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VIBRATION ANALYSIS OF A THREE-DIMENSIONAL RING GYROSCOPE

三維圓環型陀螺儀的振動分析

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摘　要

本文研究圓環的振動行為。此陀螺儀是由一圓環及其支撐結構所組成，而圓環是從 {111} 矽單晶蝕刻而成。我們將經適當設計系統的幾何參數來使圓環的平面內振動的自然頻率與平面垂直方向的自然振動頻率一致，如此可使平面節點振動的振幅與垂直面的節點振幅輸出的信號有相同的大小等級與解析度。並推導非旋轉圓環振動模態的正解，作為旋轉圓環解求析解時的特徵函數展開法所使用。經分析發現要有量測三個軸向角速率的功能，平面振動的模態數 n 與垂直面動的模態數 m 必須滿足 $n - m = \pm 1$ 的關係式。同時亦求得陀螺儀感測係數與系統參數的解析表示式。

關鍵詞： 微型陀螺儀、矽單晶、三軸角速率感測、振動自然頻率與模態。

Abstract

In this paper we study the vibration behavior of a micro-gyro, which consists of a ring and its supporting structure. This ring is made from {111} silicon wafer. The geometric parameters of the system are designed so that the in-plane natural frequencies of the ring is tuned to be equal to the out-of-plane ones; therefore the vibration amplitude of the in-pane node can be made of the same order as that of the out-of-plane node. We also derive the expression for the vibration mode shapes in the closed form. They can be used when we solve the problem of rotating ring by using the eigenfunction expansion method. It is found that the nth in-plane mode and the nth out-of-plane mode must satisfy the condition $n - m = \pm 1$ if the gyro is designated to be the three-axis one. The relationships between the sensing coefficients and the system parameters are obtained in the explicit form.

Keywords: micro-gyroscope, silicon single crystal, three-axis angular-rate sensing, natural frequency and mode shapes.

1. INTRODUCTION

Traditional vibrating gyroscopes such as Delco's hemispherical resonator gyroscope [1] made of fused quartz is of size in centimeter. It is of high accuracy, but very cost. And it can only measure single-axis angular rate. Due to the rapid development of MEMS technology many gyros of size in micrometer were designed. Typical examples include the vibrating nickel ring gyroscope [2] and the silicon ring one of British Aerospace [3]. The former use capacitors as actuators and sensors, while the latter uses electro-magnetic methods. Ayazi and Najafi [4] analyzed the same type of gyro as that of Putty [2] but with the ring made of polysilicon. There are also many other ring-type gyro patents [5,6] appeared recently with some modification in structures and fabrications. Juneau [7] showed that two-axis designs of ring-type gyros are possible.

Although there are many papers and patents talking about ring micro-gyros, most of them gave only the conceptual designs and lack rigorous analysis. In this paper the three-axis vibrating gyro is investigated. The equations of vibration are derived by using Hamilton' principle. The exact expressions for the natural frequencies of the non-rotating ring are obtained. The natural frequencies of the in-plane modes are tuned to those of the out-of-plane modes by adjusting the geometric parameters following the derived formula. When the ring is in rotating, it is found that the gyroscope effect can only occur when the nth in-plane mode and the mth out-of-plane mode satisfy the specific condition $n - m = \pm 1$. The sensing coefficients of the gyro are mportant parameters, the larger these sensing coefficients are, the better the resolution is. In this paper the relationship between the sensing coefficients and the system parameters are derived in the closed form.

2. FREQUENCY ANALYSIS

The top view of the ring gyro is shown in Fig. 1. The diameter, width, and thickness are denoted by a, h, and b, respectively. Let u, v, w be the radial, tangential, and out-of-plane displacement of the neutral line, U_r, U_θ, U_z are the corresponding displacements of any point on the ring as shown in Fig. 2. $\phi = (v - \partial u/\partial\theta)/a$ is the rotation angle about z-axis due to in-plane bending, $\phi_0 = \partial w/\partial\theta$ is the rotation angle about x-axis due to out-of-plane bending, and ϕ is the twist angle about the y-axis due to torsion.

We adopt the Euler's beam theory, the displacements [8] are given by

$$U_r = u(\theta,t) + z\phi(\theta,t)$$

$$\begin{aligned} U_\theta &= v(\theta,t) + r\phi_i - z\phi_0 \\ &= v(\theta,t) + \frac{r}{a}\left(v(\theta,t) - \frac{\partial u(\theta,t)}{\partial\theta}\right) - \frac{z}{a}\frac{\partial w(\theta,t)}{\partial\theta} \end{aligned} \tag{1}$$

$$U_z = w(\theta,t) - r\phi(\theta,t)$$

$$\phi_i = (v - \partial u/\partial\theta)/a\,, \qquad \phi_0 = \partial w/a\partial\theta$$

The strain-displacement relationships [9] are

$$\varepsilon_{\theta\theta} = \frac{v' + u}{a} + \frac{x}{a^2}(v' - u'') - \frac{z}{a}\left(\frac{w''}{a} - \phi\right)$$

$$\gamma_{r\theta} = \frac{z}{a}\left(\frac{w'}{a} + \phi'\right), \qquad \gamma_{\theta z} = -\frac{x}{a}\left(\frac{w'}{a} + \phi'\right) \tag{2}$$

The strain energy of the ring is

$$V_r = \int_0^{2\pi}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} \frac{a}{2}(E\varepsilon_{\theta\theta}^2 + G\gamma_{r\theta}^2 + G\gamma_{\theta z}^2)\, dr\cdot dz\cdot d\theta \tag{3}$$

where E is Young's Modulus, G is the shear modulus.

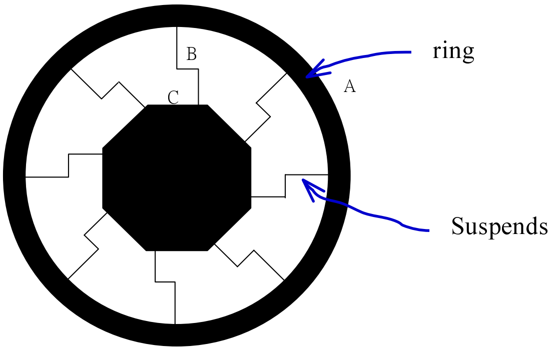

Fig. 1 The top view of the ring gyro

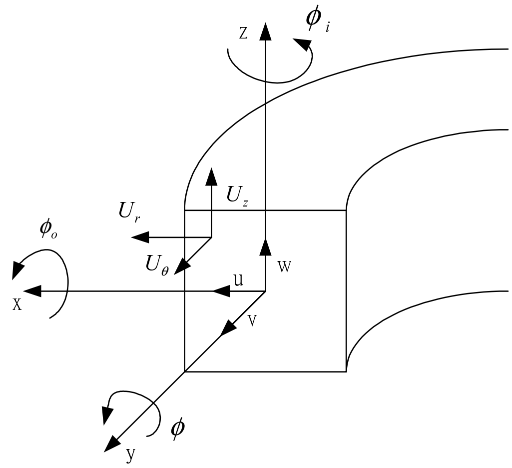

Fig. 2 Then displacements of a ring's cross-section

The position vector of any point P in the ring is

$$\vec{r}_p = (a + r + U_r)\,\vec{e}_r + U_\theta\,\vec{e}_\theta + (z + U_z)\,\vec{e}_z \tag{4}$$

Then the velocity of this point is

$$\vec{v}_p = \frac{\partial\vec{r}_p}{\partial t} = (\dot{u} + z\dot{\phi})\,\vec{e}_r + \left[\dot{v} + \frac{r}{a}(\dot{v} - \dot{u}') - \frac{z}{a}\dot{w}'\right]\vec{e}_\theta + (\dot{w} - r\dot{\phi})\,\vec{e}_z \tag{5}$$

The kinetic energy of the ring of mass density ρ is

$$T_r = \frac{\rho}{2}\int_0^{2\pi}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} a(\vec{v}_p\cdot\vec{v}_p)\, dr\, dz\, d\theta$$

$$= \frac{bh\rho}{24a}\int_0^{2\pi}\left[\begin{aligned} &12a^2\,\dot{u}^2 + (12a^2 + h^2)\,\dot{v}^2 + 12a^2\,\dot{w}^2 + a^2(b^2 + h^2)\,\dot{\phi}^2 \\ &-2h^2\,\dot{u}'\,\dot{v} + h^2\,(\dot{u}')^2 + b^2\,(\dot{w}')^2 \end{aligned}\right] d\theta \tag{6}$$

Using Hamilton's principle, $\int_{t_0}^{t_1}(\delta V_r - \delta T_r)\,dt = 0$, we get the equations of motion as

$$\ddot{u}+\frac{E}{a^2\rho}(u+v')+\frac{h^2}{12a^2}(\ddot{v}'-\ddot{u}'')+\frac{h^2E}{12a^4\rho}(u''''-v''')=0$$

$$\ddot{v}-\frac{12E}{12a^2\rho+h^2\rho}u'-\frac{h^2}{12a^2+h^2}\ddot{u}'-\frac{E}{a^2\rho}v''$$

$$+\frac{h^2E}{12a^4\rho+a^2h^2\rho}u'''=0 \tag{7}$$

$$\ddot{w}-\frac{G}{12a^4\rho}w''-\frac{(b^2G+h^2G+b^2E)}{12a^3\rho}\phi''-\frac{b^2}{12a^2}\ddot{w}''$$

$$+\frac{b^2E}{12a^4\rho}w''''=0$$

$$\ddot{\phi}+\frac{b^2E}{a^2(b^2+h^2)\rho}\phi-\frac{(b^2G+h^2G+b^2E)}{a^3(b^2+h^2)\rho}w''-\frac{G}{a^2\rho}\phi''=0$$

Since the volume of the suspends is about one thousandth of that of the ring, therefore, the effect of the suspends can be neglected when performing frequency adjustment to reconcile the in-plane natural frequency to that of the out-of-plane one.

We assume the solution in the form

$$u=A\,e^{j(\omega_1 t+n\theta)},\qquad v=B\,e^{j(\omega_1 t+n\theta)}$$

$$w=C\,e^{j(\omega_2 t+m\theta)},\qquad \phi=D\,e^{j(\omega_2 t+m\theta)} \tag{8}$$

Substituting (8) into (7) gives

$$\begin{bmatrix} M11 & M12 & 0 & 0 \\ M21 & M22 & 0 & 0 \\ 0 & 0 & M33 & M34 \\ 0 & 0 & M43 & M44 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}=0 \tag{9}$$

The non-trivial solutions of A, B, C, and D in Eg. (9) require that:

$$\det\begin{bmatrix} M11 & M12 \\ M21 & M22 \end{bmatrix}=0\text{, and }\det\begin{bmatrix} M33 & M34 \\ M43 & M44 \end{bmatrix}=0$$

Solving them we get the frequencies as

$$\omega_{11n}^2=\frac{E(12a^2(n^2+1)+h^2(2n^2+1)(n^2-1)}{2a^2[12a^2+h^2(n^2+1)]\,\rho}-\frac{\sqrt{(12a^2+h^2)[h^2(3n^2-1)^2+12a^2(n^2+1)^2])}}{2a^2[12a^2+h^2(n^2+1)]\,\rho} \tag{10a}$$

$$\omega_{12n}^2=\frac{E(12a^2(n^2+1)+h^2(2n^2+1)(n^2-1)}{2a^2[12a^2+h^2(n^2+1)]\,\rho}+\frac{\sqrt{(12a^2+h^2)[h^2(3n^2-1)^2+12a^2(n^2+1)^2])}}{2a^2[12a^2+h^2(n^2+1)]\,\rho} \tag{10b}$$

where ω_{11n} and ω_{12n} are the natural frequencies of the in-plane nth vibration modes. The difference is that ω_{11n} is the frequency of inextensible mode, while ω_{12n} is that of the extensible mode. Similarly, the frequencies of the out-of-plane mth modes are

$$\omega_{21m}=\omega_{21m}(E,G,a,h,b,m)$$

$$\omega_{22m}=\omega_{21m}(E,G,a,h,b,m) \tag{11}$$

The amplitude ratios in (8) are

$$\frac{B}{A}=j\frac{(12a^2+h^2n^4)E-a^2(12a^2+h^2n^2)\rho\omega_1^2}{n(12a^2E+h^2n^2E-a^2h^2\rho\omega_1^2)} \tag{12a}$$

$$\frac{D}{C}=\frac{-m^2[(b^2+h^2)G+b^2m^2E]+a^2(12a^2+b^2m^2)\rho\omega_2^2}{am^2[(b^2+h^2)\,G+b^2E]} \tag{12b}$$

If we choose the lowest two modes and assume that b, h << a, Eg. (7) reduces to

$$\frac{B}{A}\approx j\frac{1}{n},\quad \frac{D}{C}\approx-\frac{m^2[(b^2+h^2)G+h^2E]}{a\,[(b^2+h^2)m^2G+h^2E]} \tag{13}$$

The data of silicon wafer are E = 165GPa, G = 67.6GPa, ρ = 2330kg/m^3, a = 4000μm, h = 100μm, b = 100μm. In order to reconcile the in-plane and out-of-plane frequencies, that is, $\omega_{12n}=\omega_{22m}$ for (n, m) = (2, 3), we find that the geometric parameters are restricted to satisfy the equation

$$b=0.34\,h \tag{14}$$

and the radius has no effect on this reconciliation as shown in the Table 1.

3. ANALYSIS OF RING'S GYROSCOPES

In this section we consider the effect of Corioslis force due to the angular velocity input. We are going to use the Lagrange's equation to beam equations of vibration. The eight supporting beam are included. Their strained energy is evaluated exactly, but their kinetic energy is approximated by considering their velocity as one half of the ring at the contact point. The sensing coefficients are derived, which are the key parameters that affect the performance of the gyros.

From Eqs. (8) and (13) the displacements of the neutral line of the ring can be expressed as

Table 1 The values of b versus different values of a and h for the condition $\omega_{11n} = \omega_{2m}$ with $(n, m) = (2, 3)$

h (mm) \ a (mm)	50	90	100	200	300	400	500
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.1	0.034	0.034	0.034	0.034	0.034	0.034	0.034
0.2	0.068	0.068	0.068	0.068	0.068	0.068	0.068
0.3	0.102	0.102	0.102	0.102	0.102	0.102	0.102
0.4	0.136	0.136	0.136	0.136	0.136	0.136	0.136
0.5	0.170	0.170	0.170	0.170	0.170	0.170	0.170
0.6	0.204	0.204	0.204	0.204	0.204	0.204	0.204
0.7	0.238	0.238	0.238	0.238	0.238	0.238	0.238
0.8	0.272	0.272	0.272	0.272	0.272	0.272	0.272
0.9	0.306	0.306	0.306	0.306	0.306	0.306	0.306
1.0	0.340	0.340	0.340	0.340	0.340	0.340	0.340
1.1	0.374	0.374	0.374	0.374	0.374	0.374	0.374
1.2	0.408	0.408	0.408	0.408	0.408	0.408	0.408
1.3	0.442	0.442	0.442	0.442	0.442	0.442	0.442
1.4	0.476	0.477	0.477	0.477	0.477	0.477	0.477
1.5	0.510	0.511	0.511	0.511	0.511	0.511	0.511
1.6	0.545	0.545	0.545	0.545	0.545	0.545	0.545
1.7	0.579	0.579	0.579	0.579	0.579	0.579	0.579
1.8	0.613	0.613	0.613	0.613	0.613	0.613	0.613
1.9	0.647	0.647	0.647	0.647	0.647	0.647	0.647
2.0	0.681	0.681	0.681	0.681	0.681	0.681	0.681
2.1	0.715	0.715	0.715	0.715	0.715	0.715	0.715
2.2	0.749	0.749	0.749	0.749	0.749	0.749	0.749
2.3	0.783	0.783	0.783	0.783	0.783	0.783	0.783
2.4	0.817	0.817	0.817	0.817	0.817	0.817	0.817
2.5	0.851	0.851	0.851	0.851	0.851	0.851	0.851

$$u = X_1(t)\cos(n\theta) + X_2(t)\sin(n\theta) \tag{15a}$$

$$v = -\frac{1}{n}[X_1(t)\sin(n\theta) - X_2(t)\cos(n\theta)] \tag{15b}$$

$$w = X_3(t)\cos(m\theta) + X_4(t)\sin(m\theta) \tag{15c}$$

$$\phi = \frac{-m^2[(b^2+h^2)\,G + b^2m^2E] + a^2(12a^2+b^2m^2)\,\rho\omega_2^2}{am^2[(b^2+h^2)\,G + b^2E]} \cdot [X_3(t)\cos(m\theta) + X_4(t)\sin(m\theta)]$$

where $X_1(t)$, $X_2(t)$, $X_3(t)$ and $X_4(t)$ are generalized coordinates.

3.1 Energy of the Ring

The strain energy of the ring is the same as that of the non-rotating one, but the kinetic energy must be modified due to the inclusion of the rotation as shown in Fig. 3. The angular velocity in polar coordinates is

$$\vec{\Omega} = (\Omega_x \cos\theta + \Omega_y \sin\theta)\,\vec{e}_r + (-\Omega_x \sin\theta + \Omega_y \cos\theta)\,\vec{e}_\theta + \Omega_z\,\vec{e}_z \tag{16}$$

The kinetic energy is

$$T_r = \frac{\rho}{2}\int_0^{2\pi}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}(\vec{v}_p \cdot \vec{v}_p)\,a\,dr\,dz\,d\theta \tag{17}$$

where $\vec{v}_p = \partial\vec{r}_p/\partial t + \vec{\Omega}\times\vec{r}_p$. It can be carried out in the form

$$T_r = \int_0^{2\pi} \tau\,d\theta \tag{18}$$

where

$$\tau = \tau_c + \tau_x\,\Omega_x + \tau_y\,\Omega_y + \tau_z\,\Omega_z \tag{19a}$$

and

$$\tau_x = \tau_{xs}\sin\theta + \tau_{xc}\cos\theta \tag{19b}$$

The term τ_{xs} has the form

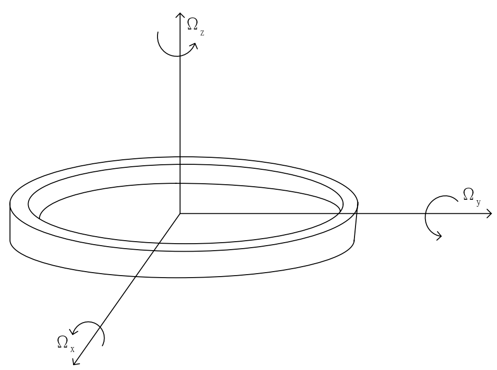

Fig. 3 Coordinate system and angular rates

$$\begin{aligned}\tau_{xs} &= \frac{abh\rho}{2}\{[-X_3[\cos((m-n)\theta+\cos\theta((m+n)\theta)]-X_4[\sin((m-n)\theta)+\sin((m+n)\theta)]]\,\dot{X}_1 \\ &+\left[X_3\left[\sin((m+n)\theta)-\sin((m-n)\theta)\right]+X_4[\cos((m-n)\theta)-\cos(m+n)\theta)]\right]\dot{X}_2 \\ &+\left[X_1\left[\cos((m-n)\theta)+\cos((m+n)\theta)\right]-X_2\left[\sin((m-n)\theta)-\sin((m+n)\theta)\right]\right]\dot{X}_3 \\ &+\left[X_1[\sin((m-n)\theta)+\sin((m+n)\theta)]+X_2[\cos((m-n)\theta)-\cos((m+n)\theta)]\right]\dot{X}_4\} \\ &+\frac{bh\rho\,[(12a^2+b^2)(b^2+h^2)\,m^2G+b^2(12a^2+b^2\,m^2)E]}{12\,[(b^2+h^2)\,m^2G+b^2E]}[\dot{X}_3\cos(m\theta)+\dot{X}_4\sin(m\theta)]\end{aligned} \tag{20}$$

The other terms like τ_{xc}, τ_y, and τ_z have the similar form as that of τ_{xs}. Consider the case where the input is the angular rate Ω_x (*i.e.*, $\Omega_x \neq 0$), and $\Omega_y = \Omega_z = 0$, then Eq. (19a) becomes

$$\tau = \tau_c + \tau_x\,\Omega_x \tag{21}$$

and if we excite the ring into the in-plane motion at $\theta = 0°$ with frequency ω_{112}, from Eq. (15a) we have $u(\theta = 0°, t) = X_1(t) \neq 0$. The Coriolis force will induce the out-of-plane vibration, thus it must be that $X_3(t) \neq 0$ and $X_4(t) \neq 0$ as can be seen from Eq. (15c). This will require that the integration $\int_0^{2\pi}\tau_{xs}\sin\theta d\theta$ will result in the non-zero terms of $\dot{X}_3(t)$ and $\dot{X}_4(t)$. Consider one term of $\dot{X}_3(t)$ like

$$X_2\int_0^{2\pi}\sin\theta\,\sin(m-n)\theta\;d\theta$$

which is non-zero only if

$$m-n=\pm 1 \tag{22}$$

Similarly, it is the same condition for non-zero term of $\dot{X}_4(t)$. It is also true for any other angular-rate input. So Eg. (22) is the condition for the gyro which can measure the three components of the angular velocity.

3.2 Strain energy of the suspensions

The structure of the suspensions of the ring is shown in Fig. 4.

The bending moments at different section of the suspension (shown in Fig. 5) are

$$\begin{aligned}&M_1 = M+Qr\,,\quad M_2 = M+QL_1-Rs \\ &M_3 = M+Q(L_1+t)-RL_2\end{aligned} \tag{23}$$

The total in-plane strain energy is

$$U_{si} = \frac{1}{2EI_x}\left(\int_0^{L_1}M_1^2\,dr+\int_0^{L_2}M_2^2\,ds+\int_0^{L_3}M_3^2\,dt\right) = \frac{1}{6EI_x}\left[\begin{matrix}3L_1\,M^2+3L_1^2\,MQ+L_1^3\,Q^2+\dfrac{(M+L_1Q)^3}{R} \\ -\dfrac{(M+L_1Q-L_2R)^3}{Q}-\dfrac{(M+L_1Q-L_2R)^3}{R} \\ +\dfrac{(M+(L_1+L_3)Q-L_2R)^3}{Q}\end{matrix}\right] \tag{24}$$

The use of Castigliano theorem

$$u_s(\theta,t)=\frac{\partial U_{si}}{\partial R}\,,\quad v_s(\theta,t)=\frac{\partial U_{si}}{\partial Q}\,,\quad \phi_{is}(\theta,t)=\frac{\partial U_{si}}{\partial M} \tag{25}$$

gives the relationship between the forces and bending moments and the displacements as

$$\begin{Bmatrix}R\\Q\\M\end{Bmatrix}=\begin{bmatrix}K_{si11}&K_{si12}&K_{si13}\\K_{si12}&K_{si22}&K_{si23}\\K_{si13}&K_{si23}&K_{si33}\end{bmatrix}\begin{Bmatrix}u_s(\theta,t)\\v_s(\theta,t)\\\phi_{is}(\theta,t)\end{Bmatrix} \tag{26}$$

where one nonzero-entry of the **K** matrix is

$$K_{si12}=\frac{3bd^3\,E(2L_1\,L_3^2+L_2\,L_3^2+L_1^2(L_2+2L_3))}{4L_2^2\,(L_1+L_2+L_3)(L_1^2-L_1\,L_3+L_3^2)(L_2\,L_3+L_1\,(L_2+3L_3))}$$

In general, $K_{sikl} = K_{sikl}$ (E, G, a, h, b, L_1, L_2, L_3). Substituting (26) into (24) yields the total in-plane strain energy in terms of generalized coordinates as

$$V_{si} = k_{ni}(X_1^2(t)+X_2^2(t)) \tag{27a}$$

where

$$k_{ni} = 4K_{si11}+K_{si22}+\frac{3(K_{si23}+K_{si32})}{a}+\frac{9K_{si33}}{a^2} \tag{27b}$$

Similarly, the total out-of-plane strain energy of the eight suspensions in terms of generalized coordinates is

$$V_{si} = k_{no}(X_3^2(t)+X_4^2(t)) \tag{28}$$

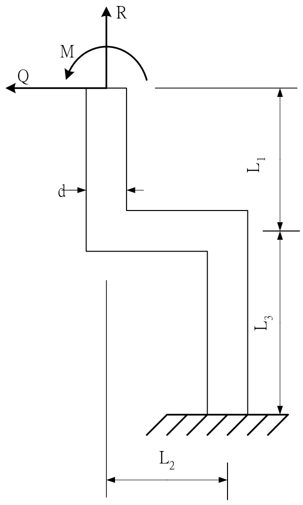

Fig. 4 The geometry of the suspension

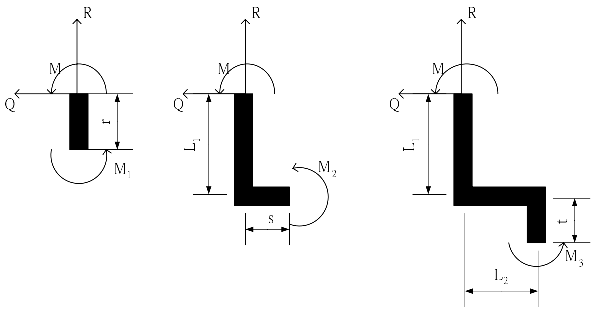

Fig. 5 Forces and bending moment diagram of the suspension

Let $\mathbf{x}$ = $(X_1(t),\ X_2(t),\ X_3(t),\ X_4(t))^T$ and we apply the Lagrange's equations to derive equations of vibration

$$\ddot{x}+\begin{bmatrix} 0 & -\lambda_1\,\Omega_z & \lambda_2\,\Omega_y & -\lambda_2\,\Omega_x \\ \lambda_1\,\Omega_z & 0 & \lambda_2\,\Omega_x & \lambda_2\,\Omega_y \\ -\lambda_3\,\Omega_y & -\lambda_3\,\Omega_x & 0 & -\lambda_4\,\Omega_z \\ \lambda_3\,\Omega_x & -\lambda_3\,\Omega_y & \lambda_4\,\Omega_z & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} \xi_1\,\omega_1 & 0 & 0 & 0 \\ 0 & \xi_1\,\omega_1 & 0 & 0 \\ 0 & 0 & \xi_2\,\omega_2 & 0 \\ 0 & 0 & 0 & \xi_2\,\omega_2 \end{bmatrix}\dot{x} + \begin{bmatrix} \omega_1^2 & 0 & 0 & 0 \\ 0 & \omega_1^2 & 0 & 0 \\ 0 & 0 & \omega_2^2 & 0 \\ 0 & 0 & 0 & \omega_2^2 \end{bmatrix} x = 0 \tag{29}$$

where λ_1, $\cdots$, λ_4 are the important parameters of the gyros and are called sensing coefficients. The larger the λ's are, the more sensitive the gyro will be and the better the resolution is. With the non-dimensional quantities $\gamma = h/a$, $\eta = b/a$, $\kappa = \mathcal{V}/a^3$ where $\mathcal{V}$ is the volume of a single suspension, the explicit form of λ_i's is

$$\lambda_1 = \frac{48(2\pi\gamma\eta + 2\kappa)}{60\pi\gamma\eta + 9\pi\gamma^3\eta + 60\kappa} \tag{30}$$

$$\lambda_2 = \frac{2}{5}\left(1+\frac{6\pi\gamma^3\eta\,(7E\eta^2+3G(\gamma^2+\eta^2))}{(E\eta^2+9G(\gamma^2+\eta^2))(\pi\gamma\eta(20+3\gamma^2)+20\kappa)}\right) \tag{31}$$

$$\lambda_3 = \lambda_{3N}/\lambda_{3D} \tag{32}$$

where

$$\lambda_{3N} = [E\eta^2+9G(\gamma^2+\eta^2)]\,\{9G(\gamma^2+\eta^2)\cdot[\pi\gamma\eta(4+\gamma^2)+4\kappa]+E\eta^2(\pi\gamma\eta(4+9\gamma^2)+4\kappa)\}$$

and

$$\begin{aligned}\lambda_{3D} &= 2\{18GE\eta^2(\gamma^2+\eta^2)[\pi\gamma\eta(4+3\gamma^2+6\eta^2)+4\kappa^2]\} \\ &+\{E^2\,\eta^4(\pi\gamma\eta(4+27\gamma^2+30\eta^2)+4\kappa^2)\} \\ &+\{27G^2(\gamma^2+\eta^2)^2(\pi\gamma\eta(12+\gamma^2+10\eta^2)+12\kappa)\}\end{aligned}$$

$$\lambda_4 = \lambda_{4N}/\lambda_{4D} \tag{33}$$

where

$$\lambda_{4N} = 36\pi\gamma\eta^3[G\gamma^2+(G+E))\eta^2][E\eta^2+9G(\gamma^2+\eta^2)]$$

$$\begin{aligned}\lambda_{4D} &= 18GE\eta^2(\gamma^2+\eta^2)[\pi\gamma\eta(4+3\gamma^2+6\eta^2)+4\kappa] \\ &+E^2\eta^4[\pi\gamma\eta\,(4+27\gamma^2+30\eta^2)+4\kappa] \\ &+27G^2(\gamma^2+\eta^2)^2[\pi\gamma\eta\,(12+\gamma^2+10\eta^2)+12\kappa]\end{aligned}$$

Now consider the case where $(n, m) = (2, 3)$, the mode shapes are shown in Fig. 6. We show that the magnitude of X_2 is mainly a measurement of Ω_z. Let $\Omega_x = \Omega_y = 0$ and $\omega_1 = \omega_2$. Assume harmonic excitation so that $X_1(t) = X_{10}\sin\omega_1 t$, the initial conditions are chosen as $X_2(0) = X_3(0) = X_4(0) = 0$, and $\dot{X}_2(0)=0$, $\dot{X}_3(0)=\dot{X}_4(0)=0$. Then the second one of Eq. (29) becomes

$$\ddot{X}_2+\lambda_1\,\Omega_z\,\dot{X}_1+\xi_1\,\omega_1\,\dot{X}_2+\omega_1^2\,X_2=0 \tag{34a}$$

The third and fourth ones of Eq. (29) are

$$\begin{cases}\ddot{X}_3-\lambda_4\,\Omega_z\,\dot{X}_4+\xi_2\,\omega_1\dot{X}_3+\omega_1^2\,X_3=0 & \text{(34b)}\\ \ddot{X}_4+\lambda_4\,\Omega_z\,\dot{X}_3+\xi_2\,\omega_1\dot{X}_4+\omega_1^2\,X_4=0 & \text{(34c)}\end{cases}$$

Equation (34a) tell us that the in-plane vibration $X_2(t)$ is induced by the Coriolis force due to Ω_z, but Eq. (34b), (34c) show that the out-of-plane vibration $X_3(t)$ and

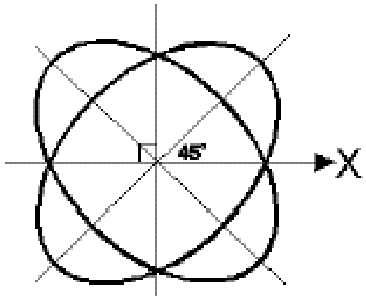

Fig. 6(a) cos2θ in-plane mode shape

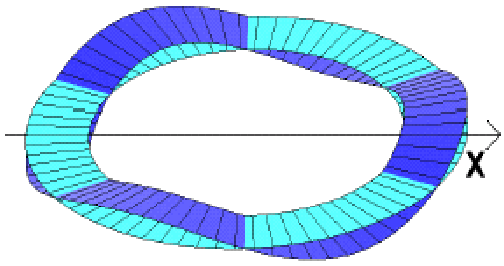

Fig. 6(b) cos3θ out-of-plane mode shape

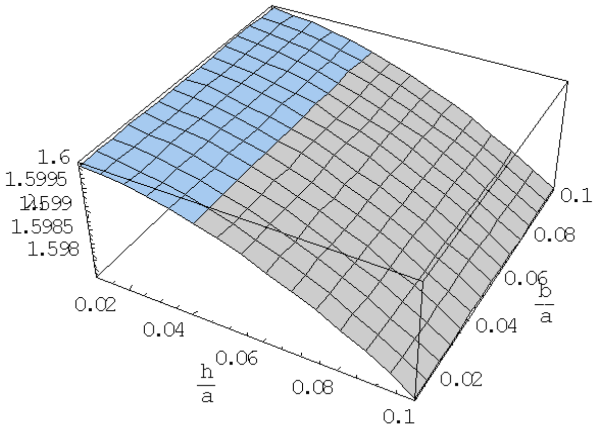

Fig. 7 The plot of λ_1 versus *h*/*a* and *b*/*a*

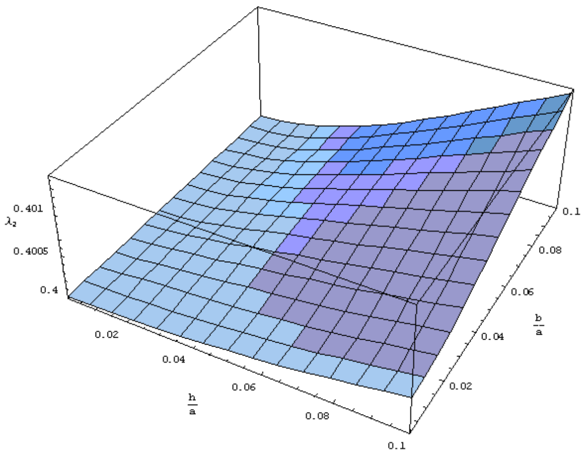

Fig. 8 The plot of λ_2 versus *h*/*a* and *b*/*a*

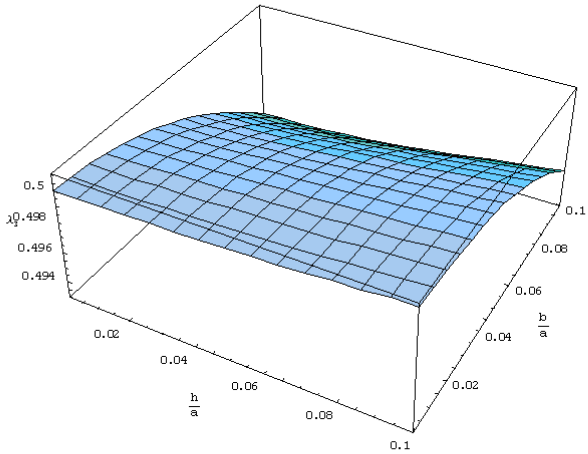

Fig. 9 The plot of λ_3 versus *h*/*a* and *b*/*a*

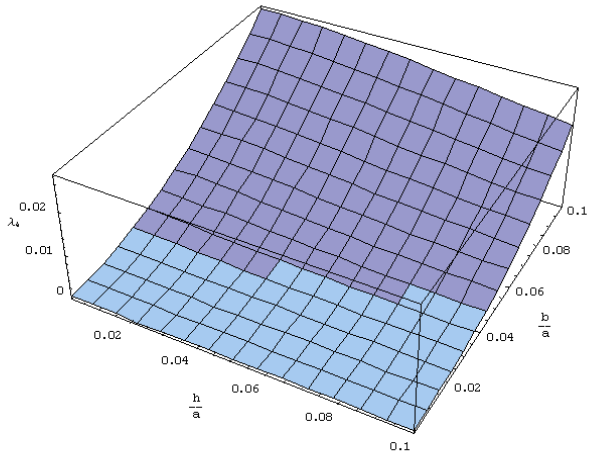

Fig. 10 The plot of λ_4 versus *h*/*a* and *b*/*a*

$X_4(t)$ can not be elicited, since the initial conditions for X_3 and X_4 are all zero.Similarly, it is easy to justify that X_3 can be used to measure manly Ω_y, and X_4 to measure Ω_x.

The effects of geometric parameters on the sensing coefficients λ_i's are shown in Figs. 7 to 10. They reveal that λ_1 decreases as h increases, λ_2 grows as both h and b increases, λ_3 raises when h increases and b decreases, λ_4 increases when h decreases and b increases. It is also found through calculation using Eqs. (30)~(33) that whatever the volume parameter κ varies in the range from 10^{-3} to 10^{-4}, it affects the sensing coefficients very small, not larger than one thousandth. So in the beginning of analysis we have make the approximate assumption that the effect of the suspensions is neglected when evaluating the natural frequencies. In practical application the small deviation of the excited frequency from the natural frequency can be controlled by frequency-locked electric circuit.

4. CONCLUSIONS

In this paper we derive the explicit expressions of the natural frequencies by solving the equations of motion. By reconciling the in-plane frequency to that of the out-of-plane one both the output signals of the in-plane and out-of-plane vibration amplitude can be made of the same order, which will enable the gyro to measure the three-axis angular rates. We also find that the *n*th in-plane mode and the *m*th out-of-plane mode must meet the condition $m - n = \pm 1$, otherwise, the out-of-plane mode can not be driven out by Coriolis force. The material properties and the geometric size have significant effects on the gyro's sensing coefficients, their relationships are also obtained in the explicit and graphic form.

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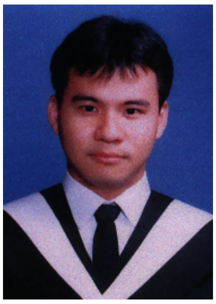

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