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SPINOR MAPS FOR SYMMETRIES IN FOUR DYNAMICAL SYSTEMS

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Abstract: In several applications the modeling dynamical systems may have internal symmetries related in certain ways to the proper orthochronous Lorentz groups $SO_o(1,n)$, $n=2,3,5$. For such symmetries the spinor representations are very effective and economical, because the dynamical system $\dot{\boldsymbol{\alpha}} = \mathbf{B}\boldsymbol{\alpha}$ for the spinor representation is linear yet low dimensional (real, complex or quaternion two-dimensional). In this paper we consider the following four dynamical systems: $\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}$ for $\mathbf{X} \in \mathbb{R}^{(n+1)\times 1}$, $\dot{\mathbf{m}} = \mathbf{A}^s{}_s\mathbf{m} + \mathbf{A}^s{}_0 - (\mathbf{m}^{\mathrm{T}}\mathbf{A}^s{}_0)\mathbf{m}$ for $\mathbf{m} \in \mathbb{R}^{n\times 1}$, $\dot{\mathbf{n}} = \mathbf{B}\mathbf{n} - \mathrm{Re}(\bar{\mathbf{n}}^{\mathrm{T}}\mathbf{B}\mathbf{n})\mathbf{n}$ for $\mathbf{n} \in \mathbb{K}^{\ell\times 1}$, and $\dot{\boldsymbol{\alpha}} = \mathbf{B}\boldsymbol{\alpha}$ for $\boldsymbol{\alpha} \in \mathbb{K}^{\ell\times 1}$, where $\mathbf{A} \in so(1,n)$, $(\mathbf{A}^s{}_s)^{\mathrm{T}} = -\mathbf{A}^s{}_s$, $\mathbf{B} \in sl(\ell,\mathbb{K})$, and $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We find that the fundamental solutions of the first and second systems are elements of $SO_o(1,n)$ and its projective group $PSO_o(1,n)$, respectively, whereas those of the fourth and third systems are elements of $SL(\ell,\mathbb{K})$

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and its normalized image, respectively. These groups characterize internal symmetries inherent in the modeling dynamical systems. When $n = 2, 3, 5$ and $\ell = 2$ we convert one system to another, especially converting the first three systems to the fourth, which is a linear system for the two-dimensional real, complex or quaternion spinor $\boldsymbol{\alpha}$. The initial conditions and invariants of the systems are also addressed. Finally, we show that the solutions of the four systems can all be expressed in terms of spinor transformations.

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1. Introduction

Let us consider the following three time-varying (*i.e.* nonautonomous) dynamical systems:

$$\left\{\begin{array}{l} \dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \\ \mathbf{A} = \left[\begin{array}{cc} \mathbf{A}^s{}_s & \mathbf{A}^s{}_0 \\ \mathbf{A}^0{}_s & 0 \end{array}\right], \ (\mathbf{A}^s{}_s)^{\mathrm{T}} = -\mathbf{A}^s{}_s, \ \ \mathbf{A}^0{}_s = (\mathbf{A}^s{}_0)^{\mathrm{T}}, \\ \mathbf{A}^s{}_s \in \mathbb{R}^{n\times n}, \ \ \mathbf{A}^s{}_0 \in \mathbb{R}^{n\times 1}, \ \ \mathbf{X} \in \mathbb{R}^{(n+1)\times 1}, \\ n = 2, 3, 5, \end{array}\right. \tag{1.1}$$

$$\left\{\begin{array}{l} \dot{\mathbf{m}} = \mathbf{A}^s{}_s\mathbf{m} + \mathbf{A}^s{}_0 - (\mathbf{m}^{\mathrm{T}}\mathbf{A}^s{}_0)\mathbf{m}, \\ (\mathbf{A}^s{}_s)^{\mathrm{T}} = -\mathbf{A}^s{}_s, \ \ \mathbf{A}^s{}_s \in \mathbb{R}^{n\times n}, \ \ \mathbf{A}^s{}_0 \in \mathbb{R}^{n\times 1}, \\ \mathbf{m} \in \mathbb{R}^{n\times 1}, \ \ n = 2, 3, 5, \end{array}\right. \tag{1.2}$$

$$\left\{\begin{array}{l} \dot{\mathbf{n}} = \mathbf{B}\mathbf{n} - \mathrm{Re}(\bar{\mathbf{n}}^{\mathrm{T}}\mathbf{B}\mathbf{n})\mathbf{n}, \\ \mathrm{tr}\,\mathbf{B} = 0, \ \ \|\mathbf{n}\| = 1, \ \ \mathbf{B} \in \mathbb{K}^{\ell\times\ell}, \ \ \mathbf{n} \in \mathbb{K}^{\ell\times 1}, \\ \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \ \ \ell = 2. \end{array}\right. \tag{1.3}$$

All the coefficients $\mathbf{A}$, $\mathbf{A}^s{}_s$, $\mathbf{A}^s{}_0$, and $\mathbf{B}$ are functions of time t. As usual, the overdot denotes the rate of change with respect to time t, the overbar the complex or the quaternion conjugate, the superscript $^{\mathrm{T}}$ the transpose, tr the trace, Re the real (scalar) part, and $\|\mathbf{n}\| := \sqrt{\bar{\mathbf{n}}^{\mathrm{T}}\mathbf{n}}$. The three dynamical systems have different levels of complexity. On the one hand, system (1.1) is linear, system (1.2) is nonlinear of the second degree, and system (1.3) is nonlinear of the third degree. On the other hand, system (1.1) is $(n+1)$-dimensional, system (1.2) is n-dimensional, and system (1.3) is (real, complex or quaternion) ℓ-dimensional.

It is well known that system (1.1) with dimension four $(n+1=4)$ is the so-called Lorentz equations in electrodynamics [1], while it is rarely known that two-dimensional Coulomb's dry friction between two relatively sliding planes can be modeled by system (1.2) with $n = 2$; see, *e.g.* [2]. System (1.2) also represents the constitutive equation of perfect elastoplasticity in plastic phase [3-5]. As regards system (1.3), it governs various models of structured media in which the anisotropic microstructure is characterized by a unit director and parameters of microstructure. For example, it was used in [6] to describe the orientation of anisotropic fluids, in [7] to govern the microstructural director in a transversely isotropic fluid, in [8] to describe the evolution of grain orientation in a double slip model for polycrystals, and in [9] to study the evolution and control of orientation of suspension particles.

These three dynamical systems seem to be closely related; to see it, let us recall a few examples. In [3] the nonlinear system (1.2), representing the constitutive law of perfect elastoplasticity, was exactly linearized to the linear system (1.1), thus revealing the internal symmetry inherent in the constitutive model and much facilitating calculations in engineering stress analysis. As shown in [3], system (1.2) is indeed a projective representation of system (1.1). In [9] the nonlinear system (1.3), representing the evolution of the suspension particle orientation, was also exactly linearized to the linear system (1.1). Thus it seems desirable to further examine the relations of these three dynamical systems in a more systematic way. In this paper we shall systematically explore their relations, and, more importantly, show that the three dynamical systems can all be reduced to

$$\left\{\begin{array}{l} \dot{\boldsymbol{\alpha}} = \mathbf{B}\boldsymbol{\alpha}, \\ \operatorname{tr}\mathbf{B} = 0, \quad \mathbf{B} \in \mathbb{K}^{\ell\times\ell}, \quad \boldsymbol{\alpha} \in \mathbb{K}^{\ell\times 1}, \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \quad \ell = 2. \end{array}\right. \tag{1.4}$$

This new dynamical system is, to the advantage, linear yet only two-dimensional. The $\boldsymbol{\alpha}$ is known as a spinor, real, complex or quaternion [10,11]. With potential applications in mind we shall write out explicit expressions for the relations among $\mathbf{X}$, $\mathbf{m}$, $\mathbf{n}$, and $\boldsymbol{\alpha}$, and also between $\mathbf{A}$ and $\mathbf{B}$. Both $\mathbf{A}$ and $\mathbf{B}$ are time-varying.

The paper is organized as follows. Section 2 studies systems (1.1) and (1.2), conversions between them, and their symmetry groups. Section 3 studies systems (1.4) and (1.3) with $\mathbb{K} = \mathbb{C}$, conversions between them, and their symmetry groups. Section 4 establishes system (1.4) with $\mathbb{K} = \mathbb{C}$ as spinor representations of the other systems. Section 5 investigates the real system (1.3) with a direct approach. Section 6 expresses the solutions of the four systems all in terms of spinor transformations. Then, taking advantage of the constraint of the state vector $\mathbf{X}$ we develop a mathematically equivalent quaternionic Hermitian matrix representation in Section 7, which is suitable for us to derive a spinor map from $SL(2,\mathbb{H})$ onto $SO_o(1,5)$ via an extension of Dirac's method. Section 8 converts system (1.1) with $n = 5$ to system (1.4) with $\mathbb{K} = \mathbb{H}$, and vice versa. Finally, we draw conclusions in Section 9.

2. Dynamical Systems (1.1) and (1.2)

In this section we study systems (1.1) and (1.2) with $n \geq 1$.

2.1. Conversion Between Systems (1.1) and (1.2)

First, suppose we are given system (1.1) for the unknown vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^s \\ X^0 \end{bmatrix} = \begin{bmatrix} X^1 \\ \vdots \\ X^n \\ X^0 \end{bmatrix}, \tag{2.1}$$

which is a function of time. From the properties of $\mathbf{A}$ stated in (1.1), it is clear that such $\mathbf{A}$ is an element of the Lie algebra $so(1,n)$ satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{g} + \mathbf{g}\mathbf{A} = \mathbf{0}, \tag{2.2}$$

where

$$\mathbf{g} = \begin{bmatrix} -\mathbf{I}_n & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1 \end{bmatrix} \tag{2.3}$$

is the Minkowski metric of the $(n+1)$-dimensional Minkowski spacetime $\mathbb{M}^{n+1} \ni \mathbf{X}$. Here $\mathbf{I}_n$ is the identity matrix of order n.

To convert (1.1) to (1.2) let us define

$$\mathbf{m} = \begin{bmatrix} m^1 \\ \vdots \\ m^n \end{bmatrix} := \frac{\mathbf{X}^s}{X^0} = \frac{1}{X^0} \begin{bmatrix} X^1 \\ \vdots \\ X^n \end{bmatrix}. \tag{2.4}$$

For this transformation to be realizable we require $X^0 \neq 0$, and for positive orientation of time we choose $X^0 > 0$. Thus, whenever given (1.1) with known $\mathbf{A}$, we can convert it into (1.2).

Conversely, suppose we are given system (1.2). The unknown vector

$$\mathbf{m} = \begin{bmatrix} m^1 \\ \vdots \\ m^n \end{bmatrix}$$

for the dynamical system is a function of time. Defining

$$\mathbf{X} = \begin{bmatrix} X^1 \\ \vdots \\ X^n \\ X^0 \end{bmatrix} := X^0 \begin{bmatrix} m^1 \\ \vdots \\ m^n \\ 1 \end{bmatrix}, \quad \frac{\dot{X}^0}{X^0} := \mathbf{m}^{\mathrm{T}}\mathbf{A}^{s}{}_{0}, \tag{2.5}$$

we can convert (1.2) to (1.1). Our earlier choice of $X^0 > 0$ conforms with the requirement $X^0 > 0$ in the left-hand side $\dot{X}^0/X^0$, that is $d(\ln X^0)/dt$, of the last equation of (2.5).

2.2. Solution and Internal Symmetry of System (1.1)

The solution of system (1.1) may be expressed in the following transition formula:

$$\mathbf{X}(t) = \mathbf{G}(t)\mathbf{X}(0), \quad \forall\, t \in I, \tag{2.6}$$

where $I := [0, w)$, $w \leq \infty$ is a time interval assigned to (1.1), and $\mathbf{G}(t)$, known as the fundamental solution of (1.1), is a transformation matrix satisfying

$$\dot{\mathbf{G}}(t) = \mathbf{A}(t)\mathbf{G}(t), \tag{2.7}$$

$$\mathbf{G}(0) = \mathbf{I}_{n+1}. \tag{2.8}$$

By (2.2) and (2.7) we find

$$\frac{d}{dt}[\mathbf{G}^{\mathrm{T}}(t)\mathbf{g}\mathbf{G}(t)] = \mathbf{0}.$$

At $t = 0$, $\mathbf{G}^{\mathrm{T}}(t)\mathbf{g}\mathbf{G}(t) = \mathbf{I}_{n+1}^{\mathrm{T}}\mathbf{g}\mathbf{I}_{n+1} = \mathbf{g}$ from (2.8); hence, we have proved that

$$\mathbf{G}^{\mathrm{T}}(t)\mathbf{g}\mathbf{G}(t) = \mathbf{g} \tag{2.9}$$

for all $t \in I$. Take determinants of both sides of the above equation, yielding $(\det \mathbf{G})^2 = 1$, so that $\mathbf{G}$ is invertible. As usual, det denotes the determinant. The 00-th component of the matrix equation (2.9) is $(G^0{}_0)^2 - \sum_{i=1}^n (G^i{}_0)^2 = 1$, from which $(G^0{}_0)^2 \geq 1$. Here $G^i{}_j, i, j = 1, \cdots, n, 0$, is the ijth mixed component of the tensor $\mathbf{G}$. Since $\det \mathbf{G} = -1$ or $G^0{}_0 \leq -1$ would violate (2.8), it turns out that

$$\det \mathbf{G} = 1, \tag{2.10}$$

$$G^0{}_0 \geq 1. \tag{2.11}$$

Hence, in view of the three characteristic properties explicitly expressed by (2.9)-(2.11), we conclude that the fundamental solution $\mathbf{G}$ of system (1.1) belongs to the proper orthochronous Lorentz group $SO_o(1, n)$, which thus characterizes completely the internal symmmetry inherent in system (1.1).

2.3. Initial Conditions and Invariants of System (1.1)

Given the initial condition $\mathbf{X}(0) = (X^1(0), \cdots, X^n(0), X^0(0))^{\mathrm{T}}$, we may calculate $\mathbf{X}^{\mathrm{T}}(0)\mathbf{g}\mathbf{X}(0) = (X^0(0))^2 - \sum_{i=1}^n (X^i(0))^2 =: r$ and obtain r. Then by (2.6) and (2.9) it is obvious that

$$\mathbf{X}^{\mathrm{T}}(t)\mathbf{g}\mathbf{X}(t) = \mathbf{X}^{\mathrm{T}}(0)\mathbf{g}\mathbf{X}(0) = r, \quad \forall\, t \in I. \tag{2.12}$$

Thus $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X}$ is an invariant of system (1.1). This simply reflects the fact that the proper orthochronous Lorentz group of transformations in Minkowski spacetime $\mathbb{M}^{n+1}$ preserves the Minkowski separation $(X^0(t))^2 - \sum_{i=1}^n (X^i(t))^2$ for all $t \in I$. Note that $(X^0)^2 - \sum_{i=1}^n (X^i)^2 = r$ is an equation of a hyperboloid — an n-dimensional pseudo-Riemannian submanifold of constant curvature which admits the Minkowski metric. The invariance indicates that all $\mathbf{X}(t)$ $\forall t \in I$ will stay in the same hyperboloid as the initial $\mathbf{X}(0)$ does. If the constant $r = 0$, the hyperboloid is a cone, while if $r > 0$, it represents two copies of Minkowskian spheres, and if $r < 0$, it is a hyperbolic space. Figure 1 shows a geometric representation of these three objects with $n = 2$. However, since $X^0 > 0$ we must select the upper branches of them as the underlying spaces of system (1.1).

2.4. Solution and Internal Symmetry of System (1.2)

Corresponding to the symmetry group $SO_o(1,n) \ni \mathbf{G}$ of system (1.1), the symmetry group of the nonlinear system (1.2) is the projection of $SO_o(1,n)$, namely the group $PSO_o(1,n)$. Partition $\mathbf{G}$ as

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}^s{}_s & \mathbf{G}^s{}_0 \\ \mathbf{G}^0{}_s & G^0{}_0 \end{bmatrix},$$

where $\mathbf{G}^s{}_s$, $\mathbf{G}^s{}_0$ and $\mathbf{G}^0{}_s$ are of order $n \times n$, $n \times 1$, and $1 \times n$, respectively. Thus, the solution $\mathbf{m}(t)$ is

$$\mathbf{m} = \frac{\mathbf{G}^s{}_s\mathbf{m}(0) + \mathbf{G}^s{}_0}{\mathbf{G}^0{}_s\mathbf{m}(0) + G^0{}_0}, \tag{2.13}$$

where $\mathbf{m}(0)$ is the initial value.

2.5. Initial Conditions and Invariants of System (1.2)

From (1.2) it follows that

$$\frac{d}{dt}\|\mathbf{m}\|^2 = 2\mathbf{m}^{\mathrm{T}}\mathbf{A}^s{}_0(1 - \|\mathbf{m}\|^2),$$

the solution of which is

$$\|\mathbf{m}(t)\|^2 = (\|\mathbf{m}(0)\|^2 - 1)\exp\left(-2\int_0^t \mathbf{m}^{\mathrm{T}}(\eta)\mathbf{A}^s{}_0(\eta)d\eta\right) + 1.$$

It is obvious that $\|\mathbf{m}(t)\| < 1 \ \ \forall\, t \geq 0$ when $\|\mathbf{m}(0)\| < 1$, that $\|\mathbf{m}(t)\| > 1 \ \ \forall\, t \geq 0$ when $\|\mathbf{m}(0)\| > 1$, and that $\|\mathbf{m}(t)\| = 1 \ \ \forall\, t \geq 0$ when $\|\mathbf{m}(0)\| = 1$. In particular, if given the initial condition $\mathbf{m}(0) = (m^1(0), \cdots, m^n(0))^{\mathrm{T}}$ such that $\|\mathbf{m}(0)\| = 1$, the projective map shown in (2.13) will leave the unit sphere $\|\mathbf{m}\| = 1$ invariant; in this case (2.13) is a sphere map, more precisely an $\mathbb{S}^{n-1}$-endomorphism [12,13], which maps the points in the sphere $\mathbb{S}^{n-1}$ to another points in the sphere $\mathbb{S}^{n-1}$.

3. Dynamical Systems (1.4) and (1.3) with $\mathbb{K} = \mathbb{C}$

In this section we study complex ℓ-dimensional systems (1.4) and (1.3) ($\ell \geq 2$, $\mathbb{K} = \mathbb{C}$, $\mathbf{B} \in \mathbb{C}^{\ell \times \ell}$, $\boldsymbol{\alpha}, \mathbf{n} \in \mathbb{C}^{\ell \times 1}$). The derivations and results apply also to the corresponding real cases ($\ell \geq 2$, $\mathbb{K} = \mathbb{R}$, $\mathbf{B} \in \mathbb{R}^{\ell \times \ell}$, $\boldsymbol{\alpha}, \mathbf{n} \in \mathbb{R}^{\ell \times 1}$).

3.1. Solution and Internal Symmetry of System (1.4)

The solution of system (1.4) may be expressed in the following transition formula:

$$\boldsymbol{\alpha}(t) = \mathbf{U}(t)\boldsymbol{\alpha}(0), \quad \forall\, t \in I, \tag{3.1}$$

where

$$\begin{aligned} \dot{\mathbf{U}}(t) &= \mathbf{B}(t)\mathbf{U}(t), & (3.2)\\ \mathbf{U}(0) &= \mathbf{I}_\ell. & (3.3) \end{aligned}$$

As given in (1.4), $\mathbf{B} \in \mathbb{C}^{\ell \times \ell}$ and $\operatorname{tr} \mathbf{B} = 0$, so $\mathbf{B} \in sl(\ell, \mathbb{C})$ and hence $\mathbf{U} \in SL(\ell, \mathbb{C})$, which means $\mathbf{U} \in \mathbb{C}^{\ell \times \ell}$ and $\det \mathbf{U} = 1$. Thus the special linear group $SL(\ell, \mathbb{C})$ charaterizes the internal symmetry of system (1.4).

3.2. Conversion Between Systems (1.4) and (1.3)

First, suppose we are given system (1.4) with $\mathbb{K} = \mathbb{C}$ for the unknown vector

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^\ell \end{bmatrix},$$

which is a function of time. From the properties of $\mathbf{B}$ stated in (1.4), it is clear that such $\mathbf{B}$ is an element of the Lie algebra $sl(\ell, \mathbb{C})$.

To convert (1.4) to (1.3) let us define

$$\mathbf{n} = \begin{bmatrix} n^1 \\ \vdots \\ n^\ell \end{bmatrix} := \frac{\boldsymbol{\alpha}}{\|\boldsymbol{\alpha}\|} = \frac{1}{\|\boldsymbol{\alpha}\|} \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^\ell \end{bmatrix}. \tag{3.4}$$

In order for this transformation to be realizable, we require $\|\boldsymbol{\alpha}\| > 0$. From (3.4) it follows that

$$\dot{\boldsymbol{\alpha}} = \|\boldsymbol{\alpha}\|\dot{\mathbf{n}} + \mathbf{n}\frac{d}{dt}\|\boldsymbol{\alpha}\|.$$

Substituting $\dot{\boldsymbol{\alpha}} = \mathbf{B}\boldsymbol{\alpha} = \mathbf{B}\|\boldsymbol{\alpha}\|\mathbf{n}$ into the above equation and then dividing by $\|\boldsymbol{\alpha}\|$, we get

$$\dot{\mathbf{n}} = \mathbf{B}\mathbf{n} - \left(\frac{d}{dt}\ln\|\boldsymbol{\alpha}\|\right)\mathbf{n}. \tag{3.5}$$

Due to $\|\mathbf{n}\| = 1$, we have

$$\bar{\mathbf{n}}^{\mathrm{T}}\dot{\mathbf{n}} + \dot{\bar{\mathbf{n}}}^{\mathrm{T}}\mathbf{n} = 0.$$

Substituting (3.5) and its transposed complex conjugate into the above equation, we obtain

$$\frac{d}{dt}\ln\|\boldsymbol{\alpha}\| = \mathrm{Re}(\bar{\mathbf{n}}^{\mathrm{T}}\mathbf{B}\mathbf{n}), \tag{3.6}$$

by which (3.5) becomes system (1.3). Thus we have converted system (1.4) into system (1.3).

Conversely, suppose we are given system (1.3) with $\mathbb{K} = \mathbb{C}$. The unknown vector

$$\mathbf{n} = \begin{bmatrix} n^1 \\ \vdots \\ n^\ell \end{bmatrix}$$

for the dynamical system is a function of time. Defining

$$\boldsymbol{\alpha} := \|\mathbf{U}\boldsymbol{\alpha}(0)\|\mathbf{n}, \tag{3.7}$$

we can convert (1.3) to (1.4). In the above $\mathbf{U}$ is the fundamental solution of (1.4) and so satisfies (3.2).

3.3. Solution and Internal Symmetry of System (1.3)

Substituting (3.1) into (3.7) then dividing both the sides by $\|\mathbf{U}\boldsymbol{\alpha}(0)\|$ and noting (3.4) we obtain the solution of $\mathbf{n}(t)$ as follows:

$$\mathbf{n} = \frac{\mathbf{U}\mathbf{n}(0)}{\|\mathbf{U}\mathbf{n}(0)\|}, \tag{3.8}$$

where $\mathbf{n}(0)$ is the initial value. This formula indicates that the internal symmetry of dynamical system (1.3) with $\mathbb{K} = \mathbb{C}$ is characterized by the normalized image of $SL(\ell, \mathbb{C})$.

3.4. Initial Conditions and Invariants

If tentatively ignoring $\|\mathbf{n}\| = 1$ and considering only the rest of (1.3) we can obtain

$$\frac{d}{dt}\|\mathbf{n}\|^2 = 2\mathrm{Re}(\bar{\mathbf{n}}^{\mathrm{T}}\mathbf{B}\mathbf{n})(1 - \|\mathbf{n}\|^2), \tag{3.9}$$

from which it is obvious that $\|\mathbf{n}(t)\| = 1 \;\; \forall\, t \geq 0$ when $\|\mathbf{n}(0)\| = 1$. Here for the the modeling dynamic system (1.3), $\mathbf{n}$ is by definition unimodular. (Recall that in some applications cited in Section 1, $\mathbf{n} \in \mathbb{R}^{\ell\times 1}$, $\ell = 2, 3$, represented the directors of structured media.) Thus $\mathbb{S}^{2\ell-1} = \{\|\mathbf{n}\| = 1 \mid \mathbf{n} \in \mathbb{C}^{\ell\times 1}\}$ is an invariant manifold for system (1.3) with $\mathbb{K} = \mathbb{C}$.

Unlike system (1.3), system (1.4) does not preserve the length of the initial value $\boldsymbol{\alpha}(0)$, because all we can do is to take advantage of the property $\det \mathbf{U} = 1$ (or $\mathrm{tr}\, \mathbf{B} = 0$), which leads us to consider the determinant of $\boldsymbol{\alpha}(t)\boldsymbol{\alpha}^{\mathrm{T}}(t) = \mathbf{U}(t)\boldsymbol{\alpha}(0)\; \boldsymbol{\alpha}^{\mathrm{T}}(0)\mathbf{U}^{\mathrm{T}}(t)$, obtaining a trivial equation $0 = (\det \mathbf{U})0(\det \mathbf{U})$. Note that the determinant of the tensor product of two vectors is always zero such as $\det(\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{T}}) = 0$ here. It is hardly surprising that system (1.4) is so simple — linear and *low*-dimensional — as to find *no* invariant structure except to preserve volume forms, namely $\det \mathbf{U} = 1$.

4. System (1.4) as Spinor Representations

The symmetry group in Section 2 is specialized to $SO_o(1,3)$ for system (1.1) with $n = 3$, while the symmetry group in Section 3 is specialized to $SL(2, \mathbb{C})$ for system (1.4) with $\ell = 2$ and $\mathbb{K} = \mathbb{C}$. In this section we intend to establish system (1.4) with $\ell = 2$ and $\mathbb{K} = \mathbb{C}$ as a complex spinor representation of system (1.1) with $n = 3$, for which we are therefore led to study the relations of the symmetry groups $SL(2, \mathbb{C})$ and $SO_o(1,3)$ and also the relations of their Lie algebras $sl(2, \mathbb{C})$ and $so(1,3)$. Once we establish system (1.4) to be a spinor representation of system (1.1), it is straightforward to obtain the relations between systems (1.4) and (1.2) via the conversion relations set up in Section 2. Thus, if the relations obtained in Sections 2 through 4 are summarized, systems (1.1), (1.2), and (1.3) will all have the very economical spinor representation (1.4). In the final subsection of the section we briefly treat the corresponding real spinor case with $n=2$, $\mathbb{K}=\mathbb{R}$ and $\ell=2$.

4.1. Algebraic Relations of Fundamental Solutions

To relate the fundamental solutions of the four-dimensional system (1.1) and the complex system (1.4) this subsection presents an algebraic approach to the spinor map from $SL(2,\mathbb{C})$ onto $SO_o(1,3)$ (see, *e.g.*, [14-16]).

Any 2×2 (complex) Hermitian matrix $\mathbf{H}$ can be written as

$$\mathbf{H}=\begin{bmatrix} X^0+X^1 & X^2-iX^3 \\ X^2+iX^3 & X^0-X^1 \end{bmatrix}, \tag{4.1}$$

where X^0, X^1, X^2 and X^3 are real numbers, and $i^2:=-1$. The determinant of $\mathbf{H}$ is the Minkowski separation $(X^0)^2-(X^1)^2-(X^2)^2-(X^3)^2$.

The group $SL(2,\mathbb{C})$ is (represented by) the set of all complex 2×2 matrices $\mathbf{U}$ with unit determinant. Elements of $SL(2,\mathbb{C})$ are often called spin transformations. Hence $\mathbf{U}\in SL(2,\mathbb{C})$ is a spin transformation. Since $\mathbf{H}$ is Hermitian, it is obvious that $\mathbf{UH\bar{U}}^{\mathrm{T}}$ is also a 2×2 Hermitian matrix. This leads us to write

$$\hat{\mathbf{H}}=\mathbf{UH\bar{U}}^{\mathrm{T}}. \tag{4.2}$$

Thus we may write

$$\begin{aligned}&\begin{bmatrix} \hat{X}^0+\hat{X}^1 & \hat{X}^2-i\hat{X}^3 \\ \hat{X}^2+i\hat{X}^3 & \hat{X}^0-\hat{X}^1 \end{bmatrix}\\ &\quad=\begin{bmatrix} U^1_{\ 1} & U^1_{\ 2} \\ U^2_{\ 1} & U^2_{\ 2} \end{bmatrix}\begin{bmatrix} X^0+X^1 & X^2-iX^3 \\ X^2+iX^3 & X^0-X^1 \end{bmatrix}\begin{bmatrix} \bar{U}^1_{\ 1} & \bar{U}^2_{\ 1} \\ \bar{U}^1_{\ 2} & \bar{U}^2_{\ 2} \end{bmatrix}. \end{aligned}\tag{4.3}$$

Taking the determinant of both sides and using $\det\mathbf{U}=\det\bar{\mathbf{U}}^{\mathrm{T}}=1$, one readily obtains $(\hat{X}^0)^2-(\hat{X}^1)^2-(\hat{X}^2)^2-(\hat{X}^3)^2=(X^0)^2-(X^1)^2-(X^2)^2-(X^3)^2$, ensuring that the Minkowski separation of the $(3+1)$-vector $\mathbf{X}=(X^1,X^2,X^3,X^0)^{\mathrm{T}}$ is preserved by the spin transformation

$\mathbf{U}:\mathbf{H}\longmapsto\hat{\mathbf{H}}$. Indeed, the transformation $\mathbf{U}:\mathbf{H}\longmapsto\hat{\mathbf{H}}$ induces a proper orthochronous Lorentz transformation $\mathbf{G}:\mathbf{X}\longmapsto\hat{\mathbf{X}}$, which is an element of $SO_o(1,3)$. Equation (4.1) may be rearranged as

$$\begin{bmatrix} H_{11} \\ H_{12} \\ H_{21} \\ H_{22} \end{bmatrix} = \mathbf{C} \begin{bmatrix} X^1 \\ X^2 \\ X^3 \\ X^0 \end{bmatrix}, \tag{4.4}$$

where

$$\mathbf{C} := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

Then (4.3) is equivalent to

$$\begin{bmatrix} \hat{H}_{11} \\ \hat{H}_{12} \\ \hat{H}_{21} \\ \hat{H}_{22} \end{bmatrix} = \mathbf{R} \begin{bmatrix} H_{11} \\ H_{12} \\ H_{21} \\ H_{22} \end{bmatrix}, \tag{4.6}$$

where

$$\mathbf{R} := \begin{bmatrix} U^1{}_1\bar{U}^1{}_1 & U^1{}_1\bar{U}^1{}_2 & U^1{}_2\bar{U}^1{}_1 & U^1{}_2\bar{U}^1{}_2 \\ U^1{}_1\bar{U}^2{}_1 & U^1{}_1\bar{U}^2{}_2 & U^1{}_2\bar{U}^2{}_1 & U^1{}_2\bar{U}^2{}_2 \\ U^2{}_1\bar{U}^1{}_1 & U^2{}_1\bar{U}^1{}_2 & U^2{}_2\bar{U}^1{}_1 & U^2{}_2\bar{U}^1{}_2 \\ U^2{}_1\bar{U}^2{}_1 & U^2{}_1\bar{U}^2{}_2 & U^2{}_2\bar{U}^2{}_1 & U^2{}_2\bar{U}^2{}_2 \end{bmatrix} = \mathbf{U}\otimes\bar{\mathbf{U}}, \tag{4.7}$$

where $\otimes$ denotes the Kronecker (tensor) product.

Applying

$$\mathbf{C}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{4.8}$$

to both sides of (4.6) and noting (4.4) yield

$$\hat{\mathbf{X}} = \mathbf{C}^{-1}\mathbf{RCX}. \tag{4.9}$$

Comparing with the proper orthochronous Lorentz transformation, $\hat{\mathbf{X}} = \mathbf{GX}$, we obtain

$$\mathbf{G} = \mathbf{C}^{-1}\mathbf{RC}. \tag{4.10}$$

Note that there are two spin transformations, namely $\pm\mathbf{U}$, which map to the same proper orthochronous Lorentz transformation $\mathbf{G}$, and hence $\pm\mathbf{U}$ are said to be a two-valued representation of $\mathbf{G}$. Indeed the procedure is a map from the group $SL(2,\mathbb{C})$ onto the group $SO_o(1,3)$ and is usually called a *spinor map*. The hierachy of the spinor maps included in $SL(2,\mathbb{C}) \longrightarrow SO_o(1,3)$ is shown in Figure 2; see also Subsection 4.4 and Appendix for related maps and systems. But, we still lack the conversion formulae from $\mathbf{B} \in sl(2,\mathbb{C})$ to $\mathbf{A} \in so(1,3)$, and vice versa. This will be formulated in the next subsection.

4.2. Conversions Between Complex system (1.4) and 4-D System (1.1)

In this subsection we convert a complex system (1.4) to a four-dimensional system (1.1), and vice versa. The conversion relations are indeed a Lie algebra isomorphism of $sl(2,\mathbb{C}) \ni \mathbf{B}$ onto $so(1,3) \ni \mathbf{A}$.

Parametrizing the spin transformation $\mathbf{U} \in SL(2,\mathbb{C})$, *cf.* (4.2),

$$\mathbf{H}(t) = \mathbf{U}(t)\mathbf{H}(0)\bar{\mathbf{U}}^{\mathrm{T}}(t), \tag{4.11}$$

differentiating (4.11) with respect to the parameter t, and using (3.2), we obtain

$$\begin{aligned}\begin{bmatrix} \dot{H}_{11} & \dot{H}_{12} \\ \dot{H}_{21} & \dot{H}_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \\ + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} \bar{B}_{11} & \bar{B}_{21} \\ \bar{B}_{12} & \bar{B}_{22} \end{bmatrix}.\end{aligned} \tag{4.12}$$

Also parametrizing (4.6) as

$$\begin{bmatrix} H_{11}(t) \\ H_{12}(t) \\ H_{21}(t) \\ H_{22}(t) \end{bmatrix} = \mathbf{R}(t) \begin{bmatrix} H_{11}(0) \\ H_{12}(0) \\ H_{21}(0) \\ H_{22}(0) \end{bmatrix}, \tag{4.13}$$

taking time derivative of (4.13) and then using (4.13) again, we have

$$\begin{bmatrix} \dot{H}_{11} \\ \dot{H}_{12} \\ \dot{H}_{21} \\ \dot{H}_{22} \end{bmatrix} = \dot{\mathbf{R}}\mathbf{R}^{-1} \begin{bmatrix} H_{11} \\ H_{12} \\ H_{21} \\ H_{22} \end{bmatrix}.$$

Comparing the above equation with (4.12) yields

$$\dot{\mathbf{R}}\mathbf{R}^{-1} = \begin{bmatrix} B_{11}+\bar{B}_{11} & \bar{B}_{12} & B_{12} & 0 \\ \bar{B}_{21} & B_{11}+\bar{B}_{22} & 0 & B_{12} \\ B_{21} & 0 & B_{22}+\bar{B}_{11} & \bar{B}_{12} \\ 0 & B_{21} & \bar{B}_{21} & B_{22}+\bar{B}_{22} \end{bmatrix} = \mathbf{B}\otimes\mathbf{I}_2 + \mathbf{I}_2\otimes\bar{\mathbf{B}}. \tag{4.14}$$

For the proper orthochronous Lorentz transformation $\mathbf{G} \in SO_o(1,3)$, taking the time derivative of (4.10), using (2.7), (4.14), and (4.10) again, we obtain

$$\mathbf{A} = \mathbf{C}^{-1} \begin{bmatrix} B_{11}+\bar{B}_{11} & \bar{B}_{12} & B_{12} & 0 \\ \bar{B}_{21} & B_{11}+\bar{B}_{22} & 0 & B_{12} \\ B_{21} & 0 & B_{22}+\bar{B}_{11} & \bar{B}_{12} \\ 0 & B_{21} & \bar{B}_{21} & B_{22}+\bar{B}_{22} \end{bmatrix} \mathbf{C}, \tag{4.15}$$

and the inverse,

$$\begin{bmatrix} B_{11}+\bar{B}_{11} & \bar{B}_{12} & B_{12} & 0 \\ \bar{B}_{21} & B_{11}+\bar{B}_{22} & 0 & B_{12} \\ B_{21} & 0 & B_{22}+\bar{B}_{11} & \bar{B}_{12} \\ 0 & B_{21} & \bar{B}_{21} & B_{22}+\bar{B}_{22} \end{bmatrix} = \mathbf{C}\mathbf{A}\mathbf{C}^{-1}. \tag{4.16}$$

Expanding formula (4.15) and using $B_{11}+B_{22}=0$, we have $\mathbf{A}$ from $\mathbf{B}$ as follows:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{10} \\ A_{21} & A_{22} & A_{23} & A_{20} \\ A_{31} & A_{32} & A_{33} & A_{30} \\ A_{01} & A_{02} & A_{03} & A_{00} \end{bmatrix} =$$
$$\begin{bmatrix} 0 & \mathrm{Im}\,(B_{22}-B_{11}) & \mathrm{Re}(B_{21}-B_{12}) & \mathrm{Re}(B_{21}+B_{12}) \\ -\,\mathrm{Im}\,(B_{22}-B_{11}) & 0 & -\,\mathrm{Im}\,(B_{21}+B_{12}) & \mathrm{Im}\,(B_{12}-B_{21}) \\ -\mathrm{Re}(B_{21}-B_{12}) & \mathrm{Im}\,(B_{12}+B_{21}) & 0 & \mathrm{Re}(B_{11}-B_{22}) \\ \mathrm{Re}(B_{21}+B_{12}) & \mathrm{Im}\,(B_{12}-B_{21}) & \mathrm{Re}(B_{11}-B_{22}) & 0 \end{bmatrix}, \tag{4.17}$$

where the prefixes Im and Re denote the imaginary and real parts, respectively. It is easy to see the 4×4 real matrix $\mathbf{A}$ as obtained in (4.17) takes the form

$$\mathbf{A} = \begin{bmatrix} 0 & A_{12} & A_{13} & A_{10} \\ -A_{12} & 0 & A_{23} & A_{20} \\ -A_{13} & -A_{23} & 0 & A_{30} \\ A_{10} & A_{20} & A_{30} & 0 \end{bmatrix} \tag{4.18}$$

and satisfies (2.2), where $\mathbf{g}$ defined by (2.3) with $n = 3$ is a Minkowski metric of the four-dimensional Minkowski spacetime $\mathbb{M}^4$; hence, the $\mathbf{A}$ thus obtained is an element of the Lie algebra $so(1,3)$.

Similarly, it follows from (4.16) that

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} A_{30} - iA_{12} & A_{10} - A_{13} + i(A_{20} - A_{23}) \\ A_{13} + A_{10} - i(A_{23} + A_{20}) & -A_{30} + iA_{12} \end{bmatrix}. \tag{4.19}$$

It is easy to check that the 2×2 complex matrix $\mathbf{B}$ as obtained in (4.19) has a zero trace ; hence, the $\mathbf{B}$ thus obtained is an element of the lie algebra $sl(2, \mathbb{C})$. Formula (4.17) enables us to obtain $\mathbf{A}$ from $\mathbf{B}$, whereas formula (4.19) $\mathbf{B}$ from $\mathbf{A}$. Formulae (4.17) and (4.19) together explicitly expressing the Lie algebra isomorphism of $sl(2,\mathbb{C})$ onto $so(1,3)$ is very important for calculations and for further study of the properties. Both the Lie algebras have six independent parameters.

4.3. Relations of Solutions

Now we are concerned with the two-dimensional complex spinor space [16,17] and the dynamical systems on the space. We return to the matrix $\mathbf{H}$ defined in (4.1) and remark that if the vector $\mathbf{X} \in \mathbb{M}^4$ then $\mathbf{H}$ may be written as the dyadic product of a complex two-dimensional vector and its conjugate transpose:

$$\mathbf{H} = \begin{bmatrix} X^0 + X^1 & X^2 - iX^3 \\ X^2 + iX^3 & X^0 - X^1 \end{bmatrix} = 2 \begin{bmatrix} \alpha^1\bar{\alpha}^1 & \alpha^1\bar{\alpha}^2 \\ \alpha^2\bar{\alpha}^1 & \alpha^2\bar{\alpha}^2 \end{bmatrix} = 2\boldsymbol{\alpha}\bar{\boldsymbol{\alpha}}^{\mathrm{T}}. \tag{4.20}$$

We are therefore led to consider a complex two-dimensional vector space $\mathbb{C}^2$ with elements $\boldsymbol{\alpha}$, on which $SL(2,\mathbb{C})$ left acts. The road (4.20) to spinors may be described as being based on the idea of taking the square root of a quadratic form and thus linearizing the quadratic map

$$\mathbf{H} \longmapsto \hat{\mathbf{H}} = \mathbf{U}\mathbf{H}\bar{\mathbf{U}}^{\mathrm{T}}$$

to be a linear one

$$\boldsymbol{\alpha} \longmapsto \hat{\boldsymbol{\alpha}} = \mathbf{U}\boldsymbol{\alpha}.$$

From (4.20) it follows that

$$\begin{aligned} X^0 &= \|\boldsymbol{\alpha}\|^2, \quad X^1 = |\alpha^1|^2 - |\alpha^2|, \\ X^2 &= 2\,\mathrm{Re}(\alpha^1\bar{\alpha}^2), \quad X^3 = -2\ \mathrm{Im}\ (\alpha^1\bar{\alpha}^2), \end{aligned} \tag{4.21}$$

where $|\alpha^1|$ is the absolute value (modulus) of the complex number α^1.

The relations of complex spinor $\boldsymbol{\alpha}$ and unit complex spinor $\mathbf{n}$ were shown in (3.4) and (3.7). Thus, we have systems (1.4) and (1.3) with $\mathbb{K} = \mathbb{C}$ and with the same $\mathbf{B} \in sl(2, \mathbb{C})$. Correspondingly, the equation in $\mathbb{M}^4$ and its projection are given by systems (1.1) and (1.2) with $n = 3$, where the relations of $\mathbf{X}$ and $\mathbf{m}$ were already shown in (2.4) and (2.5). The matrix $\mathbf{A}$ given in (4.18) satisfying (2.2) is an element of $so(1, 3)$.

Because of $\mathbf{n} \in \mathbb{C}^2$ and $\|\mathbf{n}\| = 1$, we may identify the unit complex spinor

$$\mathbf{n} = \begin{bmatrix} n^1 \\ n^2 \end{bmatrix} = \begin{bmatrix} n^1_{\mathrm{R}} + in^1_{\mathrm{I}} \\ n^2_{\mathrm{R}} + in^2_{\mathrm{I}} \end{bmatrix} \tag{4.22}$$

with the point $(n^1_{\mathrm{R}}, n^2_{\mathrm{R}}, n^1_{\mathrm{I}}, n^2_{\mathrm{I}})$ on the unit sphere $\|\mathbf{n}\|^2 = (n^1_{\mathrm{R}})^2 + (n^2_{\mathrm{R}})^2 + (n^1_{\mathrm{I}})^2 + (n^2_{\mathrm{I}})^2 = 1$ in $\mathbb{R}^4$, and hence $\mathbf{n} \in \mathbb{S}^3$.

On the other hand, from (2.4) with $n = 3$ we know that $\mathbf{m} \in \mathbb{S}^2$ if $\mathbf{X}$ is null. The map from $\mathbb{S}^3$ to $\mathbb{S}^2$ has been derived by Hopf [18],

$$\begin{aligned} m^1 &= (n^1_{\mathrm{R}})^2 + (n^1_{\mathrm{I}})^2 - (n^2_{\mathrm{R}})^2 - (n^2_{\mathrm{I}})^2, \\ m^2 &= 2n^1_{\mathrm{R}}n^2_{\mathrm{R}} + 2n^1_{\mathrm{I}}n^2_{\mathrm{I}}, \\ m^3 &= 2n^2_{\mathrm{R}}n^1_{\mathrm{I}} - 2n^1_{\mathrm{R}}n^2_{\mathrm{I}}. \end{aligned} \tag{4.23}$$

With the above identities, it is not difficult to prove that $\mathbf{n} \in \mathbb{S}^3$ if and only if $\mathbf{m} \in \mathbb{S}^2$. However, it is a two-to-one correspondence; that is, the two $\pm\mathbf{n}$ correspond to the same $\mathbf{m}$.

It deserves to note that the two $\pm\boldsymbol{\alpha}$ lead to the same $\mathbf{X}$, and the *two complex components* α^1 and α^2 suffice to determine the *four real components* X^0, X^1, X^2 and X^3 as shown in (4.21). However, from

(2.4) with $n = 3$ and (4.23) we know that neither the three components m^1, m^2 and m^3 nor the four real numbers n^1_{R}, n^2_{R}, n^1_{I}, and n^2_{I} suffice to determine the four components X^0, X^1, X^2 and X^3. But from (4.23) it is the four real numbers n^1_{R}, n^2_{R}, n^1_{I}, and n^2_{I} suffice to determine the three components m^1, m^2 and m^3. Figure 3 summarizes the relations of the Lie group homomorphism from $SL(2, \mathbb{C})$ onto $SO_o(1, 3)$, the Lie algebra isomorphism of $sl(2, \mathbb{C})$ onto $so(1, 3)$, the two linear systems (1.4) with $\mathbb{K} = \mathbb{C}$ and (1.1) with $n = 3$, and the two nonlinear systems (1.3) with $\mathbb{K} = \mathbb{C}$ and (1.2) with $n = 3$.

4.4. Spinor Map from $SL(2, \mathbb{R})$ onto $SO_o(1, 2)$

The symmetry group in Section 2 is specialized to $SO_o(1, 2)$ if $n = 2$, while the symmetry group in Section 3 is specialized to $SL(2, \mathbb{R})$ if $\ell = 2$ and $\mathbb{K} = \mathbb{R}$. In this subsection we study the relations of the symmetry groups $SL(2, \mathbb{R})$ and $SO_o(1, 2)$, their Lie algebras $sl(2, \mathbb{R})$ and $so(1, 2)$, and also the four systems (1.1)-(1.4) with $n = 2$, and $\mathbb{K} = \mathbb{R}$ and $\ell = 2$.

Deleting the X^3 component in (4.1) and following the similar procedures in the above we can obtain a spinor map from $SL(2, \mathbb{R}) \ni \mathbf{U}$ onto $SO_o(1, 2) \ni \mathbf{G}$ as given by (4.10) but with

$$\mathbf{C} := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \tag{4.24}$$

and

$$\mathbf{R} := \begin{bmatrix} (U^1{}_1)^2 & 2U^1{}_1U^1{}_2 & (U^1{}_2)^2 \\ U^1{}_1U^2{}_1 & U^1{}_2U^2{}_1 + U^1{}_1U^2{}_2 & U^1{}_2U^2{}_2 \\ (U^2{}_1)^2 & 2U^2{}_1U^2{}_2 & (U^2{}_2)^2 \end{bmatrix}. \tag{4.25}$$

Similarly, the Lie algebra isomorphism of $sl(2, \mathbb{R}) \ni \mathbf{B}$ onto $so(1, 2) \ni \mathbf{A}$ is derived as follows:

$$\begin{aligned} \mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{10} \\ A_{21} & A_{22} & A_{20} \\ A_{01} & A_{02} & A_{00} \end{bmatrix} & \\ &= \begin{bmatrix} 0 & B_{12} - B_{21} & B_{11} - B_{22} \\ B_{21} - B_{12} & 0 & B_{12} + B_{21} \\ B_{11} - B_{22} & B_{12} + B_{21} & 0 \end{bmatrix}, \end{aligned} \tag{4.26}$$

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4}(A_{11} + A_{00} + A_{10} + A_{01}) & \frac{1}{2}(A_{12} + A_{02}) \\ \frac{1}{2}(A_{21} + A_{20}) & \frac{1}{4}(A_{11} + A_{00} - A_{10} - A_{01}) \end{bmatrix}. \quad (4.27)$$

Deleting the X^3 component in (4.20) and noting $\boldsymbol{\alpha}$ is a real spinor, we have

$$X^0 = \|\boldsymbol{\alpha}\|^2, \;\; X^1 = (\alpha^1)^2 - (\alpha^2)^2, \;\; X^2 = 2\alpha^1\alpha^2. \quad (4.28)$$

Here we need $\mathbf{X}$ to be a null element in $\mathbb{M}^3$, namely $(X^0)^2 - (X^1)^2 - (X^2)^2 = 0$. It follows that

$$\alpha^1 = \pm\sqrt{\frac{1}{2}(X^0 + X^1)}, \;\; \alpha^2 = \pm\sqrt{\frac{1}{2}(X^0 - X^1)}. \quad (4.29)$$

Two points should be noted. The two $\pm\boldsymbol{\alpha}$ lead to the same $\mathbf{X}$, and the *two* components α^1 and α^2 suffice to determine the *three* components X^0, X^1, and X^2 by (4.28) because X^0, X^1, and X^2 are constrianed by $(X^0)^2 - (X^1)^2 - (X^2)^2 = 0$. Conversely, given X^0, X^1, and X^2 constrianed by $(X^0)^2 - (X^1)^2 - (X^2)^2 = 0$, we can determine up to sign α^1 and α^2 according to (4.29).

5. Real System (1.3)

In Section 3 we studied the third degree nonlinear system (1.3) via system (1.4). In view of the importance of system (1.3) with $\mathbb{K} = \mathbb{R}$ in applications as hinted in Section 1, we turn in this section our attention to it again. In Subsection 5.1 we approach it in such a direct manner that quantities remain in their physical spaces without recource to the other three systems. Then in Subsection 5.2 we convert system (1.1) with $n = 2$ to the real system (1.3). In Subsection 5.3 we show that y^0 introduced in the direct approach is nothing but X^0 of system (1.1).

5.1. A Direct Approach

In some applications system (1.3) describes the temporal changing of the director $\mathbf{n}$, which is known to possess the property $\|\mathbf{n}\| = 1$. Introduce the structural matrix (called structural tensor in [19] or conformation tensor in [20])

$$\mathbf{N} := \mathbf{n}\mathbf{n}^{\mathrm{T}}, \tag{5.1}$$

which is thus characterized by the properties:

$$\mathbf{N}^2 = \mathbf{N}, \quad \operatorname{tr}\mathbf{N} = 1, \quad \det\mathbf{N} = 0. \tag{5.2}$$

Then the differential equation of system (1.3) with $\mathbb{K} = \mathbb{R}$ can be written as

$$\dot{\mathbf{n}} = \mathbf{B}\mathbf{n} - \mathbf{B}\cdot\mathbf{N}\mathbf{n}, \tag{5.3}$$

where the dot between two matrices of the same order denotes their inner product, namely the sum of the products of all corresponding entries. Taking time derivative of both sides of (5.1) and utilizing (5.3), we obtain

$$\dot{\mathbf{N}} - \mathbf{B}\mathbf{N} - \mathbf{N}\mathbf{B}^{\mathrm{T}} + 2\mathbf{B}\cdot\mathbf{N}\mathbf{N} = \mathbf{0}. \tag{5.4}$$

If $\mathbf{U}$ satisfies (3.2) and (3.3), the change of variables

$$\mathbf{N} = \mathbf{U}\mathbf{M}\mathbf{U}^{\mathrm{T}} \tag{5.5}$$

transfers (5.4) to a differential equation for $\mathbf{M}$:

$$\mathbf{M} + 2(\mathbf{U}^{\mathrm{T}}\mathbf{B}\mathbf{U})\cdot\mathbf{M}\mathbf{M} = \mathbf{0}. \tag{5.6}$$

Upon letting

$$y^0 := \exp\left(2\int_0^t [\mathbf{U}^{\mathrm{T}}(\eta)\mathbf{B}(\eta)\mathbf{U}(\eta)]\cdot\mathbf{M}(\eta)d\eta\right), \tag{5.7}$$

(5.6) becomes

$$\frac{d}{dt}(y^0\mathbf{M}) = \mathbf{0}.$$

The solution is

$$\mathbf{M} = \frac{1}{y^0}\mathbf{M}(0), \tag{5.8}$$

where $y^0(0) = 1$ has been used. According to (5.5) and (5.8) we then have

$$\mathbf{N} = \frac{1}{y^0}\mathbf{U}\mathbf{N}(0)\mathbf{U}^{\mathrm{T}}. \tag{5.9}$$

From (5.7) and (5.8) it follows that

$$\dot{y}^0 = 2(\mathbf{U}^{\mathrm{T}}\mathbf{B}\mathbf{U}) \cdot \mathbf{M}(0).$$

Substituting $\mathbf{M}(0) = \mathbf{N}(0)$ into the above equation and integrating we have

$$y^0 = 1 + 2\int_0^t [\mathbf{U}^{\mathrm{T}}(\eta)\mathbf{B}(\eta)\mathbf{U}(\eta)] \cdot \mathbf{N}(0)d\eta = (\mathbf{U}^{\mathrm{T}}\mathbf{U}) \cdot \mathbf{N}(0) = \|\mathbf{U}\mathbf{n}(0)\|^2, \tag{5.10}$$

where $y^0(0) = \operatorname{tr}\mathbf{N}(0) = \|\mathbf{n}(0)\|^2 = 1$ has been used. Consequently, the solution of the structural matrix is

$$\mathbf{N} = \frac{\mathbf{U}\mathbf{N}(0)\mathbf{U}^{\mathrm{T}}}{\|\mathbf{U}\mathbf{n}(0)\|^2}, \tag{5.11}$$

and the solution of the director is

$$\mathbf{n} = \frac{\mathbf{U}\mathbf{n}(0)}{\|\mathbf{U}\mathbf{n}(0)\|}, \tag{5.12}$$

which is the same as the result (3.8) indirectly obtained with recourse to system (1.4).

Note that the so-called direct approach developed here is "direct" only in a relative sense; in fact. it needs to utilize $\mathbf{U}$ and y^0, which are in turn related intimately to systems (1.4) and (1.1). Recall that $\mathbf{U}$ is the fundamental solution of system (1.4) and y^0 is nothing but X^0 of $\mathbf{X}$ in system (1.1), as to be proved in Subsection 5.3. We also comment that the direct approach in itself is indeed applicable not only to the two-dimensional real system (1.3) but also to any ℓ-dimensional real system (1.3) with $\ell \geq 2$. The last comment enables us to apply the direct approach to both two- and three-dimensional problems, especially to the problems of microstructural orientation.

5.2. Converting System (1.1) to System (1.3)

Suppose we are given a $(2+1)$-dimensional system (1.1) with

$$\mathbf{A} = \begin{bmatrix} 0 & -a_3 & a_1 \\ a_3 & 0 & a_2 \\ a_1 & a_2 & 0 \end{bmatrix}. \tag{5.13}$$

Suppose the initial condition $\mathbf{X}(0)$ satisfies $\mathbf{X}^{\mathrm{T}}(0)\mathbf{g}\mathbf{X}(0) = 0$. From (2.12) we know the quantity

$$\mathbf{X}^{\mathrm{T}}(t)\mathbf{g}\mathbf{X}(t) = (X^0(t))^2 - (X^1(t))^2 - (X^2(t))^2 = 0 \tag{5.14}$$

is conserved and so that $\mathbf{X}$ is a null vector in $\mathbf{M}^3$. The parametrization,

$$X^1 := X^0 \cos\theta, \quad X^2 := X^0 \sin\theta, \tag{5.15}$$

satisfies (5.14) automatically. Through system (1.1) with $\mathbf{A}$ given by (5.13) θ is found to be governed by

$$\dot{\theta} = a_3 + b\cos(\theta + \omega), \tag{5.16}$$

where

$$b := \sqrt{a_1^2 + a_2^2}, \quad \omega := \arctan\left(\frac{a_1}{a_2}\right). \tag{5.17}$$

Now let us define a unit real spinor

$$\mathbf{n} = \begin{bmatrix} n^1 \\ n^2 \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{bmatrix}. \tag{5.18}$$

This is the simplest spinor, which may be intuitively interpreted as an entity satisfying the requirement that it changes sign under a 2π rotation; that is

$$\mathbf{n}(\theta + 2\pi) = -\mathbf{n}(\theta).$$

From (5.18) and (5.15) it follows that

$$(n^1)^2 = \frac{1}{2}\left[1 + \frac{X^1}{X^0}\right], \quad (n^2)^2 = \frac{1}{2}\left[1 - \frac{X^1}{X^0}\right]. \tag{5.19}$$

From (5.18) and (5.16) we obtain the differential equations for n^1 and n^2 as follows:

$$\begin{aligned}\dot{n}^1 &= -\frac{a_3}{2}n^2 + \frac{b}{2}[((n^2)^3 - n^2(n^1)^2)\cos\omega + 2n^1(n^2)^2\sin\omega],\\ \dot{n}^2 &= \frac{a_3}{2}n^1 + \frac{b}{2}[((n^1)^3 - n^1(n^2)^2)\cos\omega - 2(n^1)^2n^2\sin\omega],\end{aligned}$$

or, after a series of manupulations based upon $(n^1)^2 + (n^2)^2 = 1$,

$$\begin{aligned}\dot{n}^1 = -\frac{a_3}{2}n^2 - bn^2(n^1)^2\cos\omega + \frac{b}{2}n^2\cos\omega\\ + \frac{b}{2}n^1(n^2)^2\sin\omega + \frac{b}{2}(n^1 - (n^1)^3)\sin\omega,\end{aligned}$$

$$\begin{aligned}\dot{n}^2 = \frac{a_3}{2}n^1 + \frac{b}{2}n^1\cos\omega - bn^1(n^2)^2\cos\omega\\ - \frac{b}{2}(n^1)^2n^2\sin\omega - \frac{b}{2}(n^2 - (n^2)^3)\sin\omega.\end{aligned}$$

Rearranging the above two equations we obtain system (1.3) with $\mathbf{B}$ given by

$$\mathbf{B} = \frac{b}{2}\begin{bmatrix} \sin\omega & \cos\omega \\ \cos\omega & -\sin\omega \end{bmatrix} + \begin{bmatrix} 0 & \frac{-a_3}{2} \\ \frac{a_3}{2} & 0 \end{bmatrix}, \tag{5.20}$$

which, by (5.17), becomes

$$\mathbf{B} = \frac{1}{2}\begin{bmatrix} a_1 & a_2 - a_3 \\ a_2 + a_3 & -a_1 \end{bmatrix}. \tag{5.21}$$

Thus we have converted the $(2+1)$-dimensional system (1.1) to the real system (1.3).

The inverse relation of (5.21) is

$$\mathbf{A} = \begin{bmatrix} 0 & B_{12} - B_{21} & 2B_{11} \\ B_{21} - B_{12} & 0 & B_{12} + B_{21} \\ 2B_{11} & B_{12} + B_{21} & 0 \end{bmatrix}. \tag{5.22}$$

Obviously, $sl(2,\mathbb{R}) \ni \mathbf{B}$ is isomorphic to $so(1,2) \ni \mathbf{A}$.

5.3. $y^0 \equiv X^0$

Let us reflect that the success of the direct approach to solving system (1.3) lies in part in the introduction of the variables y^0 as defined in (5.7). It is thus interesting to further investigate y^0. Differentiating (5.7) yields

$$\dot{y}^0 = 2y^0(\mathbf{U}^{\mathrm{T}}\mathbf{B}\mathbf{U}) \cdot \mathbf{M}. \tag{5.23}$$

Substituting (5.5), (5.1) and (5.21) into the above equation we have

$$\dot{y}^0 = y^0[a_1((n^1)^2 - (n^2)^2) + 2a_2 n^1 n^2],$$

which, by using (5.19), further reduces to

$$\dot{y}^0 = y^0 \frac{1}{X^0}[a_1 X^1 + a_2 X^2].$$

Upon substituting

$$\dot{X}^0 = a_1 X^1 + a_2 X^2,$$

which is the third equation of (1.1) with $n = 2$ and $\mathbf{A}$ given by (5.13), we have

$$\frac{\dot{y}^0}{y^0} = \frac{\dot{X}^0}{X^0}. \tag{5.24}$$

Noting that $y^0(0) = 1$ from (5.7), we obtain $y^0 = X^0/X^0(0)$ by integration. We may set $X^0(0) = 1$; thus

$$y^0 = X^0. \tag{5.25}$$

Hence y^0 introduced for directly solving system (1.3) is nothing but the X^0 component of $\mathbf{X}$ in system (1.1).

6. Solutions Using Spinor Transformations

As shown earlier the solutions $\boldsymbol{\alpha}(t)$ and $\mathbf{n}(t)$ of systems (1.4) and (1.3) can be obtained by updating $\boldsymbol{\alpha}(0)$ and $\mathbf{n}(0)$ by the spinor transformation $\mathbf{U}(t)$ according to formulae (3.1) and (3.8), respectively. In this section we further show that the spinor transformation $\mathbf{U}(t)$ can also be used to express the solutions of systems (1.2) and (1.1).

6.1. Solution of System (1.2)

From (2.4) with $n = 2$, (5.15) and (5.18) it follows that

$$m^1 = (n^1)^2 - (n^2)^2, \quad m^2 = 2n^1n^2. \tag{6.1}$$

Substituting (5.12) into (6.1), m^1 and m^2 can be represented by another form:

$$m^1 = \frac{1}{y^0}\left\{ \frac{(U^1{}_1)^2 - (U^2{}_1)^2 + (U^1{}_2)^2 - (U^2{}_2)^2}{2} \right. \\ \left. + \frac{(U^1{}_1)^2 - (U^2{}_1)^2 - (U^1{}_2)^2 + (U^2{}_2)^2}{2} m^1(0) + [U^1{}_1U^1{}_2 - U^2{}_1U^2{}_2]m^2(0) \right\}, \tag{6.2}$$

$$m^2 = \frac{1}{y^0}\left\{ U^1{}_1U^2{}_1 + U^1{}_2U^2{}_2 + [U^1{}_1U^2{}_1 - U^1{}_2U^2{}_2]m^1(0) \right. \\ \left. + [U^1{}_1U^2{}_2 + U^1{}_2U^2{}_1]m^2(0) \right\}, \tag{6.3}$$

where

$$y^0 = \frac{(U^1{}_1)^2 + (U^2{}_1)^2 + (U^1{}_2)^2 + (U^2{}_2)^2}{2} \\ + \frac{(U^1{}_1)^2 + (U^2{}_1)^2 - (U^1{}_2)^2 - (U^2{}_2)^2}{2} m^1(0) \\ + [U^1{}_1U^1{}_2 + U^2{}_1U^2{}_2]m^2(0) \tag{6.4}$$

is derived from (5.10) and (6.1).

6.2. Solution of System (1.1)

With the aid of (5.25) and $(2.5)_1$, (6.2)-(6.4) are changed to

$$X^1 = \frac{(U^1{}_1)^2 - (U^2{}_1)^2 + (U^1{}_2)^2 - (U^2{}_2)^2}{2} X^0(0) \\ + \frac{(U^1{}_1)^2 - (U^2{}_1)^2 - (U^1{}_2)^2 + (U^2{}_2)^2}{2} X^1(0) \\ + [U^1{}_1U^1{}_2 - U^2{}_1U^2{}_2]X^2(0), \tag{6.5}$$

$$X^2 = [U^1{}_1U^2{}_1 + U^1{}_2U^2{}_2]X^0(0) + [U^1{}_1U^2{}_1 - U^1{}_2U^2{}_2]X^1(0) \\ + [U^1{}_1U^2{}_2 + U^1{}_2U^2{}_1]X^2(0), \quad (6.6)$$

$$X^0 = \frac{(U^1{}_1)^2 + (U^2{}_1)^2 + (U^1{}_2)^2 + (U^2{}_2)^2}{2}X^0(0) \\ + \frac{(U^1{}_1)^2 + (U^2{}_1)^2 - (U^1{}_2)^2 - (U^2{}_2)^2}{2}X^1(0) \\ + [U^1{}_1U^1{}_2 + U^2{}_1U^2{}_2]X^2(0). \quad (6.7)$$

6.3. Relation between G and U

After introducing the group $SO_o(1,2)$ and $SL(2,\mathbb{R})$ in Subsection 4.4, it is interesting to explore their relations from the current point of view. For this purpose, let us explore the relation between the spin transformation $\mathbf{U}$ and the proper orthochronous Lorentz transformation $\mathbf{G}$, which is the fundamental solution of system (1.1). From (2.1) and (2.6) we get

$$\begin{aligned} X^1 &= G^1{}_1X^1(0) + G^1{}_2X^2(0) + G^1{}_0X^0(0), \\ X^2 &= G^2{}_1X^1(0) + G^2{}_2X^2(0) + G^2{}_0X^0(0), \\ X^0 &= G^0{}_1X^1(0) + G^0{}_2X^2(0) + G^0{}_0X^0(0). \end{aligned}$$

Upon comparing the above formulae with formulae (6.5)-(6.7) we obtain again the spinor map from $SL(2,\mathbb{R})$ onto $SO_o(1,2)$.

7. Quaternionic Two-component Spinor Representation

In this section we turn our attention to the case with $n = 5$. In order to give a more economic lower dimensional quaternionic two-component spinor representation of the state vector $\mathbf{X}$ defined in (2.1), which is subjected to the constraint (2.12), let us first introduce the quaternion algebra.

7.1. Quaternion Algebra

The quaternions usually defined as a four-dimensional real vector space such that we can define a product $(\mathbf{x},\mathbf{y}) \longrightarrow \mathbf{x}\mathbf{y}$ satisfying the following associative and distributive laws for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{H}$ and all $a \in \mathbb{R}$ (see, *e.g.*, [23]):

$$(\mathbf{x}\mathbf{y})\mathbf{z} = \mathbf{x}(\mathbf{y}\mathbf{z}), \tag{7.1}$$

$$\mathbf{x}(\mathbf{y}+\mathbf{z}) = \mathbf{x}\mathbf{y}+\mathbf{x}\mathbf{z}, \tag{7.2}$$

$$(\mathbf{x}+\mathbf{y})\mathbf{z} = \mathbf{x}\mathbf{z}+\mathbf{y}\mathbf{z}, \tag{7.3}$$

$$a(\mathbf{x}\mathbf{y}) = (a\mathbf{x})\mathbf{y} = \mathbf{x}(a\mathbf{y}). \tag{7.4}$$

There exists a distinguished basis elements $\{1, \mathrm{i}_2, \mathrm{i}_3, \mathrm{i}_4\}$ with the following commutation relations:

$$\begin{aligned} &\mathrm{i}_2^2 = \mathrm{i}_3^2 = \mathrm{i}_4^2 = -1, \\ &\mathrm{i}_2\mathrm{i}_3 = -\mathrm{i}_3\mathrm{i}_2 = \mathrm{i}_4, \quad \mathrm{i}_3\mathrm{i}_4 = -\mathrm{i}_4\mathrm{i}_3 = \mathrm{i}_2, \quad \mathrm{i}_4\mathrm{i}_2 = -\mathrm{i}_2\mathrm{i}_4 = \mathrm{i}_3. \end{aligned} \tag{7.5}$$

Thus, if $\mathbf{x} = x^1 + x^2\mathrm{i}_2 + x^3\mathrm{i}_3 + x^4\mathrm{i}_4$ and $\mathbf{y} = y^1 + y^2\mathrm{i}_2 + y^3\mathrm{i}_3 + y^4\mathrm{i}_4$ are any two quaternions, their product is defined by

$$\begin{aligned} \mathbf{x}\mathbf{y} &= [x^1y^1 - x^2y^2 - x^3y^3 - x^4y^4] \\ &+ [x^1y^2 + y^1x^2 + x^3y^4 - x^4y^3]\mathrm{i}_2 \\ &+ [x^1y^3 + y^1x^3 + x^4y^2 - x^2y^4]\mathrm{i}_3 \\ &+ [x^1y^4 + y^1x^4 + x^2y^3 - x^3y^2]\mathrm{i}_4. \end{aligned} \tag{7.6}$$

The conjugate of $\mathbf{x}$ is denoted by $\bar{\mathbf{x}} := x^1 - x^2\mathrm{i}_2 - x^3\mathrm{i}_3 - x^4\mathrm{i}_4$, such that the product of $\mathbf{x}$ and $\bar{\mathbf{x}}$ gives the usual squared norm of $\mathbf{x}$ in $\mathbb{E}^4$,

$$\|\mathbf{x}\|^2 = \mathbf{x}\bar{\mathbf{x}} = \bar{\mathbf{x}}\mathbf{x} = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2. \tag{7.7}$$

For nonzero quaternion the inverse $\mathbf{x}^{-1}$ is thus given by

$$\mathbf{x}^{-1} = \frac{\bar{\mathbf{x}}}{\|\mathbf{x}\|^2}. \tag{7.8}$$

If we let $\mathbf{x} = x^1 + \mathbf{x}^s$, $\mathbf{y} = y^1 + \mathbf{y}^s$, it is shown in [24] that the product rule (7.6) can be represented by

$$\begin{aligned} \mathbf{x}\mathbf{y} &= x^1y^1 - \mathbf{x}^s \cdot \mathbf{y}^s + x^1\mathbf{y}^s + y^1\mathbf{x}^s + \mathbf{x}^s \times \mathbf{y}^s \\ &= x^1y^1 - \mathbf{x}^s \cdot \mathbf{y}^s + x^1\mathbf{y}^s + y^1\mathbf{x}^s + \tilde{\mathbf{x}}^s\mathbf{y}^s, \end{aligned} \tag{7.9}$$

where the cross product of $\mathbf{x}^s \times \mathbf{y}^s$ and the inner product of $\mathbf{x}^s \cdot \mathbf{y}^s$ are defined in the three-dimensional Euclidean space, and

$$\tilde{} \quad : \mathbf{x}^s \mapsto \tilde{\mathbf{x}}^s := \begin{bmatrix} 0 & -x^4 & x^3 \\ x^4 & 0 & -x^2 \\ -x^3 & x^2 & 0 \end{bmatrix} \tag{7.10}$$

is the tilde mapping, which maps each axial vector $\mathbf{x}^s := (x^2, x^3, x^4)^{\mathrm{T}}$ to a skew-symmetric matrix $\tilde{\mathbf{x}}^s$.

We also need to define the scalar product of two quaternions. This can be achieved through the inner product of the quaternion bases,

$$\mathbf{i}_j \cdot \mathbf{i}_k = \delta_{jk}, \quad j, k = 1, 2, 3, 4, \tag{7.11}$$

where $\mathbf{i}_1$ denotes the unit element 1 of the quaternion, and δ_{jk} is the Kronecker delta function. So that we have

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= x^1y^1 + \mathbf{x}^s \cdot \mathbf{y}^s = x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4, \\ \mathbf{x} \cdot \bar{\mathbf{y}} &= x^1y^1 - \mathbf{x}^s \cdot \mathbf{y}^s = x^1y^1 - x^2y^2 - x^3y^3 - x^4y^4, \end{aligned} \tag{7.12}$$

and accordingly, (7.9) can be written as

$$\mathbf{x}\mathbf{y} = \mathbf{x} \cdot \bar{\mathbf{y}} + x^1\mathbf{y}^s + y^1\mathbf{x}^s + \mathbf{x}^s \times \mathbf{y}^s. \tag{7.13}$$

For the later use we further derive the following formula for the product of three quaternions:

$$\begin{aligned} \mathbf{x}\mathbf{y}\mathbf{z} = z^1\mathbf{x} \cdot \bar{\mathbf{y}} - x^1\mathbf{y}^s \cdot \mathbf{z}^s - y^1\mathbf{x}^s \cdot \mathbf{z}^s - \mathbf{x}^s \times \mathbf{y}^s \cdot \mathbf{z}^s \\ + \mathbf{x} \cdot \bar{\mathbf{y}}\mathbf{z}^s + \mathbf{x} \cdot \mathbf{z}\mathbf{y}^s + \mathbf{z} \cdot \bar{\mathbf{y}}\mathbf{x}^s + z^1\mathbf{x}^s \times \mathbf{y}^s + x^1\mathbf{y}^s \times \mathbf{z}^s \\ + y^1\mathbf{x}^s \times \mathbf{z}^s. \end{aligned} \tag{7.14}$$

7.2. Dirac's Approach

Second, let us consider the 2×2 quaternionic Hermitian matrix $\mathbf{H}$, which can be written as

$$\mathbf{H} = \begin{bmatrix} x^0 + x^5 & x^1 - x^2\mathbf{i}_2 - x^3\mathbf{i}_3 - x^4\mathbf{i}_4 \\ x^1 + x^2\mathbf{i}_2 + x^3\mathbf{i}_3 + x^4\mathbf{i}_4 & x^0 - x^5 \end{bmatrix}, \tag{7.15}$$

where x^0, x^1, x^2, x^3, x^4 and x^5 are six real numbers. The determinant of $\mathbf{H}$ is the Minkowski separation $(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2-(x^5)^2$.

The group $SL(2,\mathbb{H})$ is the set of all quaternionic 2×2 matrices $\mathbf{U}$ with unit determinant. Elements of $SL(2,\mathbb{H})$ are often called quaternion spin transformations. Hence $\mathbf{U}\in SL(2,\mathbb{H})$ is a spin transformation. Since $\mathbf{H}$ is quaternionic Hermitian, it is obvious that $\mathbf{U}\mathbf{H}\bar{\mathbf{U}}^{\mathrm{T}}$ is also a 2×2 quaternionic Hermitian matrix. This led us to write

$$\hat{\mathbf{H}}=\mathbf{U}\mathbf{H}\bar{\mathbf{U}}^{\mathrm{T}}, \tag{7.16}$$

where the bar over U stands for its quaternion conjugate. Taking the determinant of both sides and using $\det\mathbf{U}=\det\bar{\mathbf{U}}^{\mathrm{T}}=1$, one readily obtains $(\hat{x}^0)^2-(\hat{x}^1)^2-(\hat{x}^2)^2-(\hat{x}^3)^2-(\hat{x}^4)^2-(\hat{x}^5)^2=(x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2-(x^5)^2$, ensuring the Minkowski separation of the $(5+1)$-vector $\mathbf{X}=(x^1,x^2,x^3,x^4,x^5,x^0)$ is preserved by the spin transformation $\mathbf{U}:\mathbf{H}\longmapsto\hat{\mathbf{H}}$. Indeed, the transformation $\mathbf{U}:\mathbf{H}\longmapsto\hat{\mathbf{H}}$ induces a proper orthochronous Lorentz transformation $\mathbf{G}:\mathbf{X}\longmapsto\hat{\mathbf{X}}$, which is an element of $SO_o(1,5)$.

Dirac [25] has proposed a quaternionic representation of the Lorentz transformation in $\mathbf{M}^4$ by expressing the quaternion $\mathbf{w}$ as the ratio of the other two quaternions $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{w}=\mathbf{u}\mathbf{v}^{-1}. \tag{7.17}$$

Then he considered three quantities

$$\mathbf{x}=\mathbf{u}\bar{\mathbf{v}},\quad x^+=\mathbf{v}\bar{\mathbf{v}},\quad x^-=\mathbf{u}\bar{\mathbf{u}}, \tag{7.18}$$

and defined

$$\mathbf{x}=x^1+x^2\mathbf{i}_2+x^3\mathbf{i}_3+x^4\mathbf{i}_4,\quad x^+=x^0+x^5,\quad x^-=x^0-x^5, \tag{7.19}$$

where $x^1,\ldots,x^5,x^0$ are real numbers. If $\mathbf{u}$ and $\mathbf{v}$ are replaced by $\mathbf{u}\boldsymbol{\lambda}$ and $\mathbf{v}\boldsymbol{\lambda}$, the six x's all get multiplied by $\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}}$ and their ratios are unchanged. Thus the ratios of x's are determined by $\mathbf{w}$. From (7.18) it follows that

$$\mathbf{x}\bar{\mathbf{x}}=\mathbf{u}\bar{\mathbf{v}}\mathbf{v}\bar{\mathbf{u}}=\mathbf{u}x^+\bar{\mathbf{u}}=x^+x^-, \tag{7.20}$$

which by means of (7.7) and (7.19) leads to

$$(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2=(x^0)^2-(x^5)^2. \tag{7.21}$$

It represents a null cone in the space $\mathbf{M}^6$.

Under the following linear transformations for $\mathbf{u}$ and $\mathbf{v}$,

$$\hat{\mathbf{u}} = \mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}, \quad \hat{\mathbf{v}} = \mathbf{c}\mathbf{u} + \mathbf{d}\mathbf{v}, \tag{7.22}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are arbitrary quaternions, $\mathbf{x}$, x^+, and x^- are transformed as follows:

$$\begin{aligned} \hat{\mathbf{x}} &= \mathbf{a}x^-\bar{\mathbf{c}} + \mathbf{b}x^+\bar{\mathbf{d}} + \mathbf{a}\mathbf{x}\bar{\mathbf{d}} + \mathbf{b}\bar{\mathbf{x}}\bar{\mathbf{c}}, && (7.23)\\ \hat{x}^+ &= \mathbf{c}x^-\bar{\mathbf{c}} + \mathbf{d}x^+\bar{\mathbf{d}} + \mathbf{c}\mathbf{x}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{x}}\bar{\mathbf{c}}, && (7.24)\\ \hat{x}^- &= \mathbf{a}x^-\bar{\mathbf{a}} + \mathbf{b}x^+\bar{\mathbf{b}} + \mathbf{a}\mathbf{x}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{x}}\bar{\mathbf{a}}. && (7.25) \end{aligned}$$

This, as has been demonstrated by Dirac, will result in the new six $\hat{x}$'s being linear functions of the old x's, and (7.21) still holds for the new $\hat{x}$'s.

Equations (7.23)-(7.25) together with the conjugate of (7.23) can be represented by

$$\begin{bmatrix} \hat{x}^+ & \hat{\bar{\mathbf{x}}} \\ \hat{\mathbf{x}} & \hat{x}^- \end{bmatrix} = \begin{bmatrix} \mathbf{d} & \mathbf{c} \\ \mathbf{b} & \mathbf{a} \end{bmatrix} \begin{bmatrix} x^+ & \bar{\mathbf{x}} \\ \mathbf{x} & x^- \end{bmatrix} \begin{bmatrix} \bar{\mathbf{d}} & \bar{\mathbf{b}} \\ \bar{\mathbf{c}} & \bar{\mathbf{a}} \end{bmatrix}. \tag{7.26}$$

Upon letting

$$\mathbf{U} = \begin{bmatrix} \mathbf{d} & \mathbf{c} \\ \mathbf{b} & \mathbf{a} \end{bmatrix}, \tag{7.27}$$

it can be seen that (7.26) is really of the form (7.16).

7.3. Spinor Map from $SL(2,\mathbb{H})$ to $SO_o(1,5)$

Although the product $\mathbf{xy}$ is non-commutative, by (7.9) we can derive the following relation:

$$\mathbf{xy} = \mathbf{yx} - 2 < \mathbf{yx} >= \mathbf{yx} - 2\tilde{\mathbf{y}}^s\mathbf{x}^s, \tag{7.28}$$

where $< \mathbf{yx} >$ denotes the skew-symmetric (cross product) part of $\mathbf{yx}$. With this formula we can rearrange (7.23)-(7.25) and the conjugate of (7.23) to the following forms:

$$\begin{aligned} \hat{\mathbf{x}} = &\ \mathbf{a}\bar{\mathbf{c}}x^- + \mathbf{b}\bar{\mathbf{d}}x^+ + \mathbf{a}\bar{\mathbf{d}}\mathbf{x} - 2\mathbf{a} < \bar{\mathbf{d}}\mathbf{x} > + \mathbf{b}\bar{\mathbf{c}}\bar{\mathbf{x}} - 2\mathbf{b} < \bar{\mathbf{c}}\bar{\mathbf{x}} >, && (7.29)\\ \hat{\bar{\mathbf{x}}} = &\ \mathbf{c}\bar{\mathbf{a}}x^- + \mathbf{d}\bar{\mathbf{b}}x^+ + \mathbf{d}\bar{\mathbf{a}}\bar{\mathbf{x}} - 2\mathbf{d} < \bar{\mathbf{a}}\bar{\mathbf{x}} > + \mathbf{c}\bar{\mathbf{b}}\mathbf{x} - 2\mathbf{c} < \bar{\mathbf{b}}\mathbf{x} >, && (7.30)\\ \hat{x}^+ = &\ \mathbf{c}\bar{\mathbf{c}}x^- + \mathbf{d}\bar{\mathbf{d}}x^+ + \mathbf{c}\bar{\mathbf{d}}\mathbf{x} - 2\mathbf{c} < \bar{\mathbf{d}}\mathbf{x} > + \mathbf{d}\bar{\mathbf{c}}\bar{\mathbf{x}} - 2\mathbf{d} < \bar{\mathbf{c}}\bar{\mathbf{x}} >, && (7.31)\\ \hat{x}^- = &\ \mathbf{a}\bar{\mathbf{a}}x^- + \mathbf{b}\bar{\mathbf{b}}x^+ + \mathbf{a}\bar{\mathbf{b}}\mathbf{x} - 2\mathbf{a} < \bar{\mathbf{b}}\mathbf{x} > + \mathbf{b}\bar{\mathbf{a}}\bar{\mathbf{x}} - 2\mathbf{b} < \bar{\mathbf{a}}\bar{\mathbf{x}} > . && (7.32) \end{aligned}$$

The above four equations can be combined together to a matrix representation

$$\begin{bmatrix} \hat{x}^+ \\ \hat{\bar{\mathbf{x}}} \\ \hat{\mathbf{x}} \\ \hat{x}^- \end{bmatrix} = \mathbf{J} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}, \tag{7.33}$$

where

$$\mathbf{J} := \begin{bmatrix} \mathbf{d}\bar{\mathbf{d}} & \mathbf{d}\bar{\mathbf{c}} - 2\mathbf{d} < \bar{\mathbf{c}} & \mathbf{c}\bar{\mathbf{d}} - 2\mathbf{c} < \bar{\mathbf{d}} & \mathbf{c}\bar{\mathbf{c}} \\ \mathbf{d}\bar{\mathbf{b}} & \mathbf{d}\bar{\mathbf{a}} - 2\mathbf{d} < \bar{\mathbf{a}} & \mathbf{c}\bar{\mathbf{b}} - 2\mathbf{c} < \bar{\mathbf{b}} & \mathbf{c}\bar{\mathbf{a}} \\ \mathbf{b}\bar{\mathbf{d}} & \mathbf{b}\bar{\mathbf{c}} - 2\mathbf{b} < \bar{\mathbf{c}} & \mathbf{a}\bar{\mathbf{d}} - 2\mathbf{a} < \bar{\mathbf{d}} & \mathbf{a}\bar{\mathbf{c}} \\ \mathbf{b}\bar{\mathbf{b}} & \mathbf{b}\bar{\mathbf{a}} - 2\mathbf{b} < \bar{\mathbf{a}} & \mathbf{a}\bar{\mathbf{b}} - 2\mathbf{a} < \bar{\mathbf{b}} & \mathbf{a}\bar{\mathbf{a}} \end{bmatrix}, \tag{7.34}$$

and $< \bullet$ denotes the operator of skew-symmetrization; for example, the operator $< \mathbf{y}$ acting on $\mathbf{x}$ is read as $< \mathbf{y}\mathbf{x} >= \tilde{\mathbf{y}}^s\mathbf{x}^s$.

For the later purpose we introduce another representation of the quaternions $\mathbf{x}$ and $\mathbf{y}$ with $\mathbf{x} = x^1 + i\mathbf{x}^s$ and $\mathbf{y} = y^1 + i\mathbf{y}^s$, such that their product is expressed by

$$\mathbf{x}\mathbf{y} = (x^1 + i\mathbf{x}^s)(y^1 + i\mathbf{y}^s) := x^1 y^1 - \mathbf{x}^s \cdot \mathbf{y}^s + i(x^1\mathbf{y}^s + y^1\mathbf{x}^s + \mathbf{x}^s \times \mathbf{y}^s). \tag{7.35}$$

Here i plays not only the role of an imaginary number with $i^2 = -1$, but also a symbol used to stress that the quantity been prefixed by i is the spatial part; for example, $\mathbf{x}^s$ is the spatial part of $\mathbf{x}$; conversely, x^1 is the scalar part of $\mathbf{x}$.

Now, the $\mathbf{H}$ in (7.15), with its $x^0 + x^5$ replaced by x^+ and $x^0 - x^5$ by x^- as that defined in (7.19), can be re-expressed as

$$\begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix} = \mathbf{C} \begin{bmatrix} x^1 \\ \mathbf{x}^s \\ x^5 \\ x^0 \end{bmatrix}, \tag{7.36}$$

where

$$\mathbf{C} := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}. \tag{7.37}$$

Conversely, we have

$$\mathbf{X} := \begin{bmatrix} x^1 \\ \mathbf{x}^s \\ x^5 \\ x^0 \end{bmatrix} = \mathbf{C}^{-1} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}, \tag{7.38}$$

where

$$\mathbf{C}^{-1} := \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{7.39}$$

Left multiplying both sides of (7.33) by $\mathbf{C}^{-1}$ and noting (7.36), yields

$$\hat{\mathbf{X}} = \mathbf{C}^{-1}\mathbf{J}\mathbf{C}\mathbf{X}, \tag{7.40}$$

which being compared with the proper orthochronous Lorentz transformation $\hat{\mathbf{X}} = \mathbf{G}\mathbf{X}$ gives

$$\mathbf{G} = \mathbf{C}^{-1}\mathbf{J}\mathbf{C}. \tag{7.41}$$

Now, letting

$$\mathbf{G} := \begin{bmatrix} G^1{}_1 & \mathbf{G}^1{}_s & G^1{}_5 & G^1{}_0 \\ \mathbf{G}^s{}_1 & \mathbf{G}^s{}_s & \mathbf{G}^s{}_5 & \mathbf{G}^s{}_0 \\ G^5{}_1 & \mathbf{G}^5{}_s & G^5{}_5 & G^5{}_0 \\ G^0{}_1 & \mathbf{G}^0{}_s & G^0{}_5 & G^0{}_0 \end{bmatrix}, \tag{7.42}$$

and then substituting (7.39) for $\mathbf{C}^{-1}$, (7.34) for $\mathbf{J}$ and (7.37) for $\mathbf{C}$ into (7.41), we obtain each term in $\mathbf{G}$ as follows:

$$G^1{}_1 = \frac{1}{2}(\mathbf{a}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{a}} + \mathbf{b}\bar{\mathbf{c}} + \mathbf{c}\bar{\mathbf{b}}) - \mathbf{d} < \bar{\mathbf{a}}\ 1 > -\mathbf{b} < \bar{\mathbf{c}}\ 1 >$$
$$-\mathbf{c} < \bar{\mathbf{b}}\ 1 > -\mathbf{a} < \bar{\mathbf{d}}\ 1 >= \mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c}, \tag{7.43}$$
$$\mathbf{G}^1{}_s = \frac{i}{2}(\mathbf{a}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{a}} + \mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) + \mathbf{d} < \bar{\mathbf{a}}\ i > +\mathbf{b} < \bar{\mathbf{c}}\ i >$$
$$-\mathbf{c} < \bar{\mathbf{b}}\ i > -\mathbf{a} < \bar{\mathbf{d}}\ i >$$
$$= (a_1\mathbf{d}_s - d_1\mathbf{a}_s + c_1\mathbf{b}_s - b_1\mathbf{c}_s + \mathbf{d}_s \times \mathbf{a}_s + \mathbf{b}_s \times \mathbf{c}_s)^{\mathrm{T}}, \tag{7.44}$$
$$G^1{}_5 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{d}} - \mathbf{c}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{c}}), \tag{7.45}$$

$$G^1{}_0 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{a}} + \mathbf{a}\bar{\mathbf{c}}), \tag{7.46}$$

$$\mathbf{G}^s{}_1 = \frac{i}{2}(\mathbf{d}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) + i(\mathbf{a} < \bar{\mathbf{d}}\ 1 > + \mathbf{b} < \bar{\mathbf{c}}\ 1 >$$
$$- \mathbf{c} < \bar{\mathbf{b}}\ 1 > - \mathbf{d} < \bar{\mathbf{a}}\ 1 >)$$
$$= c_1\mathbf{b}_s + d_1\mathbf{a}_s - a_1\mathbf{d}_s - b_1\mathbf{c}_s - \mathbf{a}_s \times \mathbf{d}_s - \mathbf{b}_s \times \mathbf{c}_s, \tag{7.47}$$

$$\mathbf{G}^s{}_s = \frac{1}{2}(\mathbf{a}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{a}} - \mathbf{b}\bar{\mathbf{c}} - \mathbf{c}\bar{\mathbf{b}}) + i(\mathbf{a} < \bar{\mathbf{d}}\ i > + \mathbf{d} < \bar{\mathbf{a}}\ i >$$
$$- \mathbf{b} < \bar{\mathbf{c}}\ i > - \mathbf{c} < \bar{\mathbf{b}}\ i >) = (\mathbf{a} \cdot \bar{\mathbf{d}} - \mathbf{b} \cdot \bar{\mathbf{c}})\mathbf{I}_3 + 2[\mathbf{a}_s \otimes \mathbf{d}_s] - 2[\mathbf{b}_s \otimes \mathbf{c}_s]$$
$$+ d_1\tilde{\mathbf{a}}_s + a_1\tilde{\mathbf{d}}_s - c_1\tilde{\mathbf{b}}_s - b_1\tilde{\mathbf{c}}_s, \tag{7.48}$$

$$\mathbf{G}^s{}_5 = \frac{i}{2}(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}} + \mathbf{a}\bar{\mathbf{c}} - \mathbf{c}\bar{\mathbf{a}})$$
$$= d_1\mathbf{b}_s - b_1\mathbf{d}_s - c_1\mathbf{a}_s + a_1\mathbf{c}_s + \mathbf{d}_s \times \mathbf{b}_s - \mathbf{c}_s \times \mathbf{a}_s, \tag{7.49}$$

$$\mathbf{G}^s{}_0 = \frac{i}{2}(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{c}})$$
$$= d_1\mathbf{b}_s - b_1\mathbf{d}_s + c_1\mathbf{a}_s - a_1\mathbf{c}_s + \mathbf{d}_s \times \mathbf{b}_s + \mathbf{c}_s \times \mathbf{a}_s, \tag{7.50}$$

$$G^5{}_1 = \frac{1}{2}(\mathbf{c}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{c}} - \mathbf{a}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{a}})$$
$$- \mathbf{d} < \bar{\mathbf{c}}\ 1 > + \mathbf{b} < \bar{\mathbf{a}}\ 1 > - \mathbf{c} < \bar{\mathbf{d}}\ 1 > + \mathbf{a} < \bar{\mathbf{b}}\ 1 >$$
$$= \mathbf{c} \cdot \mathbf{d} - \mathbf{a} \cdot \mathbf{b}, \tag{7.51}$$

$$\mathbf{G}^5{}_s = \frac{i}{2}(\mathbf{b}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{b}} + \mathbf{c}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{c}}) + \mathbf{d} < \bar{\mathbf{c}}\ i > - \mathbf{b} < \bar{\mathbf{a}}\ i >$$
$$- \mathbf{c} < \bar{\mathbf{d}}\ i > + \mathbf{a} < \bar{\mathbf{b}}\ i >$$
$$= (c_1\mathbf{d}_s - d_1\mathbf{c}_s - a_1\mathbf{b}_s + b_1\mathbf{a}_s + \mathbf{d}_s \times \mathbf{c}_s - \mathbf{b}_s \times \mathbf{a}_s)^{\mathrm{T}}, \tag{7.52}$$

$$G^5{}_5 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} - \mathbf{b}\bar{\mathbf{b}} - \mathbf{c}\bar{\mathbf{c}} + \mathbf{a}\bar{\mathbf{a}}), \tag{7.53}$$

$$G^5{}_0 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} - \mathbf{b}\bar{\mathbf{b}} + \mathbf{c}\bar{\mathbf{c}} - \mathbf{a}\bar{\mathbf{a}}), \tag{7.54}$$

$$G^0{}_1 = \frac{1}{2}(\mathbf{a}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{a}} + \mathbf{c}\bar{\mathbf{d}} + \mathbf{d}\bar{\mathbf{c}}) - \mathbf{d} < \bar{\mathbf{c}}\ 1 > - \mathbf{b} < \bar{\mathbf{a}}\ 1 >$$
$$- \mathbf{c} < \bar{\mathbf{d}}\ 1 > - \mathbf{a} < \bar{\mathbf{b}}\ 1 > = \mathbf{c} \cdot \mathbf{d} + \mathbf{a} \cdot \mathbf{b}, \tag{7.55}$$

$$\mathbf{G}^0{}_s = \frac{i}{2}(\mathbf{a}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{a}} + \mathbf{c}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{c}}) + \mathbf{d} < \bar{\mathbf{c}}\ i > + \mathbf{b} < \bar{\mathbf{a}}\ i >$$
$$- \mathbf{c} < \bar{\mathbf{d}}\ i > - \mathbf{a} < \bar{\mathbf{b}}\ i > = (c_1\mathbf{d}_s - d_1\mathbf{c}_s + a_1\mathbf{b}_s - b_1\mathbf{a}_s$$
$$+ \mathbf{d}_s \times \mathbf{c}_s + \mathbf{b}_s \times \mathbf{a}_s)^{\mathrm{T}}, \tag{7.56}$$

$$G^0{}_5 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} + \mathbf{b}\bar{\mathbf{b}} - \mathbf{c}\bar{\mathbf{c}} - \mathbf{a}\bar{\mathbf{a}}), \tag{7.57}$$

$$G^0{}_0 = \frac{1}{2}(\mathbf{d}\bar{\mathbf{d}} + \mathbf{b}\bar{\mathbf{b}} + \mathbf{c}\bar{\mathbf{c}} + \mathbf{a}\bar{\mathbf{a}}). \tag{7.58}$$

In above $\otimes$ between two three-dimensional vectors denotes their tensor product, and as before $[\mathbf{a}_s \otimes \mathbf{d}_s]$ and $[\mathbf{b}_s \otimes \mathbf{c}_s]$ denote, respectively, the symmetric parts of the tensors $\mathbf{a}_s \otimes \mathbf{d}_s$ and $\mathbf{b}_s \otimes \mathbf{c}_s$.

Only $G^1{}_5$, $G^1{}_0$, $G^5{}_5$, $G^5{}_0$, $G^0{}_5$ and $G^0{}_0$ are calculable directly from the quaternions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. The other terms above should be supplemented with their second equalities for directly calculable from the quaternions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$. We first derive the second equality in $\mathbf{G}^s{}_s$. Using the following formulae:

$$\mathbf{x}^s \times (\mathbf{y}^s \times \mathbf{z}^s) = (\mathbf{x}^s \cdot \mathbf{z}^s)\mathbf{y}^s - (\mathbf{x}^s \cdot \mathbf{y}^s)\mathbf{z}^s, \tag{7.59}$$

$$\mathbf{x}^s \cdot (\mathbf{y}^s \times \mathbf{z}^s) = \mathbf{y}^s \cdot (\mathbf{z}^s \times \mathbf{x}^s) = \mathbf{z}^s \cdot (\mathbf{x}^s \times \mathbf{y}^s) \tag{7.60}$$

for three-dimensional vectors $\mathbf{x}^s$, $\mathbf{y}^s$ and $\mathbf{z}^s$, we can prove that

$$\begin{aligned} & i\mathbf{a} < \bar{\mathbf{d}}\, i\mathbf{x} > +i\mathbf{d} < \bar{\mathbf{a}}\, i\mathbf{x} > \\ = \; & -\mathbf{a}_s \cdot (\mathbf{d}_s \times \mathbf{x}^s) + a_1\tilde{\mathbf{d}}_s\mathbf{x}^s + \mathbf{a}_s \cdot \mathbf{x}^s\mathbf{d}_s - \mathbf{a}_s \cdot \mathbf{d}_s\mathbf{x}^s \\ & -\mathbf{d}_s \cdot (\mathbf{a}_s \times \mathbf{x}^s) + d_1\tilde{\mathbf{a}}_s\mathbf{x}^s + \mathbf{d}_s \cdot \mathbf{x}^s\mathbf{a}_s - \mathbf{d}_s \cdot \mathbf{a}_s\mathbf{x}^s \\ = \; & a_1\tilde{\mathbf{d}}_s\mathbf{x}^s + d_1\tilde{\mathbf{a}}_s\mathbf{x}^s + 2[\mathbf{a}_s \otimes \mathbf{d}_s]\mathbf{x}^s - 2\mathbf{a}_s \cdot \mathbf{d}_s\mathbf{x}^s. \end{aligned} \tag{7.61}$$

It thus follows that

$$i\mathbf{a} < \bar{\mathbf{d}}\, i > +i\mathbf{d} < \bar{\mathbf{a}}\, i >= a_1\tilde{\mathbf{d}}_s + d_1\tilde{\mathbf{a}}_s + 2[\mathbf{a}_s \otimes \mathbf{d}_s] - 2\mathbf{a}_s \cdot \mathbf{d}_s\mathbf{I}_3, \tag{7.62}$$

and similarly,

$$-i\mathbf{b} < \bar{\mathbf{c}}\, i > -i\mathbf{c} < \bar{\mathbf{b}}\, i >= -b_1\tilde{\mathbf{c}}_s - c_1\tilde{\mathbf{b}}_s - 2[\mathbf{b}_s \otimes \mathbf{c}_s] + 2\mathbf{b}_s \cdot \mathbf{c}_s\mathbf{I}_3. \tag{7.63}$$

While the other terms in $\mathbf{G}^s{}_s$ can be computed as follows:

$$-i\mathbf{a}\bar{\mathbf{d}}i - i\mathbf{d}\bar{\mathbf{a}}i + i\mathbf{b}\bar{\mathbf{c}}i + i\mathbf{c}\bar{\mathbf{b}}i = (2a_1d_1 + 2\mathbf{a}_s \cdot \mathbf{d}_s - 2b_1c_1 - 2\mathbf{b}_s \cdot \mathbf{c}_s)\mathbf{I}_3. \tag{7.64}$$

Substituting (7.62)-(7.64) into the first equality on the right-hand side of $\mathbf{G}^s{}_s$, we obtain the second equality in (7.48) for $\mathbf{G}^s{}_s$.

There are

$$< \bar{\mathbf{a}}\; 1 >= 0, \quad < \bar{\mathbf{b}}\; 1 >= 0, \quad < \bar{\mathbf{c}}\; 1 >= 0, \quad < \bar{\mathbf{d}}\; 1 >= 0,$$

because 1 is a scalar; see the sentence follows (7.34). Thus, the second equalities of $G^1{}_1$, $G^5{}_1$ and $G^0{}_1$ in (7.43), (7.51) and (7.55) are proved.

By straightforward calculations we can prove that

$$i(\mathbf{a}\bar{\mathbf{d}} - \mathbf{d}\bar{\mathbf{a}}) = 2(a_1\mathbf{d}_s - d_1\mathbf{a}_s + \mathbf{a}_s \times \mathbf{d}_s), \tag{7.65}$$

$$i(\mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) = 2(c_1\mathbf{b}_s - b_1\mathbf{c}_s + \mathbf{c}_s \times \mathbf{b}_s). \tag{7.66}$$

Therefore the following equations hold:

$$\begin{aligned} i(\mathbf{d}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{d}} + \mathbf{c}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{c}}) = 2(d_1\mathbf{a}_s - a_1\mathbf{d}_s - \mathbf{a}_s \times \mathbf{d}_s) \\ + 2(c_1\mathbf{b}_s - b_1\mathbf{c}_s + \mathbf{c}_s \times \mathbf{b}_s), \end{aligned}$$

$$\begin{aligned} i(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}} + \mathbf{a}\bar{\mathbf{c}} - \mathbf{c}\bar{\mathbf{a}}) = 2(d_1\mathbf{b}_s - b_1\mathbf{d}_s - \mathbf{b}_s \times \mathbf{d}_s) \\ + 2(a_1\mathbf{c}_s - c_1\mathbf{a}_s - \mathbf{c}_s \times \mathbf{a}_s), \end{aligned}$$

$$\begin{aligned} i(\mathbf{d}\bar{\mathbf{b}} - \mathbf{b}\bar{\mathbf{d}}\mathbf{c}\bar{\mathbf{a}} - \mathbf{a}\bar{\mathbf{c}}) = 2(d_1\mathbf{b}_s - b_1\mathbf{d}_s - \mathbf{b}_s \times \mathbf{d}_s) \\ + 2(c_1\mathbf{a}_s - a_1\mathbf{c}_s + \mathbf{c}_s \times \mathbf{a}_s), \end{aligned}$$

and the second equalities for $\mathbf{G}^s{}_1$, $\mathbf{G}^s{}_5$ and $\mathbf{G}^s{}_0$ as that appeared in (7.47), (7.49) and (7.50) are proved.

It remains to check the validity of the second equalities for $\mathbf{G}^1{}_s$, $\mathbf{G}^5{}_s$, and $\mathbf{G}^0{}_s$. Similarly, through some calculations we have

$$\mathbf{d} < \bar{\mathbf{a}}\, i\mathbf{x} >= \mathbf{d}_s \times \mathbf{a}_s \cdot \mathbf{x}_s, \tag{7.67}$$

and

$$\mathbf{d} < \bar{\mathbf{a}}\ i >= \mathbf{d}_s \times \mathbf{a}_s \tag{7.68}$$

can be viewed as a row vector, since it linearly maps $\mathbf{x}$ to a scalar. With this formula the following relations are obvious

$$\begin{aligned} &\mathbf{b} < \bar{\mathbf{c}}\ i >= \mathbf{b}_s \times \mathbf{c}_s, \quad \mathbf{c} < \bar{\mathbf{b}}\ i >= \mathbf{c}_s \times \mathbf{b}_s, \quad \mathbf{a} < \bar{\mathbf{d}}\ i >= \mathbf{a}_s \times \mathbf{d}_s, \\ &\mathbf{d} < \bar{\mathbf{c}}\ i >= \mathbf{d}_s \times \mathbf{c}_s, \quad \mathbf{b} < \bar{\mathbf{a}}\ i >= \mathbf{b}_s \times \mathbf{a}_s, \quad \mathbf{c} < \bar{\mathbf{d}}\ i >= \mathbf{c}_s \times \mathbf{d}_s, \\ &\mathbf{a} < \bar{\mathbf{b}}\ i >= \mathbf{a}_s \times \mathbf{b}_s. \end{aligned}$$

Substituting these formulae into (7.44), (7.52) and (7.56), we obtain the second equalities for $\mathbf{G}^1{}_s$, $\mathbf{G}^5{}_s$, and $\mathbf{G}^0{}_s$. This completes the proof of equations (7.43)-(7.58).

The above procedure for obtaining the spinor map from $SL(2,\mathbb{H})$ onto $SO_o(1,5)$ are illustrated in Figure 4. This map is from the group $SL(2,\mathbb{H})$ onto the group $SO_o(1,5)$ and is a *spinor map* in the sense that the two spin transformations $\pm\mathbf{U}$ map to the same proper orthochronous Lorentz transformation $\mathbf{G}$. The above formulae require to know $\mathbf{U}(t)$. More definitely, we should know the time-varying Lie algebra $\mathbf{B}(t)$, which render $\mathbf{U}(t)$ obtainable through the integration of differential equation $\dot{\mathbf{U}}(t) = \mathbf{B}(t)\mathbf{U}(t)$. Thus, we below derive the conversion formulae from $\mathbf{B} \in sl(2,\mathbb{H})$ to $\mathbf{A} \in so(1,5)$, and vice versa.

8. The Explicit Isomorphism of $sl(2,\mathbb{H})$ and $so(1,5)$

Now, we attempt to convert the six-dimensional system (2.7) to a corresponding quaternion system with dimensions two, that is,

$$\dot{\mathbf{U}}(t) = \mathbf{B}(t)\mathbf{U}(t), \tag{8.1}$$

$$\mathbf{U}(0) = \mathbf{I}_2, \tag{8.2}$$

in which

$$\mathbf{B} = \begin{bmatrix} \mathbf{r} & \mathbf{q} \\ \mathbf{p} & \mathbf{o} \end{bmatrix} \tag{8.3}$$

is a quaternionic matrix, subject to $\mathrm{Sca}(\mathbf{r}+\mathbf{o}) = 0$. The conversion relation is indeed a Lie algebra isomorphism of $sl(2,\mathbb{H}) \ni \mathbf{B}$ onto $so(1,5) \ni \mathbf{A}$.

Parametrizing the spin transformation $\mathbf{U} \in SL(2,\mathbb{H})$,

$$\mathbf{H}(t) = \mathbf{U}(t)\mathbf{H}(0)\bar{\mathbf{U}}^{\mathrm{T}}(t), \tag{8.4}$$

differentiating (8.4) with respect to t, and using (8.1) we thus obtain

$$\begin{bmatrix} \dot{x}^+ & \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} & \dot{x}^- \end{bmatrix} = \begin{bmatrix} \mathbf{r} & \mathbf{q} \\ \mathbf{p} & \mathbf{o} \end{bmatrix}\begin{bmatrix} x^+ & \bar{\mathbf{x}} \\ \mathbf{x} & x^- \end{bmatrix} + \begin{bmatrix} x^+ & \bar{\mathbf{x}} \\ \mathbf{x} & x^- \end{bmatrix}\begin{bmatrix} \bar{\mathbf{r}} & \bar{\mathbf{p}} \\ \bar{\mathbf{q}} & \bar{\mathbf{o}} \end{bmatrix}. \tag{8.5}$$

It can be written as

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix} = \begin{bmatrix} x^+(\mathbf{r}+\bar{\mathbf{r}}) + \mathbf{q}\mathbf{x} + \bar{\mathbf{x}}\bar{\mathbf{q}} \\ x^-\mathbf{q} + x^+\bar{\mathbf{p}} + \mathbf{r}\bar{\mathbf{x}} + \bar{\mathbf{x}}\bar{\mathbf{o}} \\ x^+\mathbf{p} + x^-\bar{\mathbf{q}} + \mathbf{o}\mathbf{x} + \mathbf{x}\bar{\mathbf{r}} \\ x^-(\mathbf{o}+\bar{\mathbf{o}}) + \mathbf{p}\bar{\mathbf{x}} + \mathbf{x}\bar{\mathbf{p}} \end{bmatrix}. \tag{8.6}$$

From (7.28) it follows that

$$\begin{aligned}
\bar{\mathbf{x}}\bar{\mathbf{q}} &= \bar{\mathbf{q}}\bar{\mathbf{x}} - 2 < \bar{\mathbf{q}}\bar{\mathbf{x}} >,\\
\bar{\mathbf{x}}\bar{\mathbf{o}} &= \bar{\mathbf{o}}\bar{\mathbf{x}} - 2 < \bar{\mathbf{o}}\bar{\mathbf{x}} >,\\
\mathbf{x}\bar{\mathbf{r}} &= \bar{\mathbf{r}}\mathbf{x} - 2 < \bar{\mathbf{r}}\mathbf{x} >,\\
\mathbf{x}\bar{\mathbf{p}} &= \bar{\mathbf{p}}\mathbf{x} - 2 < \bar{\mathbf{p}}\mathbf{x} > .
\end{aligned}$$

Substituting these equations into (8.6) yields

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix} = \begin{bmatrix} x^+(\mathbf{r}+\bar{\mathbf{r}}) + \mathbf{q}\mathbf{x} + \bar{\mathbf{q}}\bar{\mathbf{x}} - 2 < \bar{\mathbf{q}}\bar{\mathbf{x}} > \\ x^-\mathbf{q} + x^+\bar{\mathbf{p}} + \mathbf{r}\bar{\mathbf{x}} + \bar{\mathbf{o}}\bar{\mathbf{x}} - 2 < \bar{\mathbf{o}}\bar{\mathbf{x}} > \\ x^+\mathbf{p} + x^-\bar{\mathbf{q}} + \mathbf{o}\mathbf{x} + \bar{\mathbf{r}}\mathbf{x} - 2 < \bar{\mathbf{r}}\mathbf{x} > \\ x^-(\mathbf{o}+\bar{\mathbf{o}}) + \mathbf{p}\bar{\mathbf{x}} + \bar{\mathbf{p}}\mathbf{x} - 2 < \bar{\mathbf{p}}\mathbf{x} > \end{bmatrix}, \tag{8.7}$$

which can be rearranged to

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{r}+\bar{\mathbf{r}} & \bar{\mathbf{q}} - 2 < \bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r}+\bar{\mathbf{o}} - 2 < \bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o}+\bar{\mathbf{r}} - 2 < \bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}} - 2 < \bar{\mathbf{p}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}. \tag{8.8}$$

Also parametrizing (7.33) as

$$\begin{bmatrix} x^+(t) \\ \bar{\mathbf{x}}(t) \\ \mathbf{x}(t) \\ x^-(t) \end{bmatrix} = \mathbf{J}(t) \begin{bmatrix} x^+(0) \\ \bar{\mathbf{x}}(0) \\ \mathbf{x}(0) \\ x^-(0) \end{bmatrix}, \tag{8.9}$$

taking the time derivative of (8.9) and then using (8.9) again, we have

$$\begin{bmatrix} \dot{x}^+ \\ \dot{\bar{\mathbf{x}}} \\ \dot{\mathbf{x}} \\ \dot{x}^- \end{bmatrix} = \dot{\mathbf{J}}\mathbf{J}^{-1} \begin{bmatrix} x^+ \\ \bar{\mathbf{x}} \\ \mathbf{x} \\ x^- \end{bmatrix}. \tag{8.10}$$

Comparing the above equation with (8.8) yields

$$\dot{\mathbf{J}}\mathbf{J}^{-1} = \begin{bmatrix} \mathbf{r}+\bar{\mathbf{r}} & \bar{\mathbf{q}}-2<\bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r}+\bar{\mathbf{o}}-2<\bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o}+\bar{\mathbf{r}}-2<\bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}}-2<\bar{\mathbf{p}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix}. \tag{8.11}$$

For the proper orthochronous Lorentz transformation $\mathbf{G} \in SO_o(1,5)$, taking the time derivative of (7.41), and using (2.7), (8.11), and (7.41) again, we obtain

$$\mathbf{A} = \mathbf{C}^{-1} \begin{bmatrix} \mathbf{r}+\bar{\mathbf{r}} & \bar{\mathbf{q}}-2<\bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r}+\bar{\mathbf{o}}-2<\bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o}+\bar{\mathbf{r}}-2<\bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}}-2<\bar{\mathbf{p}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix} \mathbf{C}. \tag{8.12}$$

Letting

$$\mathbf{A} := \begin{bmatrix} A_{11} & \mathbf{A}_{1s} & A_{15} & A_{10} \\ \mathbf{A}_{s1} & \mathbf{A}_{ss} & \mathbf{A}_{s5} & \mathbf{A}_{s0} \\ A_{51} & \mathbf{A}_{5s} & A_{55} & A_{50} \\ A_{01} & \mathbf{A}_{0s} & A_{05} & A_{00} \end{bmatrix}, \tag{8.13}$$

and then substituting (7.39) for $\mathbf{C}^{-1}$ and (7.37) for $\mathbf{C}$ into (8.12), we obtain each term in **A** as follows:

$$A_{11} = \frac{1}{2}(\mathbf{r}+\bar{\mathbf{r}}+\mathbf{o}+\bar{\mathbf{o}}) - <\bar{\mathbf{o}}\,1> - <\bar{\mathbf{r}}\,1>, \tag{8.14}$$

$$\mathbf{A}_{1s} = \frac{i}{2}(\bar{\mathbf{r}}-\mathbf{r}+\mathbf{o}-\bar{\mathbf{o}}) + <\bar{\mathbf{o}}\,i> - <\bar{\mathbf{r}}\,i>, \tag{8.15}$$

$$A_{15} = \frac{1}{2}(\mathbf{p}+\bar{\mathbf{p}}-\mathbf{q}-\bar{\mathbf{q}}), \tag{8.16}$$

$$A_{10} = \frac{1}{2}(\mathbf{p}+\bar{\mathbf{p}}+\mathbf{q}+\bar{\mathbf{q}}), \tag{8.17}$$

$$\mathbf{A}_{s1} = \frac{i}{2}(\mathbf{r}-\bar{\mathbf{r}}+\bar{\mathbf{o}}-\mathbf{o}) - i<\bar{\mathbf{o}}\,1> + i<\bar{\mathbf{r}}\,1>, \tag{8.18}$$

$$\mathbf{A}_{ss} = \frac{1}{2}(\mathbf{r}+\bar{\mathbf{r}}+\mathbf{o}+\bar{\mathbf{o}}) + i<\bar{\mathbf{o}}\,i> + i<\bar{\mathbf{r}}\,i>, \tag{8.19}$$

$$\mathbf{A}_{s5} = \frac{i}{2}(\bar{\mathbf{p}}-\mathbf{p}-\mathbf{q}+\bar{\mathbf{q}}), \tag{8.20}$$

$$\mathbf{A}_{s0} = \frac{i}{2}(\bar{\mathbf{p}}-\mathbf{p}+\mathbf{q}-\bar{\mathbf{q}}), \tag{8.21}$$

$$A_{51} = \frac{1}{2}(\mathbf{q}+\bar{\mathbf{q}}-\mathbf{p}-\bar{\mathbf{p}}) - <\bar{\mathbf{q}}\,1> + <\bar{\mathbf{p}}\,1>, \tag{8.22}$$

$$\mathbf{A}_{5s} = \frac{i}{2}(\mathbf{p}-\bar{\mathbf{p}}+\mathbf{q}-\bar{\mathbf{q}}) + <\bar{\mathbf{q}}\,i> + <\bar{\mathbf{p}}\,i>, \tag{8.23}$$

$$A_{55} = \frac{1}{2}(\mathbf{r}+\bar{\mathbf{r}}+\mathbf{o}+\bar{\mathbf{o}}), \tag{8.24}$$

$$A_{50} = \frac{1}{2}(\mathbf{r}+\bar{\mathbf{r}}-\mathbf{o}-\bar{\mathbf{o}}), \tag{8.25}$$

$$A_{01} = \frac{1}{2}(\mathbf{p}+\bar{\mathbf{p}}+\mathbf{q}+\bar{\mathbf{q}}) - <\bar{\mathbf{q}}\,1> - <\bar{\mathbf{p}}\,1>, \tag{8.26}$$

$$\mathbf{A}_{0s} = \frac{i}{2}(\mathbf{q}-\bar{\mathbf{q}}+\bar{\mathbf{p}}-\mathbf{p}) + <\bar{\mathbf{q}}\,i> - <\bar{\mathbf{p}}\,i>, \tag{8.27}$$

$$A_{05} = \frac{1}{2}(\mathbf{r}+\bar{\mathbf{r}}-\mathbf{o}-\bar{\mathbf{o}}), \tag{8.28}$$

$$A_{00} = \frac{1}{2}(\mathbf{r}+\bar{\mathbf{r}}+\mathbf{o}+\bar{\mathbf{o}}). \tag{8.29}$$

First, we note due to $\mathrm{Sca}(\mathbf{r}+\mathbf{o}) = 0$ that

$$\mathbf{r}+\bar{\mathbf{r}}+\mathbf{o}+\bar{\mathbf{o}} = 2\mathrm{Sca}(\mathbf{r}+\mathbf{o}) = 0. \tag{8.30}$$

It thus follows that $A_{55} = A_{00} = 0$. Secondly, we show through some calculations that

$$i<\bar{\mathbf{o}}\,i\mathbf{x}> + i<\bar{\mathbf{r}}\,i\mathbf{x}> = (\tilde{\mathbf{o}}_s+\tilde{\mathbf{r}}_s)\mathbf{x}^s, \tag{8.31}$$

and thus,

$$i < \bar{\mathbf{o}}\, i > + i < \bar{\mathbf{r}}\, i > = \tilde{\mathbf{o}}_s + \tilde{\mathbf{r}}_s. \tag{8.32}$$

Substituting (8.30) and (8.32) into (8.19) gives $\mathbf{A}_{ss} = \tilde{\mathbf{o}}_s + \tilde{\mathbf{r}}_s$. Now, by using

$$\begin{gathered} < \bar{\mathbf{o}}\, 1 > = 0, \quad < \bar{\mathbf{p}}\, 1 > = 0, \quad < \bar{\mathbf{q}}\, 1 > = 0, \quad < \bar{\mathbf{r}}\, 1 > = 0, \\ < \bar{\mathbf{o}}\, i > = 0, \quad < \bar{\mathbf{p}}\, i > = 0, \quad < \bar{\mathbf{q}}\, i > = 0, \quad < \bar{\mathbf{r}}\, i > = 0 \end{gathered}$$

in the other equations in (8.14)-(8.29), and noting that

$$\mathbf{x} + \bar{\mathbf{x}} = 2x^1 = 2\mathrm{Sca}(\mathbf{x}), \quad \mathbf{x} - \bar{\mathbf{x}} = 2i\mathbf{x}^s = 2i\mathrm{Vec}(\mathbf{x}),$$

we eventually obtain

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{r}_s^{\mathrm{T}} - \mathbf{o}_s^{\mathrm{T}} & p_1 - q_1 & p_1 + q_1, \\ \mathbf{o}_s - \mathbf{r}_s & \tilde{\mathbf{o}}_s + \tilde{\mathbf{r}}_s & \mathbf{p}_s + \mathbf{q}_s & \mathbf{p}_s - \mathbf{q}_s \\ q_1 - p_1 & -\mathbf{p}_s^{\mathrm{T}} - \mathbf{q}_s^{\mathrm{T}} & 0 & r_1 - o_1 \\ p_1 + q_1 & \mathbf{p}_s^{\mathrm{T}} - \mathbf{q}_s^{\mathrm{T}} & r_1 - o_1 & 0 \end{bmatrix}. \tag{8.33}$$

The above formula enables us to obtain $\mathbf{A}$ from $\mathbf{B}$. It is easy to check that such $\mathbf{A}$, satisfying (2.2) with $n = 5$, is an element of the Lie algebra $so(1,5)$.

In order to derive $\mathbf{B}$ from $\mathbf{A}$, we let

$$\mathbf{A} = \begin{bmatrix} 0 & \mathbf{A}_{1s}^{\mathrm{T}} & A_{15} & A_{10} \\ -\mathbf{A}_{1s} & \mathbf{A}_{ss} & \mathbf{A}_{s5} & \mathbf{A}_{s0} \\ -A_{15} & -\mathbf{A}_{s5}^{\mathrm{T}} & 0 & A_{50} \\ A_{10} & \mathbf{A}_{s0}^{\mathrm{T}} & A_{50} & 0 \end{bmatrix} \tag{8.34}$$

by considering the 6×6 real matrix $\mathbf{A} \in so(1,5)$. In above $\mathbf{A}_{ss}$ is a 3×3 matrix function. In the above $\mathbf{A}_{ss}$ is a 3×3 skew-symmetric matrix. Multiplying (8.12) by $\mathbf{C}$ from the left-hand side, then substituting (8.34) for $\mathbf{A}$ and (7.37) for $\mathbf{C}$ into the resultant, we obtain

$$\begin{aligned} &\begin{bmatrix} \mathbf{r} + \bar{\mathbf{r}} & \bar{\mathbf{q}} - 2 < \bar{\mathbf{q}} & \mathbf{q} & 0 \\ \bar{\mathbf{p}} & \mathbf{r} + \bar{\mathbf{o}} - 2 < \bar{\mathbf{o}} & 0 & \mathbf{q} \\ \mathbf{p} & 0 & \mathbf{o} + \bar{\mathbf{r}} - 2 < \bar{\mathbf{r}} & \bar{\mathbf{q}} \\ 0 & \mathbf{p} & \bar{\mathbf{p}} - 2 < \bar{\mathbf{p}} & \mathbf{o} + \bar{\mathbf{o}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{A}_{1s}^{\mathrm{T}} & A_{15} & A_{10} \\ -\mathbf{A}_{1s} & \mathbf{A}_{ss} & \mathbf{A}_{s5} & \mathbf{A}_{s0} \\ -A_{15} & -\mathbf{A}_{s5}^{\mathrm{T}} & 0 & A_{50} \\ A_{10} & \mathbf{A}_{s0}^{\mathrm{T}} & A_{50} & 0 \end{bmatrix}. \end{aligned} \tag{8.35}$$

Expanding the above quaternion algebraic equation generates

$$\begin{bmatrix} \mathbf{q}+\bar{\mathbf{q}} & i(\mathbf{q}-\bar{\mathbf{q}}) & \mathbf{r}+\bar{\mathbf{r}} & \mathbf{r}+\bar{\mathbf{r}} \\ \mathbf{r}+\bar{\mathbf{o}} & -i(\mathbf{r}+\bar{\mathbf{o}}+2\tilde{\mathbf{o}}_s) & \bar{\mathbf{p}}-\mathbf{q} & \bar{\mathbf{p}}+\mathbf{q} \\ \mathbf{o}+\bar{\mathbf{r}} & i(\mathbf{o}+\bar{\mathbf{r}}+2\tilde{\mathbf{r}}_s) & \mathbf{p}-\bar{\mathbf{q}} & \mathbf{p}+\bar{\mathbf{q}} \\ \mathbf{p}+\bar{\mathbf{p}} & i(\bar{\mathbf{p}}-\mathbf{p}) & -\mathbf{o}-\bar{\mathbf{o}} & \mathbf{o}+\bar{\mathbf{o}} \end{bmatrix}$$
$$= \begin{bmatrix} A_{10}-A_{15} & \mathbf{A}_{s0}^{\mathrm{T}}-\mathbf{A}_{s5}^{\mathrm{T}} & A_{50} & A_{50} \\ i\mathbf{A}_{1s} & \mathbf{A}_{1s}^{\mathrm{T}}-i\mathbf{A}_{ss} & A_{15}-i\mathbf{A}_{s5} & A_{10}-i\mathbf{A}_{s0} \\ -i\mathbf{A}_{1s} & \mathbf{A}_{1s}^{\mathrm{T}}+i\mathbf{A}_{ss} & A_{15}+i\mathbf{A}_{s5} & A_{10}+i\mathbf{A}_{s0} \\ A_{10}+A_{15} & \mathbf{A}_{s0}^{\mathrm{T}}+\mathbf{A}_{s5}^{\mathrm{T}} & A_{50} & -A_{50} \end{bmatrix}, \tag{8.36}$$

where $<\bar{\mathbf{o}}\ i>=-i\tilde{\mathbf{o}}_s$, $<\bar{\mathbf{r}}\ i>=-i\tilde{\mathbf{r}}_s$, $<\bar{\mathbf{o}}\ 1>=0$, $<\bar{\mathbf{q}}\ 1>=0$, $<\bar{\mathbf{r}}\ 1>=0$, $<\bar{\mathbf{p}}\ 1>=0$, $<\bar{\mathbf{p}}\ i>=0$, and $<\bar{\mathbf{q}}\ i>=0$ were used.

From (8.36) we obtain

$$\mathbf{B} = \begin{bmatrix} \mathbf{r} & \mathbf{q} \\ \mathbf{p} & \mathbf{o} \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} A_{50}+i(\mathrm{axial}(\mathbf{A}_{ss})+\mathbf{A}_{1s}) & A_{10}-A_{15}+i(\mathbf{A}_{s5}-\mathbf{A}_{s0}) \\ A_{10}+A_{15}+i(\mathbf{A}_{s5}+\mathbf{A}_{s0}) & -A_{50}+i(\mathrm{axial}(\mathbf{A}_{ss})-\mathbf{A}_{1s}) \end{bmatrix}, \tag{8.37}$$

in which $\mathrm{axial}(\mathbf{A}_{ss})$ denotes the axial vector of $\mathbf{A}_{ss}$, that is,

$$(\mathrm{axial}(\mathbf{A}_{ss}))_i = \frac{1}{2}\epsilon_{ijk}(\mathbf{A}_{ss})_{kj},$$

where ϵ_{ijk} is the Levi-Civita permutation symbol. It is obvious that $\mathbf{B}$ thus obtained is an element of $sl(2,\mathbb{H})$. Formula (8.37) enables us to obtain $\mathbf{B}$ from $\mathbf{A}$. Formulae (8.13) and (8.37) explicitly expressing the Lie algebras isomorphism of $sl(2,\mathbb{H})$ and $so(1,5)$ are very important for further calculations. Both the Lie algebras $sl(2,\mathbb{H})$ and $so(1,5)$ have fifteen independent parameters.

Here we are concerned with the quaternionic two-component spinor space and the dynamical systems on the space. We return to the matrix $\mathbf{H}$ defined in (7.15) and remark that if the vector $\mathbf{X}\in\mathbb{M}^6$ is null, *i.e.* $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X}=0$, then $\mathbf{H}$ may be written as the dyadic product of a quaternion two-dimensional vector and its conjugate transpose:

$$\mathbf{H} = \begin{bmatrix} x^0+x^5 & x^1-i\mathbf{x}^s \\ x^1+i\mathbf{x}^s & x^0-x^5 \end{bmatrix} = 2\begin{bmatrix} \alpha^1\bar{\alpha}^1 & \alpha^1\bar{\alpha}^2 \\ \alpha^2\bar{\alpha}^1 & \alpha^2\bar{\alpha}^2 \end{bmatrix} = 2\alpha\bar{\alpha}^{\mathrm{T}}. \tag{8.38}$$

We are therefore led to consider a quaternion two-dimensional vector space $\mathbb{H}^2$ with elements $\boldsymbol{\alpha}$, on which $SL(2,\mathbb{H})$ left acts. This is a *quaternion spin-space* and the elements are *quaternion spinors.*

From (8.38) it follows that

$$\begin{aligned} x^0 &= \|\boldsymbol{\alpha}\|^2 = \|\boldsymbol{\alpha}^1\|^2 + \|\boldsymbol{\alpha}^2\|^2, \quad x^1 = 2\,\mathrm{Sca}(\boldsymbol{\alpha}^1\bar{\boldsymbol{\alpha}}^2), \\ \mathbf{x}^s &= 2\,\mathrm{Vec}(\boldsymbol{\alpha}^2\bar{\boldsymbol{\alpha}}^1), \quad x^5 = \|\boldsymbol{\alpha}^1\|^2 - \|\boldsymbol{\alpha}^2\|^2. \end{aligned} \tag{8.39}$$

It deserves to note that the two $\pm\boldsymbol{\alpha}$ lead to the same $\mathbf{X}$, and the *two quaternion components* $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$ suffice to determine the *six real components* $x^1, \cdots, x^5, x^0$ as shown in (8.39), and that the map $SL(2,\mathbb{H}) \to SO_o(1,5)$ as shown in (7.41) is a two-to-one surjective covering. With this advantage we may identify $\mathbf{X} = (\mathbf{X}^s, X^0) = (x^1, \mathbf{x}^s, x^5, x^0)$ and consider the following equation system for quaternion spinor:

$$\dot{\boldsymbol{\alpha}} = \mathbf{B}\boldsymbol{\alpha}, \tag{8.40}$$

where $\mathbf{B}$ as defined in (8.37) is a quaternion matrix determined uniquely by $\mathbf{A}$. If we can solve (8.40) instead of (1.1), it naturally gives $\mathbf{X}$ by the above correspondence (8.39).

As that done for the complex spinor equation (1.4) with $\mathbb{K} = \mathbb{C}$ in Subsection 3.2, we can convert the above quaternion equation (8.40) to the unit quaternion equation (1.3) with $\mathbb{K} = \mathbb{H}$. The relations of quaternion spinor $\boldsymbol{\alpha}$ and unit quaternion spinor $\mathbf{n}$ is still defined by (3.4). Therefore, we have systems (1.4) and (1.3) with $\mathbb{K} = \mathbb{H}$ and with the same $\mathbf{B} \in sl(2,\mathbb{H})$. Correspondingly, the equation in $\mathbb{M}^6$ and its projection are given by systems (1.1) and (1.2) with $n = 5$. The matrix $\mathbf{A}$ given in (8.34) satisfying (2.2) is an element of $so(1,5)$.

Because of $\mathbf{n} \in \mathbb{H}^2$ and $\|\mathbf{n}\| = 1$, we may identify the unit quaternion spinor

$$\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} n_1^1 + n_1^2 \mathrm{i}_2 + n_1^3 \mathrm{i}_3 + n_1^4 \mathrm{i}_4 \\ n_2^1 + n_2^2 \mathrm{i}_2 + n_2^3 \mathrm{i}_3 + n_2^4 \mathrm{i}_4 \end{bmatrix} \tag{8.41}$$

with the point $(n_1^1, n_1^2, n_1^3, n_1^4, n_2^1, n_2^2, n_2^3, n_2^4)$ on the unit sphere $\|\mathbf{n}\|^2 = (n_1^1)^2 + (n_1^2)^2 + (n_1^3)^2 + (n_1^4)^2 + (n_2^1)^2 + (n_2^2)^2 + (n_2^3)^2 + (n_2^4)^2 = 1$ in $\mathbb{R}^8$, and hence $\mathbf{n} \in \mathbb{S}^7$. On the other hand, from (2.4) with $n = 5$ we know that $\mathbf{m} \in \mathbb{S}^4$ if $\mathbf{X}$ is null. The map from $\mathbb{S}^7$ to $\mathbb{S}^4$ is one of the Hopf bundles [26].

Figure 3 summarizes the relations of the Lie group homomorphism from $SL(2,\mathbb{H})$ onto $SO_o(1,5)$, the Lie algebra isomorphism of $sl(2,\mathbb{H})$

onto $so(1,5)$, the two linear systems (1.4) with $\mathbb{K} = \mathbb{H}$ and (1.1) with $n = 5$, and the two nonlinear systems (1.3) with $\mathbb{K} = \mathbb{H}$ and (1.2) with $n = 5$.

9. Conclusions

Due to the success of developing new approach with real, complex and quaternion spinor techniques, in this paper we have investigated the profound relationships of the four dynamical systems through differential and algebraic methods. The results are summarized in Figure 3, which, among others, shows the relations of the linear systems (1.1) with $n = 3, 5$ and (1.4) with $\mathbb{K} = \mathbb{C}, \mathbb{H}$, the two nonlinear systems (1.2) with $n = 3, 5$ and (1.3) with $\mathbb{K} = \mathbb{C}, \mathbb{H}$, the two groups $SL(2, \mathbb{K})$ with $\mathbb{K} = \mathbb{C}, \mathbb{H}$, and $SO_o(1, n)$ with $n = 3, 5$, and their Lie algebras $sl(2, \mathbb{K})$ with $\mathbb{K} = \mathbb{C}, \mathbb{H}$, and $so(1, n)$ with $n = 3, 5$. A similar result and figure can also be drawn for the relations of those with $n = 2$ and $\mathbb{K} = \mathbb{R}$, the groups $SL(2, \mathbb{R})$ and $SO_o(1, 2)$, and their Lie algebras $sl(2, \mathbb{R})$ and $so(1, 2)$ as summarized in Figure 3.

In the framework of spinors we have constructed the following two-to-one relations: $\mathbf{n} \in \mathbb{S}^1 \xrightarrow{2:1} \mathbf{m} \in \mathbb{S}^1$, $\mathbf{n} \in \mathbb{S}^3 \xrightarrow{2:1} \mathbf{m} \in \mathbb{S}^2$ and $\mathbf{n} \in \mathbb{S}^7 \xrightarrow{2:1} \mathbf{m} \in \mathbb{S}^4$. They are governed by the nonlinear differential equations, $\mathbf{n}$ is a *unit spinor* governed by the third degree nonlinear differential equations, and $\mathbf{m}$ is *the projection of internal spacetime* governed by the second degree nonlinear differential equations. The two equations have found applications of importance in physical problems. The significance of converting systems (1.2) or (1.3) to systems (1.1) or (1.4) lies in the exact linearization, and that of converting systems (1.1), (1.2), or (1.3) to system (1.4) lies in the reduction of order. We have also shown that the spinor transformation plays a dominant role in solving all the four systems.

Appendix: Related Dynamical Systems and Spinor Maps

In this appendix we briefly study dynamical systems with symmetries $SO_o(1,1) \cong \{e^{g\theta}\}$, $SO(2) \cong U(1)$, $SO(3)$, and $SU(2)$, and the spinor maps from $\{e^{g\theta}\}$ onto $SO_o(1,1)$, from $U(1)$ onto $SO(2)$, and from $SU(2)$ onto $SO(3)$. This may help comprehend the hierachy shown in Figure 2.

System (1.1) with $n = 1$ has the symmetry $SO_o(1,1)$, which is isomorphic to $\{e^{g\theta} \mid e^{g\theta} = \cosh\theta + g\sinh\theta, \theta \in \mathbb{R}, g^2 = 1\}$ as shown in [21], where $e^{g\theta}$ is called the double number [22]. As a special case of system (1.2) and also of system (1.1), the dynamical system,

$$\begin{cases} \dot{\mathbf{m}} = \mathbf{A}^s{}_s\mathbf{m}, \\ (\mathbf{A}^s{}_s)^{\mathrm{T}} = -\mathbf{A}^s{}_s, \quad \mathbf{A}^s{}_s \in \mathbb{R}^{n\times n}, \quad \mathbf{m} \in \mathbb{R}^{n\times 1}, \quad n = 2, 3, \end{cases}$$

has the internal symmetry $SO(n)$. Group $SO(2)$ is isomorphic to group $U(1) := \{e^{i\theta} \mid e^{i\theta} = \cos\theta + i\sin\theta, \theta \in \mathbb{R}, i^2 = -1\}$, where $e^{i\theta}$ is the unimodular complex number. As a special case of system (1.3) and also of system (1.4), the dynamical system,

$$\begin{cases} \dot{\mathbf{n}} = \mathbf{B}\mathbf{n}, \\ \operatorname{tr}\mathbf{B} = 0, \quad \bar{\mathbf{B}}^{\mathrm{T}} = -\mathbf{B}, \quad \mathbf{B} \in \mathbb{C}^{2\times 2}, \quad \mathbf{n} \in \mathbb{C}^{2\times 1}, \end{cases}$$

has the internal symmetry $SU(2)$.

The spinor maps from $\{e^{g\theta}\}$ onto $SO_o(1,1)$ and from $U(1)$ onto $SO(2)$ are one-to-one. Now, similar to Subsections 4.1 and 4.4, we outline the algebraic approach to the spinor map from $SU(2)$ onto $SO(3)$. Deleting the x^0 component in (4.1) and following the similar procedures in Section 4 we can obtain a well known spinor map from $SU(2) \ni \mathbf{U}$ onto $SO(3) \ni \mathbf{G}$ as given by (4.10) but with

$$\mathbf{C} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 1 & i \end{bmatrix}, \quad \mathbf{C}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{bmatrix},$$

and

$$\mathbf{R} := \begin{bmatrix} U^1{}_1\bar{U}^1{}_1 - U^1{}_2\bar{U}^1{}_2 & U^1{}_1\bar{U}^1{}_2 & U^1{}_2\bar{U}^1{}_1 \\ U^1{}_1\bar{U}^2{}_1 - U^1{}_2\bar{U}^2{}_2 & U^1{}_1\bar{U}^2{}_2 & U^1{}_2\bar{U}^2{}_1 \\ U^2{}_1\bar{U}^1{}_1 - U^2{}_2\bar{U}^1{}_2 & U^2{}_1\bar{U}^1{}_2 & U^2{}_2\bar{U}^1{}_1 \end{bmatrix}.$$

Simultaneously, the Lie algebra isomorphism of $su(2) \ni \mathbf{B}$ onto $so(3) \ni \mathbf{A}$ can be derived as follows:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 & \mathrm{Im}\,(B_{11} - B_{22}) & \mathrm{Re}(B_{21} - B_{12}) \\ -\,\mathrm{Im}\,(B_{11} - B_{22}) & 0 & -\,\mathrm{Im}\,(B_{21} + B_{12}) \\ -\mathrm{Re}(B_{21} - B_{12}) & \mathrm{Im}\,(B_{12} + B_{21}) & 0 \end{bmatrix}, \\ \mathbf{B} &= \frac{1}{2}\begin{bmatrix} iA_{12} & iA_{23} - A_{13} \\ A_{13} + iA_{23} & -iA_{12} \end{bmatrix}. \end{aligned}$$

It is obvious that $\mathbf{A}^{\mathrm{T}} + \mathbf{A} = \mathbf{0}$, and that $\bar{\mathbf{B}}^{\mathrm{T}} + \mathbf{B} = \mathbf{0}$ and $\operatorname{tr}\mathbf{B} = 0$.

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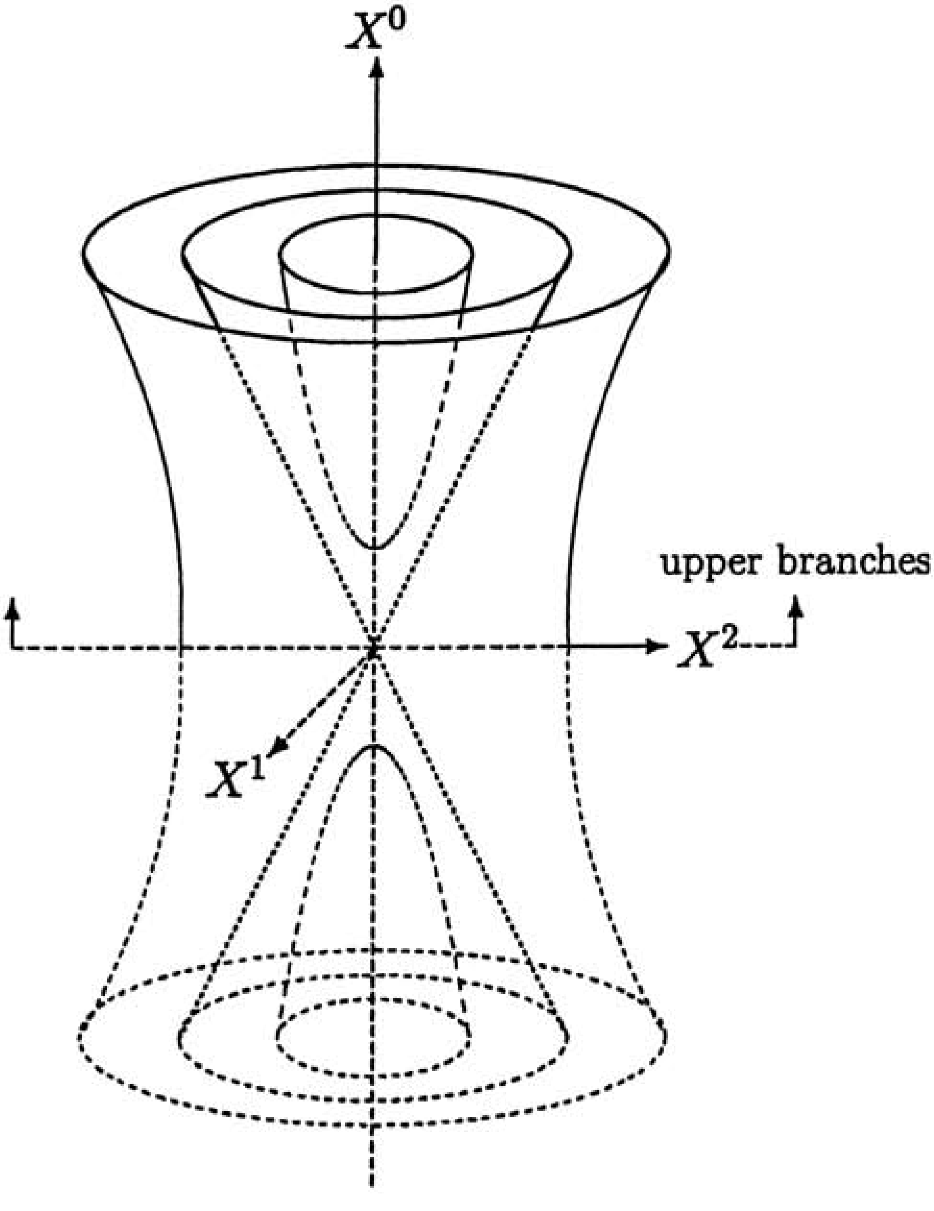

Figure 1: The three types of geometric sets of $\mathbf{X}^{\mathrm{T}}\mathbf{g}\mathbf{X} = r$ and $X^0 > 0$. Depending on the value of r the sets are the upper branches of a hyperboloid of two sheets ($r > 0$), a cone ($r = 0$), and a hyperboloid of one sheet ($r < 0$)

$$
\begin{array}{rcccl}
 & \{e^{g\theta}\} & \xrightarrow{1:1} & SO_o(1,1) & \\
 & \cap & & \cap & \\
 & SL(2,\mathbf{R}) & \xrightarrow{2:1} & SO_o(1,2) & \\
 & \cap & & \cap & \\
\text{Spinor map:} & \{SL(2,\mathbf{C}) & \xrightarrow{2:1} & SO_o(1,3)\} & \subset \{SL(2,\mathbb{H}) \xrightarrow{2:1} SO_o(1,5)\} \\
 & \cup & & \cup & \\
 & SU(2) & \xrightarrow{2:1} & SO(3) & \\
 & \cup & & \cup & \\
 & U(1) & \xrightarrow{1:1} & SO(2) & \\
 & & & & \\
\text{Spinor map:} & \text{STs } \pm\text{U} & \longmapsto & \text{POLT G} &
\end{array}
$$

Figure 2: A hierachy of spinor maps. $\subset$ means "is a subgroup of." ST is the abbreviation of the spin transformation, and POLT the proper orthochronous Lorentz transformation. Recall the isomorphisms $SO_o(1,1) \cong \{e^{g\theta}\}$ and $SO(2) \cong U(1)$

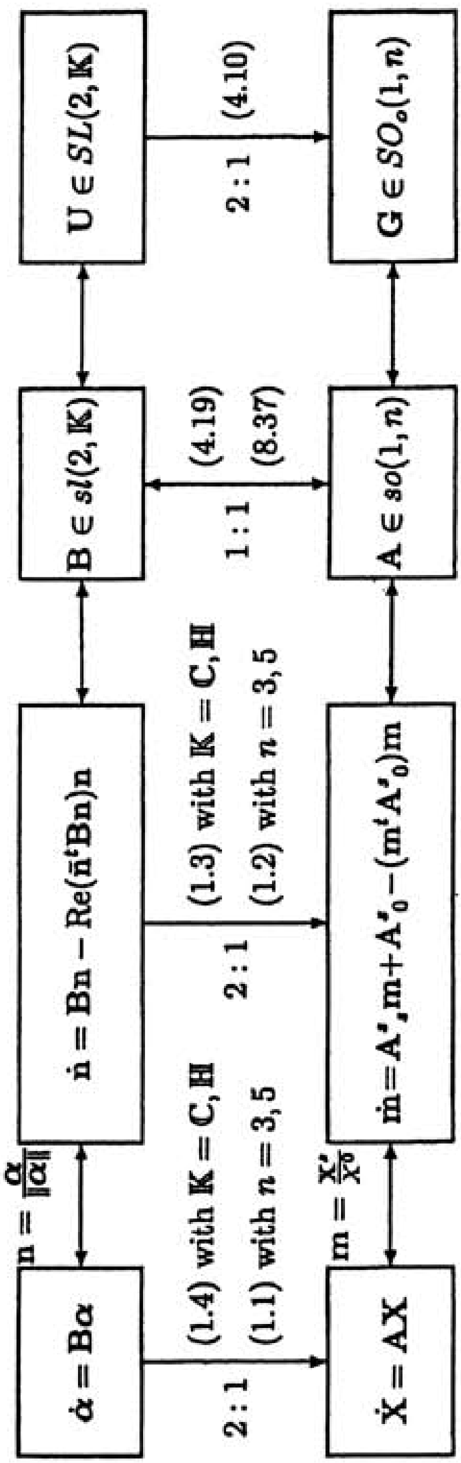

Figure 3: The relations of the four dynamical systems ($n = 3, \ell = 2, \mathbb{K} = \mathbb{C}$) and their solutions and fundamental solutions. The upper layer is 2 : 1 covering of the lower layer except for the Lie algebra isomorphism of $sl(2, \mathbb{C})$ onto $so(1, 3)$, which is 1 : 1

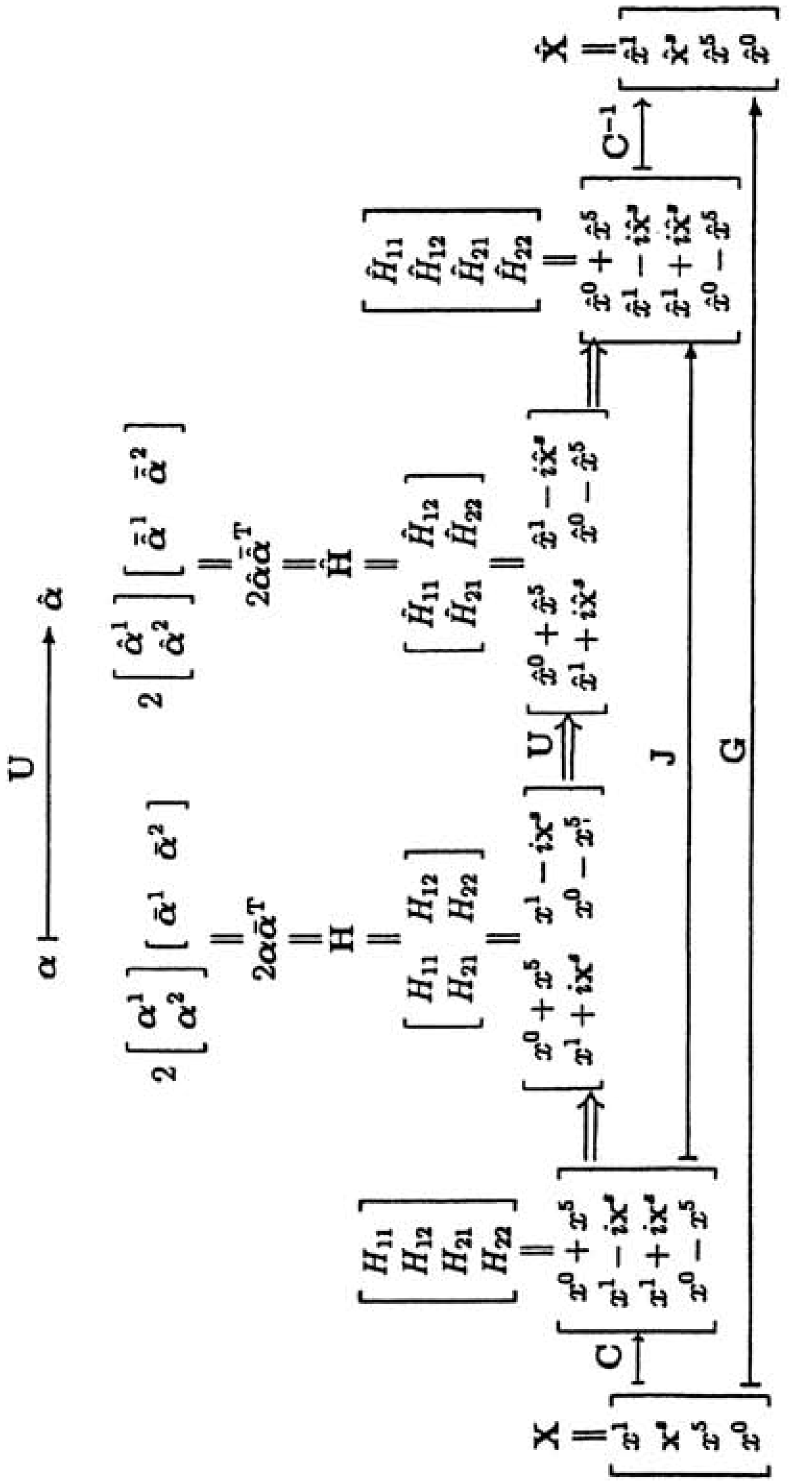

Figure 4: The algebraic procedure for constructing the spinor map from $SL(2,\mathbb{H})$ onto $SO_o(1,5)$