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Heap-ordered Trees, 2-Partitions and Continued Fractions

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This paper studies the enumerations and some interesting combinatorial properties of heap-ordered trees (HOTs). We first derive analytically the total numbers of n-node HOTs. We then show that there exists a 1–1 and onto correspondence between any two of the following four sets: the set of $(n+1)$-node HOTs, the set of 2-partitions of $\mathbf{Z}_{2n} = \{1, 2, \ldots, 2n\}$, the set of Young tableaux from $\mathbf{Z}_{2n}$ without odd-length columns, and the set of weighted paths of length $2n$. These correspondences can not only be used to obtain the above enumeration quantities through combinatorial arguments, but can also relate their generating functions to continued fractions.

1. INTRODUCTION

A *heap-ordered tree* (or *priority queue*) is an important data structure in computer applications. A heap-ordered tree (HOT) is defined recursively to be a rooted tree, the root of which contains the minimum key, and the subtrees of the root are all HOTs themselves.

Throughout this paper, all HOTs we consider will be *labeled*, in that two trees of the same 'shape' but with different labeling are considered to be distinct. In other words, the ordering of the subtrees with respect to the root is important. For an $(n+1)$-node HOT, we further assume that all the key values are distinctly chosen (labeled) from the set $\{0\} \cup \mathbf{Z}_n$, where $\mathbf{Z}_n = \{1, 2, \ldots, n\}$. It is easy to see from the definition of an HOT that the root of an HOT is always labeled with the minimum value 0.

This paper investigates the enumeration and some combinatorial properties of HOTs. The paper is organized as follows. In Section 2 we derive analytically b_n, the number of $(n+1)$-node HOTs using recurrence techniques. In Section 3, a bijection between the set of $(n+1)$-node HOTs and the set of 2-partitions of $\mathbf{Z}_{2n}$ is demonstrated. A 2-partition of $\mathbf{Z}_{2n}$ is a set of n pairs, where each element is uniquely selected from $\mathbf{Z}_{2n}$. One application of this correspondence is that b_n can be derived through combinatorial arguments. In addition, we show that the bijection also applies to the set of Young tableaux from $\mathbf{Z}_{2n}$ without odd-length columns. Finally, a bijection between the set of $(n+1)$-node HOTs and the set of weighted paths of length $2n$ is described. Moreover, the generating functions of b_n are shown to be related to continued fractions.

2. ENUMERATION OF HEAP-ORDERED TREES

Let b_n denote the numbe of $(n+1)$-node HOTs. If the leftist child of the root of an $(n+1)$-node HOT T is labeled i and the leftist subtree contains $(k+1)$ nodes, then there are $\binom{n-i}{k}$ ways to choose k nodes into the leftist subtree of T. We thus obtain the recurrence relation for b_n:

$$b_n = \begin{cases} \sum_{1 \le i \le n} \sum_{0 \le k \le n-i} \binom{n-i}{k} b_k b_{n-k-1}, & \text{if } n \ge 1; \\ 1, & \text{if } n = 0. \end{cases}$$

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Using the techniques described in [3], b_n can be simplied as

$$b_n = \begin{cases} \sum_{1 \le k \le n} \binom{n}{k} b_{k-1} b_{n-k}, & \text{if } n \ge 1; \\ 1, & \text{if } n = 0. \end{cases}$$

Let $B(x) = \sum_{n \ge 0} b_n x^n / n!$ be the exponential generating functions of b_n. The above equation gives

$$\sum_{n \ge 1} b_n x^n / n! = \sum_{n \ge 1} \sum_{1 \le k \le n} \binom{n}{k} b_{k-1} b_{n-k} x^n / n!$$
$$= \Big(\sum_{n \ge 0} b_n x^n / n!\Big)\Big(\sum_{n \ge 1} b_{n-1} x^n / n!\Big).$$

The left-hand side of the equation is $B(x) - 1$ and the right-hand side equals $B(x) \int_0^x B(t)\, dt$. Thus we have

$$B(x) - 1 = B(x) \int_0^x B(t)\, dt.$$

Solving the equation, we have

$$B(x) = (\sqrt{1 - 2x})^{-1}.$$

Expanding $B(x)$, we obtain $b_n = n! \binom{2n}{n} / 2^n$. This yields the following theorem.

THEOREM 1. *The total number of $(n+1)$-node HOTs equals*

$$b_n = \frac{n!}{2^n} \binom{2n}{n} \sim 2^{n+\frac{1}{2}} e^{-n} n^n (1 + O(n^{-1})), \qquad \textit{for } n \ge 1.$$

3. 2-Partitions and Young Tableaux

A 2-partition of $\mathbf{Z}_{2n}$ is a set of n pairs $\{(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)\}$, where p_i, $q_i \in \mathbf{Z}_{2n}$, $1 \le i \le n$, are all distinct. The algorithm below establishes a bijection Φ between the set of $(n+1)$-node HOTs and the set of 2-partition of $\mathbf{Z}_{2n}$. Given an $(n+1)$-node HOT T, this algorithm explicitly produces $\Phi(T)$, the corresponding 2-partition of $\mathbf{Z}_{2n}$:

Input: An HOT T with $(n+1)$ nodes labeled from $\{0, 1, \ldots, n\}$.
Output: A 2-partition $\Phi(T)$ of $\mathbf{Z}_{2n} = \{1, 2, \ldots, 2n\}$.
Algorithm:

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Let L = Z_2n and Π = ∅;
For j = n down to 1 do begin
    Let k = numbering(T, j);
    Delete node j from T;
    Let p be the minimum element in L and
        q be the (k + 1)st smallest element in L;
    Add (p, q) into Π and let L = L − {p, q};
end;
Φ(T) = Π.
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As for the function **numbering** used in the algorithm, an informal introduction may be helpful. For an HOT with $(k+1)$ nodes, we can inductively prove that there are

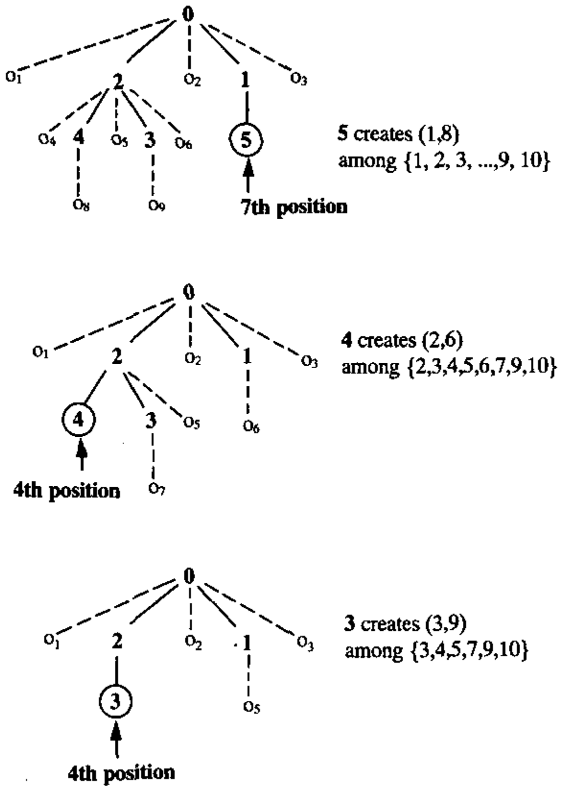

FIGURE 1. Constructing the 2-partition from a HOT.

$2k+1$ possible positions into which the new key $k+1$ can be inserted. For example, the first tree in Figure 1 contains five nodes (nodes labeled $0, 1, \ldots, 4$) before node 5 is inserted. The nine positions into which node 5 can be inserted are marked with small circle, '$\circ$'. Note that we number these positions 'level-wise' and from left to right. Actually, different numbering schemes may produce different 2-partitions for a specific HOT. The meaning of **numbering**(T, j) is indeed the position at which node j is inserted into the subtree of T containing nodes $0, 1, \ldots, (j-1)$ only.

Using the above numbering scheme, the 2-partition of $\mathbf{Z}_{10}$ corresponding to the HOT at the top of Figure 1 can be constructed as follows. As node 5 is inserted at the 7th position of the tree without 5, a pair $(1, 8)$ is thus created and added into Π. The set L then becomes $\{2, 3, 4, 5, 6, 7, 9, 10\}$. Node 4 is inserted at the 4th position of the tree without nodes 4 and 5 and thus the pair $(2, 6)$ is added to Π. The remaining three nodes $3, 2, 1$ are inserted at positions $4, 1, 0$ of their respective trees, and thus produce respectively the pairs $(3, 9)$, $(4, 5)$ and $(7, 10)$. As a result, we have $\{(1, 8), (2, 6), (3, 9), (4, 5), (7, 10)\}$ as the 2-partition corresponding to the HOT.

It is not difficult to show that the above transformation Φ which transforms an $(n+1)$-node HOT into a 2-partition of $\mathbf{Z}_{2n}$ is bijective. We thus have the following theorem.

THEOREM 2. *The number of 2-partitions of* $\mathbf{Z}_{2n}$ *equals* b_n, *the number of* $(n+1)$-*node HOTs.*

In fact, a 2-partition of $\mathbf{Z}_{2n}$ is a permutation on $\mathbf{Z}_{2n}$ containing 2-cycles only. The old theorem on *cycle indicator* by Polya [5] stated that the number of n-permutations of cycle type $(i_1, i_2, \ldots)$, where $i_1 + 2i_2 + \cdots = n$, equals $C_n(i_1, i_2, \ldots) = n!/i_1!\,1^{i_1}i_2!\,2^{i_2}\cdots$. This theorem, combined with Theorem 2, re-establishes the result in Theorem 1 that b_n equals $c_{2n}(0, n, 0, \ldots, 0) = (2n)!/n!\,2^n$.

The above bijection can be extended to an important combinatorial structure called a *Young tableau* using the Corollary on page 56 and Exercise 5.1.4–4 of [4].

COROLLARY 1. *There exists a bijection between the set of $(n+1)$-node HOTs and the set of Young tableaux from $\mathbf{Z}_{2n}$ that contains only even-length columns.*

4. CONTINUED FRACTIONS

We have derived in Section 2 that $b_n = (2n)!/n!\,2^n$ and obtained $B(x)$, the exponential generating function of b_n. In other words, the following two exponential generating functions have closed forms:

$$\sum_{n\geq 0} \frac{b_n x^{2n}}{(2n)!} = e^{x^2/2}, \qquad B(x) = \sum_{n\geq 0} \frac{b_n x^n}{n!} = (1-2x)^{-\frac{1}{2}}.$$

The closed forms for their ordinary counterparts $\hat{B}(x) = \sum_n b_n x^n$ and $\hat{B}(x^2) = \sum_n b_n x^{2n}$ are, however, difficult to obtain. The purpose of this section is to show that, instead, $\hat{B}(x)$ and $\hat{B}(x^2)$ can be elegantly expressed as continued fractions.

Let α_i, β_i and x be commutative indeterminates for $i \geq 0$. For integers $0 \leq m < n$, the *continued fraction of Jacobi* (J-fraction) is defined as

$$J_x[\alpha_k, \beta_k\colon (m,n)] = \frac{1}{1-\alpha_m x -}\,\frac{\beta_m x^2}{1-\alpha_{m+1}x -}\,\frac{\beta_{m+1}x^2}{1-\alpha_{m+2}x -}\cdots\frac{\beta_{n-1}x^2}{-1-\alpha_n x};$$

and the *continued fraction of Stieltjes* (S-fraction) is defined as

$$S_x[\beta_k\colon (m,n)] = \frac{1}{1-}\,\frac{\beta_m x}{1\ -}\,\frac{\beta_{m+1}x}{1\ -}\cdots\frac{\beta_{n-1}x}{-\ 1}\,\frac{\beta_n x}{-\ 1}.$$

The S-fractions are related to the J-fractions by an equation called a *contraction lemma* [2] (if we let $\beta_{-1} = 0$):

$$S_x[\beta_k\colon (0,\infty)] = J_x[\beta_{2k-1} + \beta_{2k}, \beta_{2k}\beta_{2k+1}\colon (0,\infty)].$$

The proposition below is a remarkable result by Rogers [6], which gives a convenient way to obtain the continued fraction representation of an ordinary generating function $\hat{G}(x) = \sum_{k\geq 0} a_k x^k$ from its exponential counterpart $G(x) = \sum_{k\geq 0} a_k x^k/k!$, provided that $G(x)$ has an addition formula. $G(x)$ is said to have an *addition formula with parameters* $\{(p_k, q_{k+1})\colon k \geq 0\}$ if $G(x+y) = \sum_{k\geq 0} p_k g_k(x) g_k(y)$, where p_k is independent of x and y and $g_k(x) = x^k/k! + q_{k+1}x^{k+1}/(k+1)! + O(x^{k+2})$.

PROPOSITION (the Stieltjes–Rogers J-fraction theorem). *The power series $G(x)$ has an addition formula with parameters $\{(p_k, q_{k+1})\colon k \geq 0\}$ iff*

$$\hat{G}(x) = J_x[q_{k+1} - q_k, p_{k+1}/p_k\colon (0,\infty)].$$

To obtain $\hat{B}(x) = \sum_k b_k x^k$, we shall show that $B(x) = (1-2x)^{-\frac{1}{2}}$ has an addition formula with parameters $\{((2k)!, (2k+1)(k+1)) : k \geq 0\}$. Since

$$\begin{aligned} B(x+y) &= (1-2x-2y)^{-\frac{1}{2}} \\ &= (1-2x)^{-\frac{1}{2}}(1-2y)^{-\frac{1}{2}}\left(1-\frac{4xy}{(1-2x)(1-2y)}\right)^{-\frac{1}{2}} \\ &= \sum_{k\geq 0}\binom{-1/2}{k}(-1)^k((2x)^k(1-2x)^{-(k+\frac{1}{2})})((2y)^k(1-2y)^{-(k+\frac{1}{2})}) \\ &= \sum_{k\geq 0}(2k)!\left(\frac{x^k}{k!}(1-2x)^{-(k+\frac{1}{2})}\right)\left(\frac{y^k}{k!}(1-2y)^{-(k+\frac{1}{2})}\right) \end{aligned}$$

and

$$\frac{x^k}{k!}(1-2x)^{-(k+\frac{1}{2})} = \frac{x^k}{k!} + (2k+1)(k+1)\frac{x^{k+1}}{(k+1)!} + O(x^{k+2}),$$

we have $p_k = (2k)!$ and $q_{k+1} = (2k+1)(k+1)$. This gives the J-fraction expression for $\hat{B}(x)$ in the theorem below. The S-fraction expression for $\hat{B}(x)$ is obtained by setting $\beta_k = k+1$.

THEOREM 3. *The two ordinary generating functions $\hat{B}(x)$ and $\hat{B}(x^2)$ can be expressed in continued fraction forms:*

$$\hat{B}(x) = J_x[4k+1, (2k+1)(2k+2) : (0, \infty)] = S_x[k+1 : (0, \infty)];$$

$$\hat{B}(x^2) = J_x[0, k+1 : (0, \infty)].$$

The continued fraction for $\hat{B}(x^2)$ in the theorem is derived from $\hat{B}(x)$:

$$\hat{B}(x^2) = S_{x^2}[k+1 : (0, \infty)] = J_x[0, k+1 : (0, \infty)].$$

5. Weighted Paths and Remarks

The above result in Section 4 on continued fractions can be re-interpreted in terms of *weighted paths* using the theory of set partitions, weighted paths and continued fractions developed by Flajolet, Françon and Viennot [1]. Indeed, their theory gives a method to construct a bijection between the set of weighted path $2n$ steps and the set of 2-partitions of $\mathbf{Z}_{2n}$ (and hence the set of $(n+1)$-node HOTs). Their theory also leads to Theorem 3, which gives the relation between the generating function of b_n and continued fractions.

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