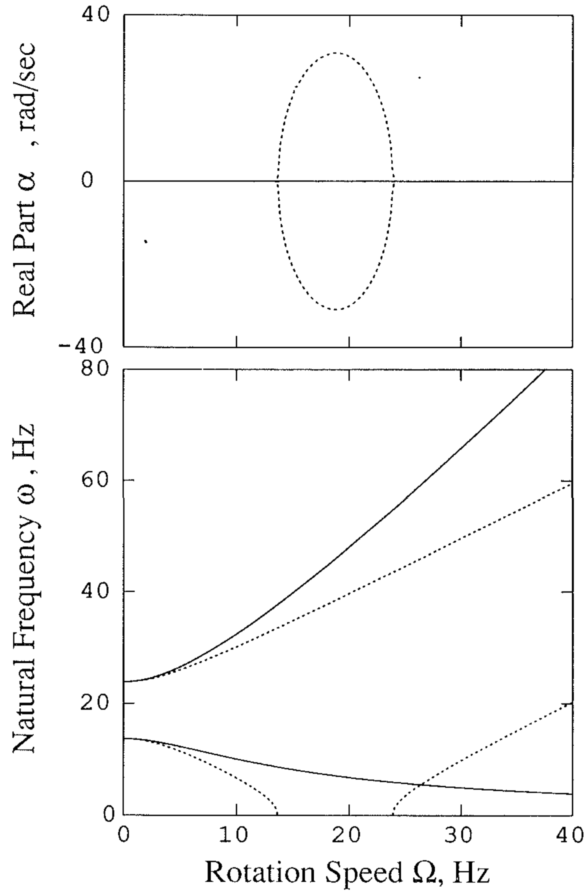

Fig. 8 Effects of transverse stiffness k_z = 5 N/m on the natural frequencies and stability of a rigid disk

by a stationary observer. The forward and backward frequency loci start with slope 1 and −1, respectively, at $\Omega = 0$. However, the slope of the forward wave approaches 2 as Ω increases. On the other hand, the natural frequency of the backward wave approaches zero asymptotically so there is no critical speed, and the backward wave will never become a reflected wave. These behaviors of the solid lines have been explained analytically by Arnold and Maunder (1961). The importance of this comparison between these two types of problems lies in the fact that instability will be induced when the rotating load system is loaded onto the stationary disk, as discussed by Stahl and Iwan (1973a,b). However, no instability will be induced when the stationary load system is loaded onto the spinning disk. An example will be given in the next section.

Effects of Load Parameters

We next consider the effects of the stationary load system spring with constant k_z on the natural frequencies and stability of the spinning flexible disk with rigid-body tilting. The clamp spring k_Θ and clamp damper c_Θ are set to 0. The dashed and the solid lines in Fig. 7 are the results for $k_z = 0$ and $k_z = 5$ N/m, respectively. By comparing the dashed lines and the solid lines, we can see that veering occurs whenever a forward or backward wave meets either another zero-nodal diameter mode, a forward or a backward wave. At the intersection point of any two curves for $k_z = 0$ at least one eigenvalue remains unchanged when k_z is introduced. When the eigenvalue of interest is well separated from the others the addition of k_z increases the natural frequency of a zero-nodal diameter mode, a forward or backward wave, but it decreases the natural frequency of a reflected wave. Furthermore, k_z causes a stationary-type instability (Ono et al., 1991) in a finite rotational speed range just beyond the critical speed. These properties are qualitatively the same as the effects of k_z in a conventional disk without rigid-body tilting (Chen and Bogy, 1992a,b). Unlike the case of a spinning flexible disk with axial translation (Chen and Bogy, 1992c), the additional rigid-body tilting mode does not eliminate the stationary-type instability induced by a load system transverse spring k_z.

Figure 8 shows the effects of $k_z = 5$ N/m on the elastically supported rigid disk, with the clamp spring $k_\Theta = 0.01$ Nm/rad. Similar to those in Fig. 6, the dashed lines are the results for a stationary disk with rotating load system, and the solid lines are the results for a spinning disk with stationary load system. By comparing with Fig. 6 we can see that for a stationary rigid disk k_z induces a stationary-type instability, as reported by Stahl and Iwan (1973a,b). On the other hand, for a spinning rigid disk k_z simply increases the natural frequencies of both the forward and backward waves, and induces no instability.

Figure 9 shows the effects of the load system mass m_z on the eigenvalues of a disk with rigid-body tilting. The dashed lines are the results for a freely spinning disk, and the solid lines are the results for $m_z = 1$ g. By comparing these results with those presented in Chen and Bogy (1992a,b) for the conventional case, we observe that the effects of m_z are qualitatively the same for both the cases, with and without rigid-body tilting. Additional calculations for the parameters c_z, k_ϕ, c_ϕ, and I_ϕ also show that the effects of these parameters are qualitatively the same for both the cases, with and without tilting. In particular, none of the pitching parameters have an effect on the zero nodal diameter modes. The dashed lines in Fig. 10 are for the freely spinning disk with rigid-body tilting and

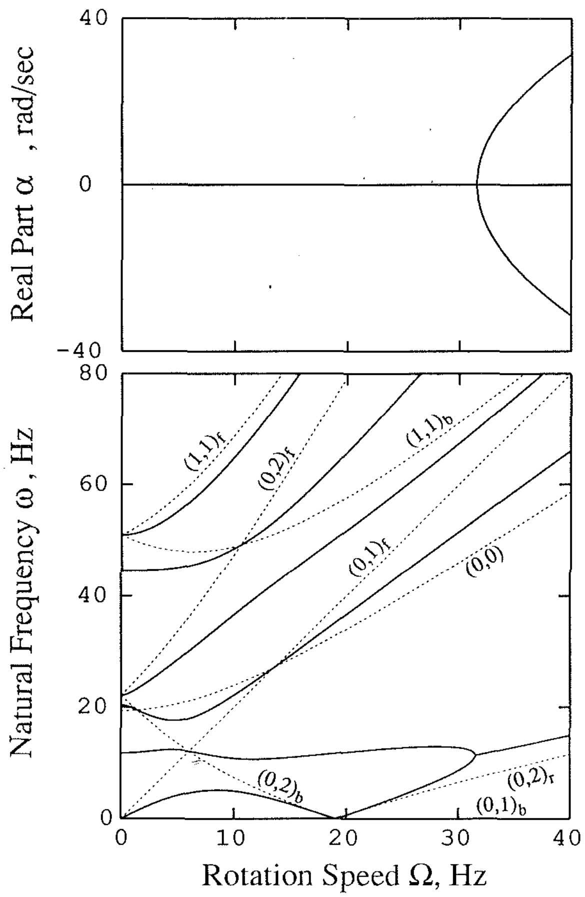

Fig. 9 Effects of transverse inertia m_z = 1 g

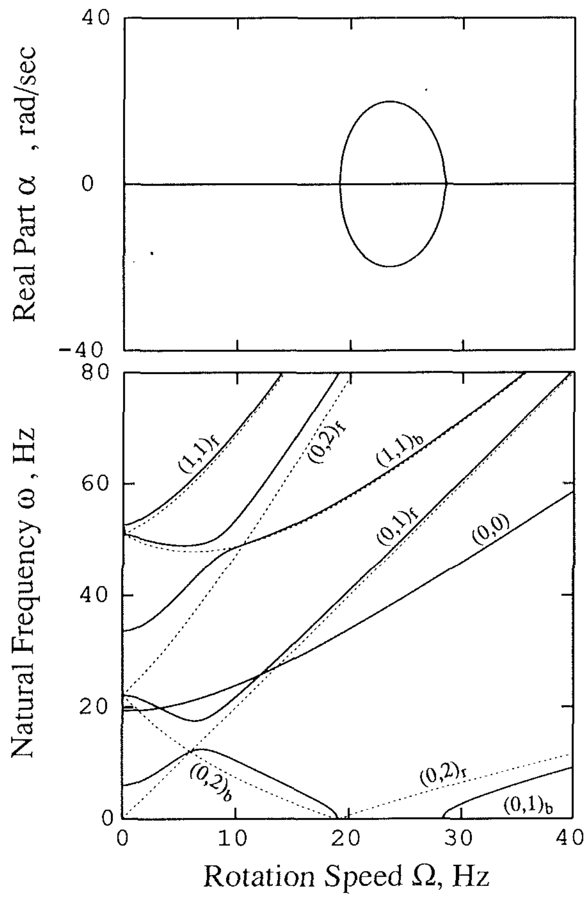

Fig. 10 Effects of pitching stiffness k_ϕ = 0.005 Nm/rad

the solid lines are for $k_\phi = 0.005$ Nm/rad. It is noted that rigid-body tilting does not eliminate the stationary-type instability induced by the pitching stiffness k_ϕ.

Derivatives of the Eigenvalues

In Chen and Bogy (1992a), we showed that for a freely spinning disk without rigid-body tilting, the eigenfunctions are orthogonal to each other. Also it was shown that expressions for the derivatives of the eigenvalues with respect to the various parameters in the stationary load system can be derived and used to predict the change of eigenvalues for small values of the parameters. In Chen and Bogy (1992c), we extended this analysis to the case when the axial translation is included, and showed that the equations of motion fall into the category of gyroscopic systems. In this section we further extend the analysis to the case when the rigid-body tilting is included.

By following the same procedure as in Chen and Bogy (1992c), we can rewrite Eqs. (1), (2), and (3) in the matrix form

$$(\mathbf{M}+\hat{\mathbf{M}})\mathbf{u}_{,tt}+(\mathbf{G}+\hat{\mathbf{G}})\mathbf{u}_{,t}+(\mathbf{K}+\hat{\mathbf{K}})\mathbf{u}=0 \tag{20}$$

where the vector $\mathbf{u}$ is defined as

$$\mathbf{u}=\begin{Bmatrix} w \\ \Theta_x \\ \Theta_y \end{Bmatrix}$$

The matrix operators $\mathbf{M}$, $\mathbf{G}$, and $\mathbf{K}$ are associated with the freely spinning disk, while $\hat{\mathbf{M}}$, $\hat{\mathbf{G}}$, and $\hat{\mathbf{K}}$ are associated with the stationary load system. It is noted that Θ_x and Θ_y are functions of time only while w is also a function of r and θ. The inner product between two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ is defined as

$$\langle \mathbf{u}_1, \mathbf{u}_2\rangle = \int_0^{2\pi}\int_a^b \overline{w}_1 w_2 r dr d\theta + \overline{\Theta}_{x1}\Theta_{x2}+\overline{\Theta}_{y1}\Theta_{y2}.$$

In the case of a freely spinning disk, Eq. (20) reduces to

$$\mathbf{M}\mathbf{u}_{,tt}+\mathbf{G}\mathbf{u}_{,t}+\mathbf{K}\mathbf{u}=0. \tag{21}$$

It can be shown that the matrix operators $\mathbf{M}$ and $\mathbf{K}$ are symmetric and $\mathbf{G}$ is skew, i.e.,

$$\langle \mathbf{M}\mathbf{u}_1, \mathbf{u}_2\rangle = \langle \mathbf{u}_1, \mathbf{M}\mathbf{u}_2\rangle$$

$$\langle \mathbf{K}\mathbf{u}_1, \mathbf{u}_2\rangle = \langle \mathbf{u}_1, \mathbf{K}\mathbf{u}_2\rangle$$

$$\langle \mathbf{G}\mathbf{u}_1, \mathbf{u}_2\rangle = -\langle \mathbf{u}_1, \mathbf{G}\mathbf{u}_2\rangle$$

Equation (20) can also be written in the first order form

$$(\mathbf{A}+\hat{\mathbf{A}})\mathbf{x}_{,t}-(\mathbf{B}+\hat{\mathbf{B}})\mathbf{x}=\mathbf{0} \tag{22}$$

by defining the state vector

$$\mathbf{x}\equiv\begin{Bmatrix} \mathbf{u}_{,t} \\ \mathbf{u} \end{Bmatrix}$$

and the matrix operators

$$\mathbf{A}\equiv\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}\end{bmatrix},\ \mathbf{B}\equiv\begin{bmatrix} -\mathbf{G} & -\mathbf{K} \\ \mathbf{K} & \mathbf{0}\end{bmatrix},$$

$$\hat{\mathbf{A}}\equiv\begin{bmatrix} \hat{\mathbf{M}} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{K}}\end{bmatrix},\ \hat{\mathbf{B}}\equiv\begin{bmatrix} -\hat{\mathbf{G}} & -\hat{\mathbf{K}} \\ \hat{\mathbf{K}} & \mathbf{0}\end{bmatrix}.$$

Now, the inner product between vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ is defined as

$$\langle \mathbf{x}_1, \mathbf{x}_2\rangle = \int_0^{2\pi}\int_a^b (\overline{w}_{1,t}w_{2,t}+\overline{w}_1 w_2) r dr d\theta$$

$$+\dot{\overline{\Theta}}_{x1}\dot{\Theta}_{x2}+\overline{\Theta}_{x1}\Theta_{x2}+\dot{\overline{\Theta}}_{y1}\dot{\Theta}_{y2}+\overline{\Theta}_{y1}\Theta_{y2}$$

Again, it can be shown that $\mathbf{A}$ is symmetric and $\mathbf{B}$ is skew.

After establishing the above definitions and following the same procedure as that presented in Chen and Bogy (1992a), we can readily show that the eigenfunctions $\mathbf{x}_{mn}$ and $\mathbf{x}_{pq}$ of a freely spinning disk with rigid-body tilting are orthogonal with respect to the operator $\mathbf{A}$, and that the derivative of eigenvalues with respect to a load parameter ρ_k is

$$\lambda_{mn,k} = -\frac{\langle \mathbf{x}_{mn},(\lambda_{mn}\hat{\mathbf{A}}_{,k}-\hat{\mathbf{B}}_{,k})\mathbf{x}_{mn}\rangle}{\langle \mathbf{x}_{mn},\ \mathbf{A}\mathbf{x}_{mn}\rangle} \tag{23}$$

where

$$\mathbf{x}_{mn} \equiv \begin{Bmatrix} i\omega_{mn}w_{mn} \\ i\omega_{mn}\Theta_{xmn} \\ i\omega_{mn}\Theta_{ymn} \\ w_{mn} \\ \Theta_{xmn} \\ \Theta_{ymn} \end{Bmatrix}.$$

By taking into account the fact that Θ_{xmn} and Θ_{ymn} are zero unless $n = 1$, a simple calculation (Chen and Bogy, 1992a) leads to

$$\langle \mathbf{x}_{mn},\ \mathbf{A}\mathbf{x}_{mn}\rangle = \begin{cases} 4\pi\omega_{mn}(\omega_{mn}\pm n\Omega)\displaystyle\int_a^b R_{mn}^2(r)rdr & n\neq \pm 1 \\ 4\pi\omega_{mn}(\omega_{mn}+n\Omega)\left[\displaystyle\int_a^b R_{mn}^2(r)rdr - \frac{4\left(\displaystyle\int_a^b R_{mn}(r)r^2dr\right)^2}{b^4-a^4}\right] & n=\pm 1 \end{cases}. \tag{24}$$

For $n \neq \pm 1$, the term $\langle \mathbf{x}_{mn}, \mathbf{A}\mathbf{x}_{mn}\rangle$ is positive for both forward and backward travelling waves, and is negative for the reflected wave, as explained in Chen and Bogy (1992a). At the critical speed, however, the term $\langle \mathbf{x}_{mn}, \mathbf{A}\mathbf{x}_{mn}\rangle$ vanishes because ω_{mn} becomes zero. For $n = \pm 1$ the term $\langle \mathbf{x}_{mn}, \mathbf{A}\mathbf{x}_{mn}\rangle$ can be shown to be non-negative by using the Schwartz inequality

$$\left(\int_0^{2\pi}\int_a^b \overline{w}_1 w_2 rdrd\theta\right)^2 \leq \int_0^{2\pi}\int_a^b \overline{w}_1 w_1 rdrd\theta \quad \int_0^{2\pi}\int_a^b \overline{w}_2 w_2 rdrd\theta$$

where we choose $w_1 = R_{m1}(r)e^{i\theta}$ and $w_2 = re^{i\theta}$.

With Eq. (23) we now can calculate the derivative of the eigenvalue with respect to the transverse stiffness parameter k_z. We first derive $\partial\hat{\mathbf{A}}/\partial k_z$ and $\partial\hat{\mathbf{B}}/\partial k_z$, and substitute the results into Eq. (23) to obtain

$$\frac{\partial\lambda_{mn}}{\partial k_z} = \frac{i\omega_{mn}[R_{mn}(\xi)-\xi\Theta_{ymn}]^2}{\rho h\langle \mathbf{x}_{mn},\ \mathbf{A}\mathbf{x}_{mn}\rangle} \tag{25}$$

Since $\Theta_{xmn} = 0$, $\Theta_{ymn} = 0$, and $R_{mn}(r) = R_{mn}^0(r)$ for $n \neq 1$, this expression is the same as that for the conventional disk without rigid-body tilting presented in Chen and Bogy (1992a) for the modes with other than one nodal diameter. This means that for $n \neq 1$ the effects of k_z on the eigenvalues are the same for both cases, with and without rigid-body tilting. For the one-nodal diameter modes, on the other hand, k_z increases the natural frequencies but by different amounts for the cases with and without rigid-body tilting.

By following similar procedures, we obtain the derivatives of the eigenvalues with respect to c_z and m_z by replacing $i\omega_{mn}$ in Eq. (25) by $-\omega_{mn}^2$ and $-i\omega_{mn}^3$, respectively, and similar conclusions follow accordingly. The derivatives with respect to the pitching parameters are the same as those of their transverse counterparts, except that there is a multiplicative factor n^2/ξ^2 in the numerators. This confirms the numerical results, which showed that pitching parameters have no effect on the zero-nodal diameter modes because $n = 0$.

Conclusions

The effects of rigid-body tilting on the dynamics of a flexible spinning disk in contact with a stationary load system are examined both numerically and analytically. The results of these analyses can be summarized as follows:

(1) To an observer moving with the tilting frame, the natural frequencies of one-nodal diameter modes increase due to the inertial coupling between the rigid-body tilting and the elastic deformation, while the natural frequencies of the modes with other than one-nodal diameter are not affected.

(2) In the conventional case without rigid-body tilting, neglecting the centrifugal force in the calculation will not affect the stability property of the freely spinning disk. However, in the present case with rigid-body tilting, the centrifugal force is very important in stabilizing the system.

(3) Another type of problem in which a stationary flexible disk is subject to a rotating load system is also considered. In the conventional case without rigid-body tilting, these two types of problems are mathematically identical except for the additional stress terms arising from the centrifugal force. In the case with rigid-body tilting, however, these two types of problem differ not only by the centrifugal effect, but also by the terms arising from the gyrodynamic effect.

(4) The rigid-body tilting does not eliminate any instability induced by any parameter in the stationary load system. It may be futile trying to improve the stability property of a spinning disk by allowing it to tilt.

(5) The system of equations of motion is again proven to be gyroscopic in the presence of rigid-body tilting. As a result, the techniques developed in previous papers (Chen and Bogy, 1992a,b,c) are extended to the current case to obtain the derivatives of eigenvalues with respect to various load system parameters. These derivatives provide a theoretical understanding for the numerical results obtained.

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