

The Iris-Loaded Wave Guide as a Boundary Value Problem. II

WOLFGANG KROLL

Department of Physics, National Taiwan University, Taipei, Taiwan

(Received 1 May, 1965)

In the previously published paper we sketched a theory for the iris-loaded waveguide. In this paper now we derive the complete set of integral equations of this theory.

In order to obtain from this theory a solution as proposed by Chu and Hansen one has to make approximations whose justification seems rather difficult.

In the case of large separation of the irises we obtain easily good approximate solutions from our theory.

THE INTEGRAL EQUATIONS

In our previous paper⁽¹⁾ we consider here an E-wave in a waveguide of radius b with irises at distances l from each other. The radius of the holes in the irises we call a .

The Cartesian component E_z of the electric field satisfies

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{\partial^2 E_z}{\partial z^2} + k^2 E_z = 0,$$

with k as the propagation constant in vacuo. Because E_z must vanish for $r=b$ we solve this equation by

$$J_0(\tau_\nu r) \sinh \sqrt{\tau_\nu^2 - k^2} z.$$

Here because of $E_z(\beta, z) = 0$, τ_ν is to be found from

$$J_0(\tau_\nu b) = 0.$$

Denoting by $F_1(\mathbf{y}) + F_2(r)$ the value of E_z in the plane of the zeroth iris, where F_1 corresponds to an even and F_2 to an odd mode, we find as the solution of Maxwell's equations

$$\begin{aligned} -ikE_z &= e^{ihn} \sum \frac{J_0(\tau_\nu r)}{\sinh \sqrt{(\quad)} l} \left[\left(e^{ihl} \int_0^a t(F_1 + F_2) J_0(\tau_\nu t) dt + \int_a^b t(F_1 - F_2) J_0(\tau_\nu t) dt \right) \right. \\ &\quad \left. \sinh \sqrt{(\quad)} z' + \int_0^b t(F_1 + F_2) J_0(\tau_\nu t) dt \sinh \sqrt{(\quad)} (l - z') \right], \\ H_\varphi &= e^{ihn} \sum \frac{J_1(\tau_\nu r)}{\tau_\nu \sinh \sqrt{(\quad)} l} \left[\left(e^{ihl} \int_0^a t(F_1 + F_2) J_0(\tau_\nu t) dt + \int_a^b t(F_1 - F_2) J_0(\tau_\nu t) dt \right) \right. \\ &\quad \left. \sinh \sqrt{(\quad)} z' + \int_0^b t(F_1 + F_2) J_0(\tau_\nu t) dt \sinh \sqrt{(\quad)} (l - z') \right], \end{aligned}$$

(1) W. Kroll, Chin. J. Phys. **2**, 63, (1964).

$$ikE_r = e^{ihn} \sum \frac{\sqrt{(\quad)} J_1(r, r)}{r, \sinh \sqrt{(\quad)} l} \left[\left(e^{ihl} \int_0^a t(F_1 + F_2) J_0(r, t) dt + \int_a^b t(F_1 - F_2) J_0(r, t) dt \right) \right. \\ \left. \cosh \sqrt{(\quad)} z' - \int_0^b t(F_1 + F_2) J_0(r, t) dt \cosh \sqrt{(\quad)} (l - z') \right]$$

Here h is the propagation constant in the waveguide. $\sqrt{(\quad)}$ is an abbreviation for $\sqrt{r_v^2 - k^2}$. $z' + nl = z$ with n as the number of the corrugation.

The normalization integral

$$N_\nu = \int_0^b t J_0^2(r, t) dt = \int_0^b t J_1^2(r, t) dt$$

in the denominator is omitted.

The continuity of E , through the zeroth iris-plane leads to the condition.

$$(1 + e^{-ihl}) (\cosh \sqrt{(\quad)} l - 1) \int_a^b t F_1 J_0(r, t) dt + (1 - e^{-ihl}) (\cosh \sqrt{(\quad)} l + 1) \times \\ \int_a^b t F_2 J_0(r, t) dt + 2 (\cosh \sqrt{(\quad)} l - \cos(hl)) \int_0^a t (F_1 + F_2) J_0(r, t) dt = 0.$$

Expanding $F_1(r)$ is a series of Bessel functions we find by the use of this relation.

$$F_1(r) + F_2(r) = -2 \sin^2 \frac{hl}{2} \int_0^a t (F_1 + F_2) \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l - 1} dt \\ - (1 - e^{-ihl}) \int_a^b t F_2 \sum \frac{J_0(r, t) J_0(r, r)}{\cosh \sqrt{(\quad)} l - 1} dt \quad : r < a \\ (1 + e^{-ihl}) F_1(r) + (1 - e^{-ihl}) F_2(r) = -4 \sin^2 \frac{hl}{2} \int_0^a t (F_1 + F_2) \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l - 1} dt \\ - 2(1 - e^{-ihl}) \int_a^b t F_2 \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l - 1} dt \quad : r > a$$

To these equations comes in addition the condition that E_r , the radial electric component must vanish on the iris-screens. This condition we use in the form

$$\int_r^b E(r, 0) dr = 0 \quad : r > a.$$

We introduce the notation

$$F_1(r) + F_2(r) = F(r) \quad : r < a \\ (1 + e^{-ihl}) F_1(r) = G_1(r), \quad (1 - e^{-ihl}) F_2(r) = G_2(r) \quad : r > a.$$

and can then write the continuity-condition for E_r in the form

$$(\cosh \sqrt{(\quad)} l - 1) \int_a^b t G_1 J_0(r, t) dt = 2 (\cos(hl) - \cosh \sqrt{(\quad)} l) \int_0^a t F J_0(r, t) dt \\ - (\cosh \sqrt{(\quad)} l + 1) \int_a^b t G_2 J_0(r, t) dt.$$

The three equations take then the form

$$F(r) = -2 \sin^2 \frac{hl}{2} \int_0^a tF \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l - 1} dt - \int_a^b tG_2 \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} Z - 1} dt : r < a \quad (1)$$

$$G_1(r) + G_2(r) = -4 \sin^2 \frac{hl}{2} \int_0^a tF \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l - 1} dt - 2 \int_a^b tG_2 \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} Z - 1} dt : r > a \quad (2)$$

$$2 \sin^2 \frac{hl}{2} \int_0^a tF \sum \frac{I_0(r, r) J_0(r, t)}{r_v^2 \sinh \sqrt{(\quad)} l \sqrt{(\quad)} (\cosh \sqrt{(\quad)} l + 1)} dt + \int_a^b tG_2 \sum \frac{J_0(r, r) J_0(r, t)}{r_v^2 \sinh \sqrt{(\quad)} l \sqrt{(\quad)} (\cosh \sqrt{(\quad)} l + 1)} dt = 0 : r > a. \quad (3)$$

These are the fundamental equations of the theory. From them we have to find the functions $F(r)$, $G_1(r)$ and $G_2(r)$ and also the propagation constant h .

The equations can be written also in another form by elimination of G_2 . Using the continuity condition for E_r , we get

$$F(y) = 2 \cos^2 \frac{hl}{2} \int_0^a tF \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l + 1} dt + \int_a^b tG_1 \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l + 1} dt : r < a \quad (1')$$

$$G_1(y) + G_2(r) = 4 \cos^2 \frac{hl}{2} \int_0^a tF \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l + 1} dt + \int_a^b tG_1 \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l + 1} dt : y > a \quad (2')$$

$$2 \cos^2 \frac{hl}{2} \int_0^a tF \sum \frac{I_0(r, r) J_0(r, t)}{r_v^2 \sinh \sqrt{(\quad)} l \sqrt{(\quad)} (\cosh \sqrt{(\quad)} l - 1)} dt + \int_a^b tG_1 \sum \frac{J_0(r, r) J_0(r, t)}{r_v^2 \sinh \sqrt{(\quad)} l \sqrt{(\quad)} (\cosh \sqrt{(\quad)} Z - 1)} dt = 0 : r > a. \quad (3')$$

From the last equation of the two sets we see that $G_2(r)$ will be small for small values of $\sin \frac{hl}{2}$ and $G_1(r)$ for small values of $\cos \frac{hl}{2}$.

VARIATION PRINCIPLE

The equations

$$F(r) = -2 \sin^2 \frac{hl}{2} \int_0^a tF \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} Z - 1} dt - \int_a^b tG_2 \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l - 1} dt : r < a$$

and

$$F(r) = 2 \cos^2 \frac{hl}{2} \int_0^a tF \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l + 1} dt + \int_a^b tG_2 \sum \frac{J_0(r, r) J_0(r, t)}{\cosh \sqrt{(\quad)} l + 1} dt : r > a.$$

can be derived from the variation principles:

$$\delta L = \delta \int_0^a r F dr \left[F(r) + 2 \sin^2 \frac{hl}{2} \int_0^a t F G_{k-}(r, t) dt + 2 \int_a^b t G_2 G_{k-}(r, t) dt \right] = 0,$$

and

$$\delta M = \delta \int_0^a r F dr \left[F(r) - 2 \cos^2 \frac{hl}{2} \int_0^a t F G_{k+}(r, t) dt - 2 \int_a^b t G_1 G_{k+}(r, t) dt \right].$$

Here

$$G_{k-} = \sum \frac{J_0(r_1 r) J_0(r_2 t)}{\cosh \sqrt{(\quad)} l - 1}$$

and

$$G_{k+} = \sum \frac{J_0(r_1 r) J_0(r_2 t)}{\cosh \sqrt{(\quad)} l + 1}.$$

When h takes such values that $G_1(r)$ resp. $G_2(r)$ can be neglected, we can write the variation-principle in the form

$$\delta L = \delta \frac{\int_0^a r F^2 dr}{\int_0^a r F dr \int_0^a t F G_k(r, t) dt}.$$

Here $G_k(r, t)$ stands for G_{k+} resp. G_{k-} . We obtain then $2 \cos^2 \frac{hl}{2}$ resp. $-2 \sin^2 \frac{hl}{2}$ as the extremal values of L , provided that $|\cos \frac{hl}{2}|$ resp. $|\sin \frac{hl}{2}|$ is small.

SOLUTION FOR LARGE SEPARATION

When l is large as compared with b , in the equations (1) and (1') we need only the first few terms in the series G_{k-} and G_{k+} , the higher terms going to zero exponentially.

Assuming

$$r_1 < k < r_2$$

we need only the first term. Confining further the consideration to small values of $(\sin \frac{hl}{2})$ resp. $|\cos \frac{hl}{2}|$, the equations take the form

$$F(r) = \frac{2 \sin^2 \frac{hl}{2} J_0(r_1 r)}{N_1 (\cos \sqrt{k^2 - r_1^2} l - 1)} \int_0^a t F J_0(r_1 t) dt$$

resp.

$$F(r) = \frac{2 \cos^2 \frac{hl}{2} J_0(r_1 r)}{N_1 (\cos \sqrt{k^2 - r_1^2} l + 1)} \int_0^a t F J_0(r_1 t) dt.$$

The solution of these equations is

$$F(r) = AJ_0(r_1 r); \quad \frac{(\cos hl - 1) \int_0^a t J_0^2(r_1 t) dt}{N_1 (\cos \sqrt{k^2 - r_1^2} l - 1)} = 1$$

resp.

$$F(r) = BJ_0(r_1 r); \quad \frac{(\cos hl + 1) \int_0^a t J_0^2(r_1 t) dt}{N_1 (\cos \sqrt{k^2 - r_1^2} l + 1)} = 1$$

with

$$I_1 = \int_0^a t J_0^2(r_1 t) dt.$$

We find then the frequency-condition

$$\cos hl = 1 - \frac{N_1}{I_1} (1 - \cos \sqrt{k^2 - r_1^2} l) \quad : \quad \left| \sin \frac{hl}{2} \right| \ll 1$$

$$\cos hl = -1 + \frac{N_1}{I_1} (1 + \cos \sqrt{k^2 - r_1^2} l) \quad : \quad \left| \cos \frac{hl}{2} \right| \ll 1.$$

The condition resembles that obtained by Slater⁽²⁾. The difference is due to our approximation. In the limiting case $a=0$ resp. $a=b$ the two forms agree completely. For $a=0$ there is of course no propagation in the waveguide, because in our equation $I_1=0$. In the limit $a=b$ or absence of irises we get, as it must be

$$h = \sqrt{k^2 - r_1^2}$$

For $\left| \sin \frac{hl}{2} \right| \ll 1$ we find in the exterior part of the corrugation essentially only an even mode and for $r=a$ on the screen the field E_z changes discontinuously by the factor $\frac{2}{(1+e^{-ihl})}$ according to equation (2).

For $\left| \cos \frac{hl}{2} \right| \ll 1$ there is in the exterior part of the corrugation essentially only an odd mode and E_z jumps for $r=a$ on the screen through the factor $\frac{2}{(1-e^{-ihl})}$ according to equation (2').

SOLUTION FOR SMALL SEPARATION OF THE IRISES

When l is small as compared with b one has to make the computation with a finite thickness of the irises. We want however to see what approximations one has to make in order to obtain a solution as proposed by Chu and Hansen⁽³⁾. The situation is much more complicated and the analysis is not quite satisfactory.

We may say that for $l \ll b$ there are many terms in the series for G_k . The exponential decrease of the terms becomes effective only in very high order terms.

(2) J. C. Slater, *Microwave Electronics*, p. 183, (New York 1950).

(3) F. I. Chu and W. W. Hansen *J. Appl. Phys.* 18, 996, (1947).

We try then to approximate G_{k-} by replacing $(\cosh z - 1)^{-1}$ by the first term in the expansion in partial fractions

$$\frac{1}{\cosh z - 1} = \frac{2}{z^2}.$$

Then we obtain

$$G_{k-} = \frac{2}{l^2} \sum \frac{J_0(r_\nu r) J_0(r_\nu t)}{N_\nu(r_\nu^2 - k^2)},$$

This is now the well-known expansion of the Green-function.

$$G_k(r, t) = \frac{\pi}{2} \begin{cases} J_0(kr) \left(\frac{N_0(kb)}{J_0(kb)} J_0(kt) - N_0(kt) \right) & : r < t \\ J_0(kt) \left(\frac{N_0(kb)}{J_0(kb)} J_0(kr) - N_0(kr) \right) & : r > t. \end{cases}$$

Confining the consideration to small values of $(\sin \frac{hl}{2})$ I we get

$$F(r) = -f(h) \int_0^a t F G_k(r, t) dt : r < a$$

with

$$f(h) = \frac{4}{l^2} \sin^2 \frac{hl}{2}.$$

This equation now is easily solved. We find

$$F(r) = A J_0(\sqrt{k^2 - f(h)} r) - \frac{J_0(\sqrt{k^2 - f(h)} a)}{\sqrt{k^2 - f(h)} J_1(\sqrt{k^2 - f(h)} a)} = \frac{N_0(kb) J_0(ka) - N_0(ka) J_0(kb)}{k(N_0(kb) J_1(ka) - N_1(ka) J_0(kb))}.$$

$G_2(r)$ is negligible and $G_1(r)$ is proportional to

$$N_0(kb) J_0(kr) - N_0(kr) J_0(kb).$$

Therefore as far as the electric field is concerned, the solution is similar to that obtained by Chu and Hansen. However the frequency-condition is modified in remarkable points.

From this consideration it seems that a solution as proposed by Chu and Hansen can be obtained only by replacing the denominator in G_{k-} by the first term of an expansion in partial fractions and it is difficult to see how this replacement can be justified mathematically.

The support of this work by the Chinese Council on Science Development is gratefully acknowledged.