行政院國家科學委員會補助專題研究計畫成果報告

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※ 交互作用粒子系統的流力極限 (5) ※

※ HYDRODYNAMIC LIMIT OF ※

※ INTERACTING PARTICLE SYSTEMS (5) ※

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計畫類別：■個別型計畫　□整合型計畫

計畫編號：NSC 90－2115－M－002－007

執行期間： 90 年 8 月 1 日至 91 年 7 月 31 日

計畫主持人：張志中

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■赴國外出差或研習心得報告一份

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□國際合作研究計畫國外研究報告書一份

執行單位：台灣大學數學系

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行政院國家科學委員會專題研究計畫成果報告

交互作用粒子系統的流力極限 (5)

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Email: ccchang@math.ntu.edu.tw

一、中文摘要

我們建立一個次流力尺度的超指數型估計以助於導出 2 維格子點空間 Z^2 上對稱簡單互斥過程中一個位置被粒子佔據的時間的大離差估計。此外，我們也對數個位置被粒子佔據的時間差的大離差估計做了初步地探討。

關鍵詞：對稱簡單互斥過程、粒子佔據的時間（差）、大離差估計、超指數型估計

Abstract

We establish a sub-hydrodynamic scaling super-exponential estimate, which is then applied to derive the occupation time large deviations of two-dimensional symmetric simple exclusion process (SEP). Furthermore, we take the initiative in studying the large deviations of occupation time difference of SEP.

Keywords: symmetric simple exclusion process, occupation time (difference), large deviations estimate, super-exponential estimate

二、緣由與目的

Consider the nearest neighbor symmetric simple exclusion process (SEP) on two-dimensional lattice Z^2. The configurations of this process are denoted by the Greek letter η so that $\eta(x)$ is equal to 1 or 0 if site x in Z^2 is occupied or not for η.

For each α in [0,1], denote by $\nu(\alpha)$ the Bernoulli product measure on the configuration space Ω with marginals given by

$$\nu(\alpha)\{\ \eta,\ \eta(x)=1\ \}=\alpha$$

for x in Z^2. A simple computation shows that $\{\nu(\alpha),\ 0\leqq\alpha\leqq 1\}$ is a one-parameter family of reversible invariant measures. In this project we study SEP accelerated by T starting from the reversible measure $\nu(\alpha)$ for a fixed α in (0,1).

Consider the occupation time of the origin:

$$V_t = \left(\int_0^t \eta_s(0)\,ds\right)/t\ .$$

It has been shown that V_t converges in probability to α as $t\to\infty$ and that $c(t)(V_t-\alpha)$ converges in distribution to a non-degenerate mean zero Gaussian variable, where c(t) is equal to the square root of t/log t ([3]). Landim ([4]) proved a large deviations principle for V_t in dimension $d\neq 2$ and that in dimension 2 the correct order is t/log t. To solve the case when d=2 we establish a sub-hydrodynamic scaling super-exponential estimate ([1]).

Furthermore, we make an initial investigation on the large deviations of occupation time difference of SEP ([5]).

三、結果與討論

The results stated in this report are taken from joint works [1] and [5] with C. Landim and T.Y. Lee.

Given $T > 0$, on the configuration space $\Omega = \{0,1\}^{\mathbb{Z}^2}$, consider the *accelerated* symmetric simple exclusion process (SEP) generated by L_T given by

$$(L_T f)(\eta) = \frac{T}{2} \sum_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|=1}} [f(\sigma^{x,y}\eta) - f(\eta)] ,$$

where the summation is carried over all nearest neighbor sites $x, y, |x-y| = 1$, of $\mathbb{Z}^2$. In this formula, f is a local function and $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if } z \neq x,\, y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x\,. \end{cases}$$

For each $0 \leq \alpha \leq 1$, denote by ν_α the Bernoulli product measure on Ω with marginals given by

$$\nu_\alpha\{\eta,\, \eta(x) = 1\} = \alpha$$

for $x \in \mathbb{Z}^2$. Clearly, $\{\nu_\alpha, 0 \leq \alpha \leq 1\}$ is a one-parameter family of reversible invariant measures. For $0 \leq \alpha \leq 1$, denote by $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ the probability on the path space $D(\mathbb{R}_+, \Omega)$ corresponding to SEP starting from ν_α.

Define the occupation time of the origin:

$$O_T = \int_0^1 \eta_s(0)\, ds\, .$$

The ultimate goal is to establish the large deviation principle of O_T under $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ as $T \to \infty$, which has been done partially in the project of last year (NSC-89-2115-M-002-022). It turns out that one needs to study the polar empirical measure $\mu^{1,T} = \mu^{1,T}(\eta)$ on $\mathbb{R}_+$ defined for any given configuration η by

$$\mu^{1,T}(\eta) = \frac{1}{2\pi \log T} \sum_{x\in\mathbb{Z}^2_*} \eta(x) \frac{1}{|x|^2} \delta_{\sigma_T(x)} ,$$

where $\sigma_T(x) = \log|x|/\log T$, δ_r is the Dirac measure concentrated on r and $\mathbb{Z}^2_* = \mathbb{Z}^2 - \{0\}$. The strategy is to on the one hand derive the large deviations estimate of $\mu^{1,T}$, and on the other hand show that O_T and the time average of $\mu^{1,T}$−measure of the interval $[0,\varepsilon]$ are super-exponentially close in the scale $T/\log T$:

Lemma 3.1 *For any $\delta > 0$ and $t > 0$,*

$$\limsup_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha \Big[\Big| \int_0^t ds W(\eta_s, T, \varepsilon) \Big| > \delta \Big] = -\infty, \tag{3.1}$$

where

$$W(\eta_s, T, \varepsilon) = \eta_s(0) - \mu^{1,T}(\eta_s)([0,\varepsilon]).$$

Now it is clear that through a contraction argument we may yield the large deviations estimate of O_T.

However, the proof of the large deviations principle for the polar empirical measure relies on a super-exponential estimate, similar to Lemma 3.1, which permits to replace (average of) local functions by functions of the polar empirical density. To state this result, we need some notation.

For $r > 0$, θ in $[0, 2\pi)$, $0 < \varepsilon < r$ and a configuration η, let

$$\iota_+ = \iota_+(\varepsilon, r, T) = \frac{1}{\log T} \log \frac{T^r + T^\varepsilon}{T^r} ,$$
$$\iota_- = \iota_-(\varepsilon, r, T) = \frac{1}{\log T} \log \frac{T^r}{T^r - T^\varepsilon}$$

and denote by $\Psi^{r,\theta}_{\varepsilon,T} : \mathbb{R}_+ \times [0, 2\pi) \to \mathbb{R}_+$ the function defined by

$$\Psi^{r,\theta}_{\varepsilon,T}(r', \theta') = \frac{\mathbf{1}\{|\theta' - \theta| \leq q(\varepsilon)\}}{2(\iota_+ + \iota_-)q(\varepsilon)} \times \mathbf{1}\{r - \iota_- \leq r' \leq r + \iota_+\}\, .$$

Denote, furthermore, by $M^{r,\theta}_{T,\varepsilon}(\eta)$ the average number of particles in the polar cube

$[r-\iota_-, r+\iota_+] \times [\theta - q(\varepsilon), \theta + q(\varepsilon)]$:

$$M^{r,\theta}_{T,\varepsilon}(\eta) = \ll \Psi^{r,\theta}_{\varepsilon,T}, \mu^T \gg$$
$$= \frac{1}{2(\iota_+ + \iota_-)q(\varepsilon)\log T} \sum_{\substack{T^r - T^\varepsilon \le |z| \le T^r + T^\varepsilon \\ \theta - q(\varepsilon) \le \Theta(z) \le \theta + q(\varepsilon)}} \frac{\eta(z)}{|z|^2}.$$

In the previous formula, the sum is carried over all sites z in $\mathbb{Z}^2_*$ satisfying $T^r - T^\varepsilon \le |z| \le T^r + T^\varepsilon$, $\theta - q(\varepsilon) \le \Theta(z) \le \theta + q(\varepsilon)$. Notice that $(\iota_+ + \iota_-)\log T = \log\{T^r + T^\varepsilon / T^r - T^\varepsilon\}$ so that the previous sum is an average up to smaller order terms.

Denote by $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$. Fix a continuous function $H\colon \mathbb{R}_+ \to \mathbb{R}$ with compact support in $(0, 1/2)$, $1 \le j \le 2$, $\varepsilon > 0$ and let

$$W^{H,\varepsilon}_{T,\pm e_j}(\eta) = \frac{1}{\log T} \sum_{1 < |x| < T^{1/2}} \frac{H(\sigma_T(x))}{|x|^2} \times \Big\{ \eta(x)\eta(x \pm e_j) \frac{(x \cdot e_j)^2}{|x|^2} - \Big(M^{\sigma_T(x),\Theta(x)}_{T,\varepsilon}(\eta) \Big)^2 \Big\} .$$

Lemma 3.2 (super-exponential estimate) *For any* $\delta > 0$, $j = 1, 2$ *and* $t > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log T}{T} \log \mathbb{P}_\alpha \Big[\Big| \int_0^t ds\, W^{H,\varepsilon}_{T,\pm e_j}(\eta_s) \Big| > \delta \Big] = -\infty.$$

Proof: The same proof of Lemma 3.1 gives that for any $0 < a < 1/4$,

$$\limsup_{\varepsilon \to 0} \limsup_{T \to \infty} \sup_{T^a \le |x| \le T^{1/2 - a}} \frac{\log T}{T} \log \mathbb{P}_\alpha \Big[\Big| \int_0^t ds\, Y(\eta_s, x, T, \varepsilon) \Big| > \delta \Big] = -\infty,$$

where

$$Y(\eta_s, x, T, \varepsilon) = \eta_s(x) - M^{\sigma_T(x),\Theta(x)}_{T,\varepsilon}(\eta_s) .$$

Introducing intermediary terms, as in Lemma 3.2 of [4], we may deduce the statement of the lemma from this result. □

Remark 3.3 $M^{\sigma_T(x),\Theta(x)}_{T,\varepsilon}(\eta)$ *is not an average over a macroscopic polar cube. This lemma is thus not replacing local functions by empirical density over macroscopic regions and must be interpreted as a super-exponential one-block estimate. However, we will see that almost no technical tools are needed in the proof of the lower bound and that a convexity argument for simple exclusion processes permits to go from microscopic boxes to macroscopic boxes in the proof of the upper bound.*

Let $\mathcal{V}$ be a local function on Ω satisfying $\nu_\alpha[\mathcal{V}(\eta)\,|\,\bar\eta] = 0$, where $\nu_\alpha[\cdot\,|\,\bar\eta]$ represents the ν_α-expectation conditioned on the average number of particles $\bar\eta$. Typical examples are $\eta(0) - \eta(e_1)$ and $\eta(e_1) + \eta(-e_1) + \eta(e_2) + \eta(-e_2) - 4\eta(0)$.

Denote the occupation time difference $\mathcal{V}_T$ associated with $\mathcal{V}$ by

$$\mathcal{V}_T = \sqrt{\log T} \Big(\int_0^1 \mathcal{V}(\eta_s)\, ds \Big) \in \mathbb{R} .$$

The large deviations principle of the joint distribution $(O_T, \mathcal{V}_T)$ can be proved, basically, by the methed similar to the one given in [1] with some natural modifications. Moreover, with some heuristic argument, we believe that under $\mathbb{P}_\alpha$ the occupation time O_T and its difference $\mathcal{V}_T$ obey a large deviations principle with rate function

$$\mathbf{I}_\alpha(c, d) = \frac{d^2}{32c(1-c)} + \Upsilon_\alpha(c) .$$

where Υ_α is the rate function of O_T ([1]) given by

$$\Upsilon_\alpha(c) = \frac{\pi}{4} \Big[\sin^{-1}(2c - 1) - \sin^{-1}(2\alpha - 1) \Big]^2,$$

which is ∞ when $c < 0$, or $c > 1$. Of course, a rigorous treatment remains to be done and this will be the main concern of the project of next year.

四、計畫成果自評

The large deviations results obtained in this project are summarized in [1] and [5]. [1] has been submitted, and [5] remains to be completed.

五、參考文獻

[1] Chang, C.C., Landim, C., Lee, T.Y.: Occupation time large deviations of two dimensional symmetric simple exclusion process. Submitted. (2001)

[2] Kipnis, C., Olla, S., and Varadhan, S.R.S.: Hydrodynamics and large deviations for simple exclusion processes. Comm. Pure Appl. Math. **XLII**, 115--137 (1989)

[3] Kipnis C., Varadhan S. R. S.; Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusion. Commun. Math. Phys. **106**, 1--19. (1986)

[4] Landim, C.: Occupation time large deviations for the symmetric simple exclusion process. Ann. Probab. **20**, 206--231 (1992)

[5] Chang, C.C., Landim, C., Lee, T.Y.: In preparation. (2002)

國外差旅心得報告

在國科會的補助下，本人於九十一年三月二十九日啓程前往馬里蘭大學訪問李宗祐教授。當地同日下午抵達華盛頓機場，再轉乘地鐵抵達馬里蘭大學。

三月三十日（週六）晨至數學系安排給我的研究室，便隨即與李宗祐教授展開討論。我們先前已合作了一篇論文導出格子點空間 Z^2 上對稱互斥過程中一個位置 (one site) 被粒子佔據的時間 (occupation time) 的大離差估計 (large deviation estimate)。這次我們決定延續前篇的成果與方法，去研究二個或數個位置被粒子佔據的時間的差異 (occupation time difference) 的大離差估計。

此後數日，我們就機率的方法、與微分方程的方法對這個問題的看法與解法的各種可能性進行各個方向的深入探討，獲得一些重要的成果，並已開始著手撰寫論文。

四月九日（週二）本人飛離馬里蘭大學，於四月十日晚返回台灣。