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在多期模型下或有求償權的 Super replicating 投資組合

(1/2)

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Boyle and Vorst (1992) work in the framework of the binomial model and derive replicating portfolios for long and short positions in call and put options, assuming proportional transactions costs on trades in the stock and no transactions costs on trades in the bond. However, even when the market is arbitrage-free and a given contingent claim has a unique replicating portfolio, there may exist super replicating portfolios of lower cost. Nevertheless, Bensaid, Lesne, Pages and Scheinkman (1992) gave conditions under which the cost of the replicating portfolio does not exceed the cost of any super replicating portfolio. These results were generalized by Stettner (1997) and Rutkowski (1998) to the case of asymmetric transaction costs. Palmer (2001a) provided a further slight generalization and gave a simpler proof. These results have the consequence that there is no super replicating portfolio for long calls and puts of lower cost than the replicating portfolio. However, this is not true for short calls and puts. Since the negative of the cost of the least cost super replicating portfolio for such a position is a lower bound for the call or put price, it is important to determine this least cost.

Previously in Palmer (2001b), I had determined the least cost super replicating portfolio for any contingent claim in a one-period binomial model. Together with Guan-Yu Chen and Yuan-Chung Sheu of the Department of Applied Mathematics of National Chiao Tung University, now I have shown that there are only finitely many possibilities for a least cost super replicating portfolio for any contingent claim in a two-period binomial model. In particular, for short positions in calls and puts, we show that there are just four possibilities enabling the least cost super replicating portfolio to be easily determined. It turns out that such a portfolio is, in general, path dependent. This work has been written up in Chen, Palmer and Sheu (2004) and submitted to the journal *Applied Mathematical Finance* for publication. I now summarize this work.

We consider an n-period binomial model of a financial market with two securities: a stock and a bond. If the current stock price is S, then at the end of the next period it is either Su or Sd, where $0 < d < u$. The bond yields a constant rate of return r over each time period meaning that a dollar now is worth $R = 1 + r$ after one period. We assume that when the stock price is S, buying one share incurs a transaction cost of λS and that selling one share incurs a transaction cost of μS, where

$$\lambda \geq 0, \quad 0 \leq \mu < 1.$$

As is usual, we assume that there are no transaction costs when a portfolio is established at time 0.

Let us denote by $\phi = \{(\Delta_i, B_i),\ i = 0, 1, 2, \cdots, n\}$ a *self-financing portfolio* where Δ_i stands for the number of shares and B_i the number of bonds held at time i. The initial value or cost of the portfolio ϕ is given by $V_0(\phi) = \Delta_0 S_0 + B_0$.

A *contingent claim* is a two-dimensional random variable $X = (g, h)$ where g represent the number of shares and h the value of bonds held at time n. We say that a portfolio $\phi = \{(\Delta_i, B_i),\ i = 0, 1, 2, \cdots, n\}$ replicates the claim X that is settled by delivery if it is self-financing and $\Delta_n = g$ and $B_n = h$. If we just have

$$\mu < 1,$$

then in an n-period model every contingent claim has a replicating portfolio but it may not be unique. (See, e.g., Palmer (2001).)

We say a self-financing portfolio ϕ is a *super replicating portfolio* for a contingent claim $X = (g, h)$ settled by delivery at time n if at time n we have $\Delta_n \geq g$ and $B_n \geq h$. For a given contingent claim, our aim is to find super replicating portfolio with the least cost.

First we show that we can reduce the problem of finding least cost super replicating portfolios for a two period claim to that of finding least cost super replicating portfolios for a class of one period claims.

Lemma. *The infimum of the cost of a super replicating portfolio for the two-period contingent claim* $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}), (\Delta_{dd}, B_{dd})\}$ *is equal to the infimum over* (Δ_u, Δ_d) *of* $C(\Delta_u, \Delta_d)$, *the least cost of super replicating portfolios for the one-period contingent claim* $\{(\Delta_u, b_u(\Delta_u)/R), (\Delta_d, b_d(\Delta_d)/R)\}$ *with initial stock price* S, *where*

$$b_u(\Delta_u) = \max\{B_{uu} + e(\Delta_u - \Delta_{uu})Su^2,\ B_{ud} + e(\Delta_u - \Delta_{ud})Sud\}$$

$$b_d(\Delta_d) = \max\{B_{ud} + e(\Delta_d - \Delta_{ud})Sud,\ B_{dd} + e(\Delta_d - \Delta_{dd})Sd^2\},$$

with

$$e(\Delta) = -\Delta + \mu\Delta^+ + \lambda\Delta^-.$$

In our first theorem we prove the existence of a least cost super replicating portfolio (note: this was already proved in Bensaid, Lesne, Pages and Scheinkman (1992)) and show that we need only consider the function $C(\Delta_u, \Delta_d)$ in a certain rectangle. Before stating the theorem, we introduce some notation.

Let $[\alpha_u, \beta_u]$ be the smallest closed interval containing all the solutions Δ_u of

$$f_u(\Delta_u) = B_{uu} + e(\Delta_u - \Delta_{uu})Su^2 - B_{ud} - e(\Delta_u - \Delta_{ud})Sud = 0 \tag{0.1}$$

and also Δ_{uu} and Δ_{ud}. Similarly, let $[\alpha_d, \beta_d]$ be the smallest closed interval containing all the solutions of

$$f_d(\Delta_d) = B_{ud} + e(\Delta_d - \Delta_{ud})Sud - B_{dd} - e(\Delta_d - \Delta_{dd})Sd^2 = 0. \tag{0.2}$$

and also Δ_{ud} and Δ_{dd}. Let Π be the rectangle in the (Δ_u, Δ_d)-plane given by

$$\Pi = \{(\Delta_u, \Delta_d) : \alpha_u \leq \Delta_u \leq \beta_u,\ \alpha_d \leq \Delta_d \leq \beta_d\}.$$

Theorem 1. *For a general two-period contingent claim $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}), (\Delta_{dd}, B_{dd})\}$, there exists a least cost super replicating portfolio; more precisely, the function $C(\Delta_u, \Delta_d)$ takes its minimum in the rectangle Π at some point (Δ_u, Δ_d) and a least cost super replicating portfolio for the one-period claim $\{(\Delta_u, b_u(\Delta_u)/R), (\Delta_d, b_d(\Delta_d)/R)\}$ with initial stock price S yields a least cost super replicating portfolio for the two-period claim.*

In the next theorem, we use linear programming to show that we need only consider finitely many possibilities in order to determine a super replicating portfolio with least cost. To state the theorem, we need to define the two quantities

$$\begin{aligned} a_u = a_u(\Delta_u, \Delta_d) \quad &= (\Delta_d - \Delta_u)S\overline{u} + \tfrac{b_d(\Delta_d) - b_u(\Delta_u)}{R}, \\ a_d = a_d(\Delta_u, \Delta_d) \quad &= (\Delta_d - \Delta_u)S\overline{d} + \tfrac{b_d(\Delta_d) - b_u(\Delta_u)}{R}, \end{aligned}$$

where

$$\overline{u} = \begin{cases} u(1+\lambda) & \text{if } \Delta_u \geq \Delta_d \\ u(1-\mu) & \text{if } \Delta_u \leq \Delta_d \end{cases}, \quad \overline{d} = \begin{cases} d(1-\mu) & \text{if } \Delta_u \geq \Delta_d \\ d(1+\lambda) & \text{if } \Delta_u \leq \Delta_d \end{cases}.$$

Theorem 2. *For a general two-period contingent claim with terminal holdings $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}), (\Delta_{dd}, B_{dd})\}$, there always exists a least cost super replicating portfolio with initial holdings (Δ, B) and holdings (Δ_u, B_u), (Δ_d, B_d) at the end of the first period which represent a least cost super replicating portfolio for the one period claim $\{(\Delta_u, b_u(\Delta_u)/R), (\Delta_d, b_d(\Delta_u)/R)\}$ and such that at least two distinct conditions from the following list are satisfied:*

$$\Delta_u = \Delta_{uu}, \quad \Delta_u = \Delta_{ud}, \quad \Delta_d = \Delta_{ud}, \quad \Delta_d = \Delta_{dd},$$

$$\Delta_u = \Delta_d,$$

$$a_u(\Delta_u, \Delta_d) = 0, \quad a_d(\Delta_u, \Delta_d) = 0,$$

$$f_u(\Delta_u) = B_{uu} + e(\Delta_u - \Delta_{uu})Su^2 - B_{ud} - e(\Delta_u - \Delta_{ud})Sud = 0$$
$$f_d(\Delta_d) = B_{ud} + e(\Delta_d - \Delta_{ud})Sud - B_{dd} - e(\Delta_d - \Delta_{ud})Sd^2 = 0$$

Theorem 2 narrows down the search for a least cost super replicating portfolio to a finite number of possibilities. However, the number of possibilities is still quite large.

Now we consider a restricted class of claims for which the number of possibilities can be reduced to a manageable number. In fact, we determine the initial holdings of the least cost super replicating portfolios for a claim in the two period model with

$$\Delta_{uu} = \Delta_{ud} < \Delta_{dd}, \quad B_{uu} = B_{ud}.$$

This includes short calls and puts with the exercise price between Sud and Sd^2. Note we could treat the case $\Delta_{uu} < \Delta_{ud} = \Delta_{dd}$, $B_{ud} = B_{dd}$ similarly. This would include short calls and puts with the exercise price between Su^2 and Sud.

Theorem 3. *Consider a two period binomial model with parameters S, u, d, R, μ and λ. For every contingent claim $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}), (\Delta_{dd}, B_{dd})\}$ satisfying*

$$\Delta_{uu} = \Delta_{ud} < \Delta_{dd}, \quad B_{uu} = B_{ud},$$

there always exists a least cost super replicating portfolio which belongs to one of the following four types (note that in all cases transactions are carried out at the terminal nodes so that the final share holdings are Δ_{uu}, Δ_{uu}, Δ_{uu}, Δ_{dd} in states uu, ud, du and dd respectively):

(I) the initial holdings are $(\Delta_{dd}, B_{dd}/R^2)$ and the only additional share transaction is selling $(\Delta_{dd} - \Delta_{uu})$ shares in state u (this type arises only if $R \leq d(1+\lambda)$ and $B_{uu} - B_{dd} - Sud(1-\mu)(\Delta_{dd} - \Delta_{uu}) < 0$);

(II) the initial holdings are (δ, B), where $\delta \leq \Delta_{uu}$ and (δ, B) is such that $BR - B_{uu}/R$ is just enough to carry out the only additional share transaction of buying back $(\Delta_{uu} - \delta)$ shares of stocks in state u; there are two possibilities:

(a) $\delta = \Delta_{uu}$ and the terminal holdings in the du state are (Δ_{uu}, B_{uu}) (this case only arises if $B_{uu} - B_{dd} - Sd^2(1+\lambda)(\Delta_{dd} - \Delta_{uu}) \geq 0$);

(b) $\delta < \Delta_{uu}$ and the terminal holdings in the dd state are (Δ_{dd}, B_{dd}) (this case only arises if $B_{uu} - B_{dd} - Sd^2(1+\lambda)(\Delta_{dd} - \Delta_{uu}) < 0$);

(III) the initial holdings are $(\alpha, B/R)$, where $\alpha > \Delta_{uu}$ and (α, B) are the initial holdings in a replicating portfolio for the one-period portion $\{d, du, dd\}$, and the only additional share transaction is selling $(\alpha - \Delta_{uu})$ shares in state u (this case only arises if $R \leq d(1+\lambda)$);

(IV) a replicating portfolio for the whole two-period model.

Finally we show that even when the least cost super replicating portfolio is unique, it may be path dependent.

Theorem 4. *Consider a two period binomial model with parameters S, u, d, R, μ and λ satisfying*

$$d(1+\lambda) < u(1-\mu), \quad R(1-\mu) < d(1+\lambda) < R.$$

For every contingent claim with terminal holdings $\{(\Delta_{uu}, B_{uu}), (\Delta_{ud}, B_{ud}), (\Delta_{dd}, B_{dd})\}$ *satisfying* $\Delta_{uu} = \Delta_{ud} < \Delta_{dd}$, $B_{uu} = B_{ud}$ *and*

$$B_{dd} - B_{uu} - Sd^2(1+\lambda)(\Delta_{uu} - \Delta_{dd}) > 0,$$

there exists a unique least cost super replicating portfolio. Moreover, this portfolio is path-dependent.

I gave a talk on this work at the 11th Annual Conference on Pacific Basin Finance, Economics and Accounting held at the Grand Hotel, Taipei on November 21-22, 2003.

Now I report on other related work:

In Palmer (2001a), I gave an example of a two-period contingent claim in which some of the "*probabilities*" did not lie between 0 and 1 and yet numerical simulations suggested that no super replicating portfolio costs less than the replicating portfolio. However in that paper I was only considering path-independent portfolios. It turns out that there is a path dependent portfolio of lower cost than the replicating portfolio. Nevertheless, using the techniques developed in Chen, Palmer and Sheu (2004), I have now been able to produce another example of a two-period contingent claim in which some of the "probabilities" do not lie between 0 and 1 and can now prove that no super replicating portfolio (including path dependent ones) costs less than the replicating portfolio. So it seems that whether or not the replicating portfolio is the least cost super replicating portfolio cannot be determined in a simple way from the "probabilities" associated with it.

Another problem I am thinking about is that of *cash settlement*. Most of the papers in the literature assume settlement by delivery. I am convinced that my work on two-period claims with Chen and Sheu can be extended to cash settlement. However, unlike settlement by delivery, we should be able to find the least cost super replicating portfolio for all claims, not just short calls and puts.

Bensaid, Lesne, Pages and Scheinkman (1992) characterised certain claims for which the cost of the replicating portfolio does not exceed the cost of any super replicating portfolio. These claims include portfolios of long calls and puts but it was left open whether or not there were other claims satisfying their conditions. I think now that I can prove that a portfolio satisfying the conditions they gave (with transactions costs set to zero) in fact include only long calls and puts together with the stock and the bond.

Bensaid, Lesne, Pages and Scheinkman (1992) also considered the case where settlement is up to the seller. This is essentially the same assumption as made by

Edirsinghe, Naik and Uppal (1993) and Reiss (1999). In Palmer (2001a), I proved the results in Bensaid, Lesne, Pages and Scheinkman (1992) concerning settlement by delivery using my "probability" technique. Now I plan to use this same technique to prove the theorem in Bensaid, Lesne, Pages and Scheinkman (1992) concerning settlement up to the seller.

Lastly I am studying the more recent contributions of Perrakis and LeFoll (1997, 2000, 2004).

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