A Generalized Sampling Theorem

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A generalized sampling theorem of Someya and Shannon is presented and its examples are also given.

In the theory of information, or in the statistical theory of communication, the sampling theorem of Someya[1] and Shannon[2] plays an important role in case of treating the "Continuous channel" of the information systems. In this paper the authors generalize[3] the sampling theorem in such a way that one can find some applications not only in the information theory but also in the physical analysis.

Let f(t) belong to L_p, and let the following reciprocity relations hold:

$$F(s)=L_s \cdot f \equiv \int K(s,\ t) f(t)\, dt. \tag{1}$$

and

$$f(t) = L_t^{-1} \cdot F \equiv \int \widetilde{K}(t,\ s) F(s)\, ds. \tag{2}$$

Further we assume that

$$F(s) = 0, \qquad \text{for } s \leqslant a \text{ and } \beta \leqslant s \tag{3}$$

and that $F(s)$ can be expanded by an orthonormal set of functions $\{\phi_n(s);\ \int \phi_m(s)\phi_n(s)ds = \delta_{m,n},\ m, n = \text{integers}\}$, which is complete in the interval $\alpha \leqslant s \leqslant \beta$; *i. e.*

$$F(s) = \sum_n a_n \phi_n(s), \qquad \text{for } \alpha \leqslant s \leqslant \beta \tag{4}$$

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(1) I. Someya: *Transmission of Wave-Forms* (Syûkyôsya, Tôkyô, 1949) (染谷勳：波形傳送 (1949), 修敎社)

(2) C. E. Shannon: Bell System Tech. Journ. 27, 379-423; 623-656. (1948) C. E. Shannon and W. Weaver: *The Mathematical Theory of Communication* (Univ. of Illinois Press, Urbana, 1949).

(3) É. I. Takizawa: *Information Theory and its Exercises* (Hirokawa Syoten, Tôkyô, 1966) p. 219-221, 274-276. (瀧澤英一：情報の理論と演習 (1966)，廣川書店)

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The expression (4), being multiplied by $\phi_m(s)$ and integrated over s, gives

$$\int_\alpha^\beta F(s)\phi_m(s)ds=\sum_n a_n\cdot\int_\alpha^\beta \phi_m(s)\phi_n(s)ds=\sum_n a_n\delta_{m,n}=a_m. \tag{5}$$

From (2), (3), (4), and (5), we obtain

$$f(t)=L_t^{-1}\cdot F=\sum_n a_n\cdot L_t^{-1}\cdot\phi_n=\sum_n\left(\int_\alpha^\beta F(s)\phi_n(s)ds\right)\cdot\int\widetilde{K}(t,s)\phi_n(s)ds. \tag{6}$$

In the expression (6), if the reciprocity relations (1) and (2), $f(t)$, and $\{\phi_n(s)\}$ are given, we can calculate a,, and $L_t^{-1}\cdot\phi_n$ by means of (5) and (2), respectively. It means that the function f(t) is expressible in a series of $L_t^{-1}\cdot\phi_n$ with expansion coefficients a_n ($n=$ integers). The expression (6) forms a generalized sampling theorem.

If we can take

$$\widetilde{K}\left(\frac{n}{r},s\right)=\phi_n(s),\qquad \text{for } \alpha\leqslant s\leqslant\beta \tag{7}$$

with a constant r, *i.* e. if the integral kernel $\widetilde{K}(\frac{n}{r}, s)$ can be put equal to $\phi_n(s)$, then we obtain

$$a_n=\int_\alpha^\beta F(s)\widetilde{K}\left(\frac{n}{r},s\right)ds=f\left(\frac{n}{r}\right). \tag{8}$$

The expression (6) is simplified into

$$f(t)=\sum f\left(\frac{n}{r}\right)\cdot\int\widetilde{K}(t,s)\phi_n(s)ds, \tag{9}$$

and the points at the variable t:

$$t=\frac{n}{r},\ (n=\text{integers}) \tag{10}$$

are called *sampling points,* and the function of t:

$$L_t^{-1}\cdot\phi_n=\int\widetilde{K}(t,s)\phi_n(s)ds=\int\widetilde{K}(t,s)\widetilde{K}\left(\frac{n}{r}\right)ds,\qquad (n=\text{integers}) \tag{11}$$

is a *sampling function.*

The importance of the expressions (6) and (9) in the analysis is made clear in the following examples.

[Example 1]

We take Fourier transform for (1). Let

$f(t)\in L_2$,

$$F(s)=L_s\cdot f=\int_{-\infty}^{+\infty}K(s,t)f(t)dt=\int_{-\infty}^{+\infty}\exp[ist]\cdot f(t)\,dt, \tag{12}$$

$$\{\phi_n(s)\}=\{\exp[is\frac{\pi n}{\beta}];\ n=\text{integers}\}, \tag{13}$$

and

$$a=-\beta, \qquad r=\frac{\beta}{2\pi} \tag{14}$$

then the reciprocity relation (2) and the expression (4) lead us to

$$f(t)=L_t^{-1}\cdot F=\int_{-\infty}^{+\infty}\tilde{K}(t,s)F(s)\,ds=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[-its]\cdot F(s)ds$$
$$=\frac{1}{2\pi}\int_{-\beta}^{\beta}\exp[-its]\cdot F(s)ds. \tag{15}$$

and

$$F(s)=\sum_{n=-\infty}^{+\infty}a_n\cdot\exp[is\frac{\pi n}{\beta}], \tag{16}$$

$$\frac{1}{2\beta}\int_{-\beta}^{\beta}F(s)\bullet\exp[-is\frac{\pi n}{\beta}]\,ds. \tag{17}$$

In this case, the expression (6) becomes

$$f(t)=\frac{1}{4\pi\beta}\sum_{n=-\infty}^{+\infty}\left(\int_{-\beta}^{\beta}F(s)\bullet\exp[-is\frac{\pi n}{\beta}]ds\right)\cdot\int_{-\beta}^{\beta}\exp[-its]\cdot\exp[is\frac{\pi n}{\beta}]\,ds$$
$$=\sum_{n=-\infty}^{+\infty}f\left(\frac{\pi n}{\beta}\right)\cdot\frac{\sin(\beta t-\pi n)}{\beta t-\pi n} \tag{18}$$
$$=\sum_{n=-\infty}^{+\infty}f\left(\frac{n}{2r}\right)\cdot\frac{\sin[\pi(2rt-n)]}{\pi(2rt-n)}, \tag{19}$$

which is Shannon's sampling theorem. The expressions (18) and (19) give $f(t)$ in terms of the values of $f(t)$ at sampling points $t=\pi n/\beta=n/(2r)$ (n=integers), with sampling function $\sin t/t$.

[Example 23

We shall take Hankel transform of order ν ($\nu\geqq-1/2$), with

$$\int_0^{+\infty}f(t)dt<+\infty,$$

absolutely convergent. The reciprocity relations (1),(2), and Fourier-Bessel expansion (4) are chosen to be

$$F(s)=L_s\cdot f=\int_0^{+\infty}K(s,s)f(t)dt=\int_0^{+\infty}t\,J_\nu(st)f(t)\,dt. \tag{20}$$

$$f(t)=L_t^{-1}\cdot F=\int_0\tilde{K}(t,t)F(s)\,ds=\int_0^{+\infty}s\,J_\nu(ts)F(s)\,ds. \tag{21}$$

and

$$F(s) = \sum_{n=1}^{\infty} a_n J_\nu(j_n s), \qquad \text{for } 0 \leqslant s \leqslant 1 \tag{22}$$

$$\{\phi_n(s)\} = \{J_\nu(j_n s);\ n=1, 2, 3, \ldots\}, \tag{23}$$

with

$$a_n = \frac{2}{J_{\nu+1}^2(j_n)} \int_0^1 s\ J_\nu(j_n s)\ F(s)\ ds, \tag{24}$$

where $j_n\,(n=1, 2, 3, \ldots)$ are the positive zeros of $J_\nu(t)$ arranged in ascending order of magnitude. Then the expression (6) becomes

$$f(t) = \sum_{n=1}^{+\infty} \frac{2}{J_{\nu+1}^2(j_n)} \left(\int_0^1 s\ F(s)\ J_\nu(j_n s)\ ds\right) \cdot \int_0^1 s\ J_\nu(ts)\ J_\nu(j_n s)\ ds$$

$$= \sum_{n=1}^{+\infty} \frac{2}{J_{\nu+1}^2(j_n)} f(j_n) \cdot \int_0^1 s\ J_\nu(ts)\ J_\nu(j_n s)\ ds, \tag{25}$$

with sampling points $t=j_n\,(n=1, 2, 3, \ldots)$, and sampling function $\int_0^1 s\ J_\nu(ts)\ J_\nu(j_n s) ds$.

[Example 31

We shall take Laplace transform for (1) and (2):

$$F(s) = \int K(s,\ t) f(t)\ dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} f(t)\ dt, \tag{26}$$

$$f(t) = \int \widetilde{K}(t,\ s)\ F(s)\ ds = \int_0^\infty e^{-ts} F(s)\ ds. \tag{27}$$

The expression for (4) is chosen to be expanded in Laguerre polynomials $L_n(s)$:

$$F(s) = \sum_{n=0}^{+\infty} a_n L_n(s), \qquad (0 \leqslant s) \tag{28}$$

with

$$a_n = \int_0^\infty e^{-s} L_n(s)\ F(s)\ ds. \qquad (n=\text{integers}) \tag{29}$$

Then the expression for (6) becomes

$$f(t) = \sum_{n=0}^{+\infty} \left(\int_0^\infty e^{-s} L_n(s)\ F(s)\ ds\right) \cdot \int_0^\infty e^{-ts} L_n(s)\ ds$$

$$= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} \cdot \int_0^\infty (-)^k s^k e^{-s} F(s)\ ds\right) \cdot \int_0^\infty e^{-ts} L_n(s)\ ds$$

$$= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} \cdot \left[\frac{\partial^k}{\partial t^k} \int_0^\infty e^{-ts} F(s)\ ds \right]_{t=1} \right) \cdot \frac{1}{t} \left(\frac{t-1}{t}\right)^n$$

$$= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \binom{n}{k} \cdot f^{(k)}(1) \right) \cdot \frac{1}{t} \left(\frac{t-1}{t} \right)^n \tag{30}$$

If we take, instead of (28), an expansion in generalized Laguerre polynomials:

$$F(s) = \sum_{n=0}^{-\infty} b_n L_n^{(\alpha)}(s), \qquad (0 \leqslant s) \tag{31}$$

which are defined by:

$$L_n^{(\alpha)}(s) = \frac{s^{-\alpha} e^s}{n!} \frac{d^n}{ds^n} [s^{n+\alpha} e^{-s}], \ (n=1, 2, 3, \ldots);$$
$$\text{and } L_0^{(\alpha)}(s) = 1, \qquad (a > -1) \tag{32}$$

with orthogonality relations:

$$\int_0^{+\infty} e^{-s} s^\alpha L_m^{(\alpha)}(s) L_n^{(\alpha)}(s) ds = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{m,n}, \qquad (a > -1) \tag{33}$$

then we obtain a similar but a little complicated expression corresponding to (30).

[Example 41

We shall take Mellin transform for (1) and (2) :

$$F(s) = \int K(s, t) f(t) dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-t} f(t) dt, \tag{34}$$

$$f(t) = \int \widetilde{K}(t, s) F(s) ds = \int_0^{+\infty} s^{t-1} F(s) ds. \tag{35}$$

By means of the shifted Legendre functions $P_n^*(s) = P_n(2s-1)$, which are orthogonal in the interval $0 \leqslant s \leqslant 1$, we have an expansion for (4) as follows:

$$F(s) = \sum_{n=0}^{+\infty} a_n P_n^*(s) = \sum_{n=0}^{+\infty} a_n P_n(2s-1), \qquad (0 \leqslant s \leqslant 1) \tag{36}$$

with P_n (z) Legendre functions, and

$$a_n = (2n+1) \int_0^1 F(s) P_n^*(s) ds = (2n+1) \int_0^1 F(s) P_n(2s-1) ds. \ (n = \text{integers}) \tag{37}$$

Accordingly, the expression (6) becomes

$$f(t) = \sum_{n=0}^{+\infty} \left[(2n+1) \int_0^1 F(s) P_n^*(s) ds \right] \cdot \int_0^1 s^{t-1} P_n^*(s) ds$$

$$= \sum_{n=0}^{+\infty} \left[(2n+1) \int_0^1 F(s) P_n(2s-1) ds \right] \cdot \int_0^1 s^{t-1} P_n(2s-1) ds$$

$$= \sum_{n=0}^{+\infty} \left[(2n+1) \sum_{r=0}^{\leqslant n/2} (-)^r \frac{(2n-2r)!}{2^n \cdot r! (n-r)! (n-2r)!} \cdot \int_0^1 F(s) (2s-1)^{n-2r} ds \right] \times$$

$$\times \left[\sum_{r=0}^{\leqslant n/2} (-)^r \frac{(2n-2r)!}{2^n \cdot r! (n-r)! (n-2r)!} \cdot \int_0^1 s^{t-1} (2s-1)^{n-2r} ds \right], \tag{38}$$

where

$$\int_0^1 F(s)\,(2s-1)^{n-2r}ds=\sum_{k=0}^{n-2r}(-)^k\binom{n-2r}{k}\,2^{n-2r-k}\cdot f(n-2r-k+1),$$

and

$$\int_0^1 s^{t-1}(2s-1)^{n-2r}ds=\sum_{k=0}^{n-2r}(-)^k\binom{n-2r}{k}\,2^{n-2r-k}\cdot\frac{1}{t+n-2r-k}.$$

[Example 5]

If the function $F(s)$ can be expanded by an orthogonal set of functions $\{\phi_n(s);\ n=0,1,2,\ldots\}$ in the interval $\alpha\leqslant s\leqslant\beta$ with respect to the weight function $w(s)$, i.e. if

$$F(s)=\sum_n a_n\phi_n(s), \tag{39}$$

$$\int_\alpha^\beta w(s)\phi_m(s)\phi_n(s)ds=\delta_{m,n}, \tag{40}$$

and

$$a_n=\int_\alpha^\beta w(s)\phi_n(s)F(s)\,ds, \tag{41}$$

then it can be readily shown that under the assumption (7) we have

$$f(t)=\sum_{n=0}^{+\infty}f\left(\frac{n}{r}\right)\cdot\int_\alpha^\beta\widetilde{K}(t,s)w(s)\phi_n(s)\,ds, \tag{42}$$

instead of (9), the original $F(s)$ in (1) and (2) being replaced by $w(s)F(s)$.

If the Mellin transform is taken and Laguerre polynomials $\phi_n(s)=L_n(s)$ are used to expand F(s), with $w(s)=e^{-s}$ in the interval $0\leqslant s<+\infty$, we have

$$\begin{aligned}
f(t)&=\int_0^{+\infty}e^{-s}s^{t-1}F(s)\,ds\\
&=\sum_{n=0}^{+\infty}\left[\int_0^{+\infty}e^{-s}L_n(s)F(s)ds\right]\cdot\left[\int_0^{+\infty}e^{-s}s^{t-1}L_n(s)\,ds\right]\\
&=\sum_{n=0}^{+\infty}\left[\sum_{r=0}^{n}(-)^r\frac{1}{r!}\binom{n}{r}\cdot\int_0^{+\infty}e^{-s}s^rF(s)\,ds\right]\cdot\left[\sum_{r=0}^{n}(-)^r\frac{1}{r!}\binom{n}{r}\times\right.\\
&\quad\left.\times\int_0^{+\infty}e^{-s}s^{t-r-1}\,ds\right]\\
&=\Gamma(t)\cdot\sum_{n=0}^{+\infty}\left[\sum_{r=0}^{n}(-)^r\frac{1}{r!}\binom{n}{r}\cdot f(r+1)\right]\cdot\left[\sum_{r=0}^{n}(-)^r\frac{1}{r!}\binom{n}{r}\times\right.\\
&\quad\left.\times(t+r-1)(t+r-2)\ldots t\right],
\end{aligned} \tag{43}$$

where $\Gamma(t)$ represents the gamma function.

[Example 6]

Similarly, if Jacobi's polynomials $\phi_n(s)=G_n(p,q,s)/\sqrt{h_n}$ with $(1-s)^{p-q}s^{q-1}$, are used to expand $F(s)$ in the interval $0\leqslant s\leqslant 1$, *i. e.*

$$F(s)=\sum_n a_n \frac{G_n(p,q,s)}{\sqrt{h_n}}, \qquad (0\leqslant s\leqslant 1)$$

and

$$a_n=\int_0^1 w(s)\frac{G_n(p,q,s)}{\sqrt{h_n}}F(s)\,ds,$$

then we obtain

$$f(t)=\int_0^1 (1-s)^{p-q}s^{q-1}s^{t-1}F(s)\,ds$$

$$=\sum_{n=0}^{+\infty}\frac{1}{h_n}\left[\int_0^1 (1-s)^{p-q}s^{q-1}G_n(p,q,s)F(s)\,ds\right]\times$$

$$\times\left[\int_0^1 (1-s)^{p-q}s^{q-1}s^{t-1}G_n(p,q,s)\,ds\right]$$

$$=\sum_{n=0}^{+\infty}\frac{1}{h_n}\frac{\Gamma^2(q+n)}{\Gamma^2(p+2n)}\left[\sum_{r=0}^{n}(-)^r\binom{n}{r}\frac{\Gamma(p+2n-r)}{\Gamma(q+n-r)}\cdot f(n-r+1)\right]\times$$

$$\times\left[\sum_{r=0}^{+\infty}(-)^r\binom{n}{r}\frac{\Gamma(p+2n-r)}{\Gamma(q+n-r)}\bullet B(t+n+q-r-1,\ p-q+1)\right],$$

by means of the orthogonality relation:

$$\int_0^1 (1-s)^{p-q}s^{q-1}G_m(p,q,s)G_n(p,q,s)\,ds=h_n\delta_{m,n}, \qquad (p-q>-1,\ q>0)$$

with

$$h_n=\frac{n!}{(2n+p)}\frac{\Gamma(n+p)\Gamma(n+q)\Gamma(n+p-q-1)}{\Gamma^2(p+2n)},$$

$$B(x,\ y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$