REPORTS ON REPRESENTATION THEORY OF CONFORMAL SUPERALGEBRAS

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In this note we report on our works on the representation theory of conformal superalgebras [5] [8] funded by NSC-grant 89-2115-M-006-002 and 89-2115-M-002-040 of the R.O.C. We like to express our gratitude to the National Science Council for supporting this project.

1. Conformal superalgebras and Modules over conformal superalgebras

1.1. Conformal Superalgebras. We first recall the notions of a conformal superalgebra, which is a straightforward adaption of a conformal algebra to the super case.

A *conformal superalgebra* is a left $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module R with a $\mathbb{C}$-bilinear product $a_{(n)}b$ for each $n \in \mathbb{Z}_+$ such that the following axioms hold ($a, b, c \in R; m, n \in \mathbb{Z}_+$ and $\partial^{(j)} = \frac{1}{j!}\partial^j$):

(C0) $a_{(n)}b = 0$, for $n >> 0$,
(C1) $(\partial a)_{(n)}b = -na_{(n-1)}b$,
(C2) $a_{(n)}b = (-1)^{p(a)p(b)} \sum_{j=0}^{\infty}(-1)^{j+n+1}\partial^{(j)}(b_{(n+j)}a)$,
(C3) $a_{(m)}(b_{(n)}c) = \sum_{j=0}^{\infty} \binom{m}{j}(a_{(j)}b)_{(m+n-j)}c + (-1)^{p(a)p(b)}b_{(n)}(a_{(m)}c)$.

Above $p(a)$ denotes the parity of a. Note that (C1) and (C2) implies that $a_{(n)}\partial b = \partial(a_{(n)}b) + na_{(n-1)}b$, and thus ∂ is a derivation of all products.

The simplest non-trivial example of a conformal algebra, that is a conformal algebra with trivial odd part, is the *Virasoro conformal algebra* $R(\mathfrak{V}) = \mathbb{C}[\partial] \otimes L$ with products

$$L_{(0)}L = \partial L, \quad L_{(1)}L = 2L, \quad L_{(j)}L = 0, \quad \text{for } j > 1.$$

Another example is provided by the current conformal algebra, which is defined as follows: Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. We let $R(\tilde{\mathfrak{g}}) = \mathbb{C}[\partial] \otimes \mathfrak{g}$ with non-zero products defined by $a_{(0)}b = [a, b]$,for $a, b \in \mathfrak{g}$.

Starting with a simple Lie algebra $\mathfrak{g}$ and an odd indeterminate θ generating the Grassmann superalgebra $\Lambda(1)$ we may construct $R(\widetilde{\mathfrak{g} \otimes \Lambda(1)})$. This superalgebra, usually denoted by $R(\tilde{\mathfrak{g}}_{\text{super}})$, is called the supercurrent conformal algebra. It is

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generated over $\mathbb{C}[\partial]$ by $\{a, a^\theta | a \in \mathfrak{g}\}$, where $a^\theta = a \otimes \theta$. The non-zero products are then $a_{(0)}b = [a,b]$ and $a_{(0)}b^\theta = [a,b]^\theta$, for $a, b \in \mathfrak{g}$.

Let R be a conformal superalgebra. Suppose that R contains as a subalgebra a Virasoro conformal algebra. Then we will call R a *superconformal algebra*. There are several examples of superconformal algebras and they can be found in [7]. We refer the interested reader to [7] for their explicit description.

1.2. **Modules over Conformal superalgebras.** Next we recall the representation theory of conformal superalgebras, i.e. the theory of conformal modules.

A module M over a conformal superalgebra R is defined as follows [1], [6]. It is a (left) $\mathbb{Z}_2$-graded $\mathbb{C}[\partial]$-module equipped with a family of $\mathbb{C}$-linear maps $a \to a^M_{(n)}$ of R to $\mathrm{End}_{\mathbb{C}}M$, for each $n \in \mathbb{Z}_+$, such that the following properties hold for $a, b \in R$ and $m, n \in \mathbb{Z}_+$:

(M0) $a^M_{(n)}v = 0$, for $v \in M$ and $n >> 0$,

(M1) $[a^M_{(m)}, b^M_{(n)}] = \sum_{j=0}^m \binom{m}{j}(a_{(j)}b)^M_{(m+n-j)}$,

(M2) $(\partial a)^M_{(n)} = [\partial, a^M_{(n)}] = -na^M_{(n-1)}$.

Introducing the generating series

$$a_\lambda b = \sum_{n=0}^\infty a_{(n)}b\frac{\lambda^n}{n!} \quad \text{and} \quad a^M_\lambda = \sum_{n=0}^\infty a^M_{(n)}\frac{\lambda^n}{n!},$$

which lie in $\mathbb{C}[\lambda] \otimes_{\mathbb{C}} R$ and $\mathbb{C}[\lambda] \otimes \mathrm{End}_{\mathbb{C}}(M)$ due to (C0) and (M0), respectively.

The modules over the Virasoro, current and supercurrent conformal algebras have been classified in [1]. The extension problem for modules over the Virasoro, current and their semidirect sums also have solved in [3]. The extension problem over the supercurrent conformal algebras was solved in [8].

2. The extension problem for the sueprcurrent algebra

In this section we report on the results obtained in [8]. Let us consider extensions of modules of the form

$$0 \to E_1 \to E \to E_2 \to 0,$$

where E_1 and E_2 are finite irreducible modules over the supercurrent conformal algebra $R(\tilde{\mathfrak{g}}_{\text{super}})$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra. Let V be a finite-dimensional irreducible module over $\mathfrak{g}$. We may define an action of $R(\tilde{\mathfrak{g}}_{\text{super}})$ on the space $M(V) = \mathbb{C}[\partial] \otimes V$ by

$$a\lambda v = av, \quad a^\theta_\lambda v = 0.$$

Then $M(V)$ is an irreducible module over $R(\tilde{\mathfrak{g}}_{\text{super}})$.

Another way to construct an irreducible module over $R(\tilde{\mathfrak{g}}_{\text{super}})$ is as follows. Let $\alpha \in \mathbb{C}$ and let $\mathbb{C}c_\alpha$. Define an action on of $R(\tilde{\mathfrak{g}}_{\text{super}})$ on $\mathbb{C}c_\alpha$ by $a\lambda c_\alpha = a^\theta_\lambda c_\alpha = 0$ and $\partial c_\alpha = \alpha c_\alpha$.

The following theorem was proved in [1].

Theorem 2.1. *Every finite irreducible module over $R(\tilde{\mathfrak{g}}_{\mathrm{super}})$ is either $M(V)$, where V is a finite-dimensional irreducible module over $\mathfrak{g}$, or else it is $\mathbb{C}c_\alpha$.*

Thus to solve the extension problem amounts to classifying E above with E_i being either $M(V)$ or $\mathbb{C}c_\alpha$. The case when both E_1, E_2 are of the same parity follows immediately from [3] so that we need not consider this case. We will put $M(V)_0$, if we regard $M(V)$ as an even space, and $M(V)_1$, if we regard it as an odd space. Similarly we write $\mathbb{C}\epsilon_\alpha$, if we regard $\mathbb{C}c_\alpha$ as an odd space.

The following three theorems describes the complete solution to the extension problem for the supercurrent conformal algebra [8].

Theorem 2.2. *There are no non-trivial extensions in the case when $E_1 = M_1(U)$ and $E_2 = \mathbb{C}c_\alpha$.*

Theorem 2.3. *A non-trivial extension in the case $E_1 = \mathbb{C}c_\alpha$ and $E_2 = M_1(U)$ exists only if $U \cong \mathfrak{g}$. Identifying E with $\mathbb{C}c_\alpha \oplus M_1(\mathfrak{g})$, the non-trivial extension is given by*

$$a_\lambda c_\alpha = 0, \quad a_\lambda c_\alpha = 0, \quad a^\theta_\lambda b = (a|b)c_\alpha,$$

where $(\cdot|\cdot)$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$.

Theorem 2.4. *Let U and V be two finite-dimensional irreducible $\mathfrak{g}$-modules. Identifying E with the $\mathbb{C}[\partial]$-module $M(U)_0 \oplus M(V)_1$, every non-trivial extension with $E_1 = M(U)_0$ and $E_2 = M(V)_1$ is given by*

$$a_\lambda u = au, \quad a_\lambda v = av, \quad a^\theta_\lambda u = \varphi(a \otimes u), \quad a^\theta_\lambda v = 0,$$

where $\varphi : \mathfrak{g} \otimes U \to V$ is a $\mathfrak{g}$-module homomorphism.

3. Representation theory of general superconformal algebras

The list of superconformal algebras given in [7] comprises of $W(N)$, $S(N)$, $K(N)$ and the exceptional algebra CK_6 [2], where $N \in \mathbb{Z}_+$. In [1] it was shown that the representation theory of superconformal algebras reduces to the representation theory of the corresponding annihilation subalgebras, which we will denote by $W(N)_+$, $S(N)_+$, $K(N)_+$ and $(CK_6)_+$. For the definition of the annihilation subalgebra the reader is referred to [1]. These algebras have a natural $\mathbb{Z}$-gradation of finite depth such that its zero-th graded components are respectively, $gl(1, N)$, $sl(1, N)$, $cso(N)$ and $gl(4)$ (see [5]).

Theorem 3.1. [5] *There exists a bijection between finite irreducible conformal modules over $W(N)$, $S(N)$, $K(N)$ and CK_6 with finite-dimensional irreducible representations of the finite-dimensional Lie superalgebras $gl(1, N)$, $sl(1, N)$, $cso(N)$ and $gl(4)$.*

To construct this bijection we proceed as follows. Let $\mathfrak{g}$ be any of the algebras $W(N)_+$, $S(N)_+$, $K(N)_+$ and $(CK_6)_+$ above. Then $\mathfrak{g} = \bigoplus_{j\geq -d}\mathfrak{g}_j$, where d is a positive integer. In first two cases $d = 2$, while in the last two cases $d = 1$. Let V be a finite-dimensional irreducible representation of $\mathfrak{g}_0$. We may extend V trivially to a module over $\bigoplus_{j\geq 0}\mathfrak{g}_j$. Then induce the module to a module over $\mathfrak{g}$ and call this module $M(V)$. Then $M(V)$ has a unique maximal submodule. The unique irreducible quotient is a finite irreducible module over the conformal superalgebra of $W(N)$, $S(N)$, $K(N)$ and (CK_6).

Even though Theorem 3.1 gives a correspondence between finite conformal modules over superconformal algebras and finite-dimensional irreducible modules over certain finite-dimensional Lie superalgebras, the explicit description of these finite conformal modules remain a rather difficult problem in general. In [5] we studied in more detail the cases of the $N = 2$, $N = 3$ and the two (small and big) $N = 4$ superconformal algebras. In these cases the corresponding finite-dimensional Lie (super)algebras are $gl(1)$, $gl(2)$, $gl(2)$ and $gl(2) \oplus sl(2)$, respectively. In these cases we can reduce the problem essentially to decomposing certain infinite-dimensional representations with respect to these finite-dimensional Lie algebras. As the Lie algebras involved are rather simple, we can obtain a very explicit description of the corresponding irreducible finite conformal modules. Our description is so explicit that we can read off the characters in a straight forward way. The interested reader is referred to [5] for more details.

Recently, together with Ruibin Zhang, we have studied in details the structure of conformal modules over the superconformal algebras of $W(N)$ and $S(N)$ types. In particular we have determined for which highest weight module V (of the corresponding finite-dimensional Lie superalgebra) the finite conformal module $M(V)$ remains irreducible. In turns out that generically the $M(V)$ is irreducible and only for a finite number of highest weight $M(V)$ becomes reducible. This problem has also been studied independently by Kac and Rudakov.

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