ELSEVIER

Statistics & Probability Letters 52 (2001) 401–411

STATISTICS & PROBABILITY LETTERS

www.elsevier.nl/locate/stapro

Some classes of orthogonal $2^{n_1} \times 3^{n_2}$ mixed factorial designs of median-resolution

C.T. Liao *

Division of Biometry, Institute of Agronomy, National Taiwan University, No. 1, Sec. 4, Roosevett Rd., Taipei 10764, Taiwan

Received November 1999; received in revised form November 2000

Abstract

In a $2^{n_1} \times 3^{n_2}$ mixed factorial experiment, we consider the orthogonal designs of median-resolution which can be used to estimate the grand mean, all main effects and certain classes of low-order interactions involving at least one of few specified factors, assuming that the other effects are negligible. Some classes of the desired designs constructed from the class of parallel-flats designs are proposed in this paper.

Keywords: Parallel-flats designs; Fractional factorial designs; Orthogonal arrays; Design resolutions

1. Introduction

During the initial stages of experimentation, the 2-level, 3-level or mixed 2-level and 3-level factorial designs was commonly used to estimate some factorial effects which were often specified according to the knowledge of the investigator and the *hierarchical ordering principle*; see Wu and Hamada (2000). The specified set of factorial effects may include the grand mean, all main effects and some low-order interactions. Wu and Chen (1992) presented a graph-aided method designed to meet this requirement for the 2-level factorial designs. The results were extended for the 3-level case in Sun and Wu (1994). Some of the earliest published works using the similar approach can be seen in Taguchi (1987). However, the designs they discussed were confined within the class of regular 2-level or 3-level fractional factorial designs. So these methods can only yield orthogonal designs with the run sizes that are powers of 2 or 3.

The class of *parallel-flats designs* (PFDs) was first introduced by Connor and Young (1961). A general theory of PFDs was presented in Srivastava et al. (1984) which was further developed in Srivastava (1987). Then Srivastava and Li (1996) discussed the construction of orthogonal designs of *median-resolution* from

* Corresponding author. Fax: +886-2-236-20879.
E-mail address: ctliao@ccms.ntu.edu.tw (C.T. Liao).

PII: S0167-7152(01)00007-4

PFDs. They presented three classes of orthogonal 2-level factorial designs which can be used to estimate the grand mean, all main effects and some specified 2-factor interactions which can be classified into the following three categories.

(I) 2-factor interactions within some subgroups of factors.
(II) 2-factor interactions between some subgroups of factors.
(III) 2-factor interactions within some subgroups of factors and 2-factor interactions between some subgroups of factors.

Note that some of the designs in Srivastava and Li (1996) are not the designs with run size being a power of 2. Liao (1999) extended the results for the 3-level case.

In this paper, we propose some classes of orthogonal $2^{n_1} \times 3^{n_2}$ mixed factorial designs of median-resolution constructed from PFDs. The designs obtained are similar to the designs of category (III) described above. In the next section, we introduce the notation used in the article; review the definition of PFDs; and describe a lemma for the orthogonality of $2^{n_1} \times 3^{n_2}$ mixed factorial designs. In Section 3, we presented some classes of orthogonal designs based on the lemma.

2. Preliminaries

Let $F_1, F_2, \ldots, F_{n_1}$ denote the n_1 2-level factors; and $F_{n_1+1}, F_{n_1+2}, \ldots, F_{n_1+n_2}$ denote the n_2 3-level factors. The expression $F_1^{e_{11}} F_2^{e_{12}} \ldots F_{n_1}^{e_{1n_1}} F_{n_1+1}^{e_{21}} F_{n_1+2}^{e_{22}} \ldots F_{n_1+n_2}^{e_{2n_2}}$ will represent a general factorial effect with e_{1i}'s being over GF[2], Galois field of order 2; and e_{2i}'s being over GF[3], Galois field of order 3. If e_{1i} is not equal to 0 then the 2-level factor F_i appears in the factorial effect; otherwise, it does not. Similarly, if e_{2i} is not equal to 0 then the 3-level factor F_{n_1+i} appears in the factorial effect; otherwise, it does not. The vector $\boldsymbol{e} = [\boldsymbol{e}_1, \boldsymbol{e}_2]$, where $\boldsymbol{e}_1 = [e_{11}, e_{12}, \ldots, e_{1n_1}]$ and $\boldsymbol{e}_2 = [e_{21}, e_{22}, \ldots, e_{2n_2}]$, is called the *defining vector* for the general factorial effect $F_1^{e_{11}} F_2^{e_{12}} \ldots F_{n_1}^{e_{1n_1}} F_{n_1+1}^{e_{21}} F_{n_1+2}^{e_{22}} \ldots F_{n_1+n_2}^{e_{2n_2}}$. Without loss of generality, the first nonzero entry in $\boldsymbol{e}_2$ is assumed to be 1 for any effects involving 3-level factors. Note that any factorial effects involving 3-level factors have 2 degrees of freedom. The defining vector $\boldsymbol{e} = [0, 0, \ldots, 0]$ is for the grand mean μ. For more details regarding the definition of factorial effects refer to Liao and Iyer (1998).

2.1. Parallel-flats designs

For $i = 1, 2, \ldots, f$, let D_{1i} be the set consisting of all $\boldsymbol{t}_1 = [t_{11}, t_{12}, \ldots, t_{1n_1}]$ satisfying

$$\boldsymbol{t}_1' = \boldsymbol{z}_{1i} + \boldsymbol{B}_1 \boldsymbol{v}_1,$$

where $\boldsymbol{B}_1$ is an $n_1 \times k_1$ matrix of rank k_1; $\boldsymbol{z}_{1i}$ is any $n_1 \times 1$ vector; and $\boldsymbol{v}_1$ ranges over all possible vectors of length k_1 over GF[2] (there are 2^{k_1} such vectors).

Similarly, for $i = 1, 2, \ldots, f$, let D_{2i} be the set consisting of all $\boldsymbol{t}_2 = [t_{21}, t_{22}, \ldots, t_{2n_1}]$ satisfying

$$\boldsymbol{t}_2' = \boldsymbol{z}_{2i} + \boldsymbol{B}_2 \boldsymbol{v}_2,$$

where $\boldsymbol{B}_2$ is an $n_2 \times k_2$ matrix of rank k_2; $\boldsymbol{z}_{2i}$ is any $n_2 \times 1$ vector; and v_2 ranges over all possible vectors of length k_2 over GF[3] (there are 3^{k_2} such vectors).

Furthermore, let D_i be the set consisting of the treatment combinations for the $2^{n_1} \times 3^{n_2}$ mixed factorial experiment given by

$$D_i = \{[\boldsymbol{t}_1, \boldsymbol{t}_2] \mid \boldsymbol{t}_1 \in D_{1i}, \boldsymbol{t}_2 \in D_{2i}\}.$$

A PFD determined by the pairs $(\boldsymbol{B}_1, \boldsymbol{Z}_1)$ and $(\boldsymbol{B}_2, \boldsymbol{Z}_2)$ for the $2^{n_1} \times 3^{n_2}$ factorial experiment is obtained by taking all the D_i together for $i = 1, 2, \ldots, f$. Here $\boldsymbol{Z}_1$ is the $n_1 \times f$ matrix whose columns are $z_{11}, z_{12}, \ldots, z_{1f}$ and $\boldsymbol{Z}_2$ is the $n_2 \times f$ matrix whose columns are $z_{21}, z_{22}, \ldots, z_{2f}$. Clearly, the PFD has $N = f \times 2^{k_1} \times 3^{k_2}$ treatment combinations.

2.2. Orthogonality for a PFD

Let $\boldsymbol{\beta}$ denote the vector of factorial effects that are not assumed to be zero. We consider some specific type of $\boldsymbol{\beta}$ which consists of grand mean μ, all main effects and some low-order interactions. This $\boldsymbol{\beta}$ has the *hierarchical property* considered in Liao (2000). Note that the hierarchical property stated here is somewhat different from the term *tree structure* used in the literature. A $\boldsymbol{\beta}$ is said to have hierarchical property if whenever one factorial effect occurs in the $\boldsymbol{\beta}$, then any factorial effects involving any subset of the set consisting of all of the factors involved in this factorial effect also occur in the $\boldsymbol{\beta}$. In general, the tree structure just indicates that at least one of the factorial effects involving some of the factors of the included factorial effect occurs in the $\boldsymbol{\beta}$. We thus apply the following lemma to construct the desired designs.

Lemma 2.1 (see Liao, 2000). *In a $2^{n_1} \times 3^{n_2}$ experiment, let a PFD be determined by the pairs $(\boldsymbol{B}_1, \boldsymbol{Z}_1)$ and $(\boldsymbol{B}_2, \boldsymbol{Z}_2)$. For a given factorial effects vector $\boldsymbol{\beta}$, which is assumed to have hierarchical property, we can define the set of generalized defining vectors R as follows*:

$$R = \left\{ [\boldsymbol{e}_{11} \oplus \boldsymbol{e}_{21}, \boldsymbol{e}_{12} \oplus \gamma \boldsymbol{e}_{22}] \;\middle|\; \begin{array}{l} [\boldsymbol{e}_{11}, \boldsymbol{e}_{12}] \textit{ and } [\boldsymbol{e}_{21}, \boldsymbol{e}_{22}] \textit{ are defining vectors corresponding to two distinct effects} \\ \textit{of } \boldsymbol{\beta} \textit{ and } \gamma = 1, 2 \textit{ over } GF[3]. \end{array} \right.$$

where the notation $\oplus$ denotes the componentwise addition of two vectors over GF[2] *or* GF[3] *without any confusion. Then the PFD is orthogonal for the $\boldsymbol{\beta}$ if and only if*
(1) *for $[\boldsymbol{e}_1, \boldsymbol{0}] \in R$, $\boldsymbol{e}_1 \boldsymbol{B}_1 = \boldsymbol{0}$ implies that $\boldsymbol{e}_1 \boldsymbol{Z}_1$ is an* OA(1);
(2) *for $[\boldsymbol{0}, \boldsymbol{e}_2] \in R$, $\boldsymbol{e}_2 \boldsymbol{B}_2 = \boldsymbol{0}$ implies that $\boldsymbol{e}_2 \boldsymbol{Z}_2$ is an* OA(1);
(3) *for $[\boldsymbol{e}_1, \boldsymbol{e}_2] \in R$, $\boldsymbol{e}_1 \boldsymbol{B}_1 = \boldsymbol{0}$ and $\boldsymbol{e}_2 \boldsymbol{B}_2 = \boldsymbol{0}$ imply that*

$$\begin{bmatrix} \boldsymbol{e}_1 \boldsymbol{Z}_1 \\ \boldsymbol{e}_2 \boldsymbol{Z}_2 \end{bmatrix}$$

is an OA(2).

Note that the notation OA(d) denotes an *orthogonal array of strength d*.

3. Orthogonal designs

In this section, we construct the desired orthogonal PFDs by appropriately choosing the rows of an OA(2) as the rows of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ matrices. Let $L_{N_0}(2^{d_1} \times 3^{d_2})$ denote the mixed 2-level and 3-level OA(2). Formally, $L_{N_0}(2^{d_1} \times 3^{d_2})$ is a $(d_1 + d_2) \times N_0$ matrix in which all possible combinations of levels in any 2 rows appear the same number of times (Rao, 1947, 1973). There are some $L_{N_0}(2^{d_1} \times 3^{d_2})$ with economical run size are available in the literature. For example, $L_{12}(2^4 \times 3)$, $L_{18}(2 \times 3^7)$, $L_{36}(2^{11} \times 3^{12})$ and $L_{36}(2^4 \times 3^{13})$, see Wang and Wu (1991). For convenience, we denote the rows of $L_{N_0}(2^{d_1} \times 3^{d_2})$ as $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_{d_1}$; $\boldsymbol{l}_{d_1+1}, \boldsymbol{l}_{d_1+2}, \ldots, \boldsymbol{l}_{d_1+d_2}$.

Furthermore, we also describe the pair $(\boldsymbol{B}_1, \boldsymbol{Z}_1)$ as follows:

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{b}_{11} \\ \boldsymbol{b}_{12} \\ \vdots \\ \boldsymbol{b}_{1n_1} \end{bmatrix}, \qquad \boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{a}_{11} \\ \boldsymbol{a}_{12} \\ \vdots \\ \boldsymbol{a}_{1n_1} \end{bmatrix},$$

where the vectors $\boldsymbol{b}_{1i}$ and $\boldsymbol{a}_{1i}$ are of size $1 \times k_1$ and $1 \times f$, respectively. Each pair of $(\boldsymbol{b}_{1i}, \boldsymbol{a}_{1i})$ corresponds to factor i, for $i = 1, 2, \ldots, n_1$. Similarly, the vectors $\boldsymbol{b}_{2i}(1 \times k_2)$ and $\boldsymbol{a}_{2i}(1 \times f)$ represent the ith rows of $\boldsymbol{B}_2$ and $\boldsymbol{Z}_2$, respectively, for $i = 1, 2, \ldots, n_2$. Based on Lemma 2.1, we have the following classes of orthogonal designs.

Class 1. For a given $L_{N_0}(2^{d_1} \times 3^{d_2})$, there is an orthogonal PFD with $N = 2N_0$ for the following $\boldsymbol{\beta}$ of $2^{n_1} \times 3^{n_2}$ factorial experiments.

$$\begin{aligned} \boldsymbol{\beta} = \{\mu\} \cup \{F_1, F_2, \ldots, F_{n_1}\} & \cup \{F_{n_1+1}, F_{n_1+2}, \ldots, F_{n_1+n_2}\} \\ & \cup \{F_1 F_j \,|\, F_j \in G_1\} \\ & \cup \{F_1 F_{n_1+j} \,|\, j = 1, 2, \ldots, n_2\}, \end{aligned}$$

where G_1 is a set consisting of some 2-level factors. Let the number of factors in G_1 be $|G_1| = n_{11}$. Then $n_{11} \leqslant d_1$, $n_1 \leqslant 2d_1 - n_{11} + 1$ and $n_2 \leqslant d_2$.

Consider the PFD, with $f = N_0$, constructed from the following steps.

Step 1: For this $\boldsymbol{\beta}$, all 2-factor interactions involve the 2-level factor F_1. Partition the remaining $(n_1 - 1)$ 2-level factors $F_2, F_3, \ldots, F_{n_1}$ into two distinct sets S_0 and S_1 such that S_0 contains G_1; and neither S_0 nor $G_1 \cup S_1$ contains more than d_1 factors.

Step 2: For $\boldsymbol{B}_1$ matrix, let $\boldsymbol{b}_{11} = 1$ and $\boldsymbol{b}_{1i} = \alpha$ for $F_i \in S_\alpha$, $\alpha = 0, 1$. For $\boldsymbol{B}_2$ matrix, let $\boldsymbol{b}_{2i} = 0$ for all i.

Step 3: For $\boldsymbol{Z}_1$ matrix, let $\boldsymbol{a}_{11}$ be the zero row over GF[2]. Then assign each of the remaining $\boldsymbol{a}_{1i}$'s one of the first d_1 rows of $L_{N_0}(2^{d_1} \times 3^{d_2})$ such that the $\boldsymbol{a}_{1i}$'s corresponding to the factors in each set of S_0 and $S_1 \cup G_1$ are all distinct. This leads to that $n_1 \leqslant 2d_1 - n_{11} + 1$. For $\boldsymbol{Z}_2$ matrix, let $\boldsymbol{a}_{2i} = \boldsymbol{l}_{d_1+i}$, for $i = 1, 2, \ldots, n_2$, where $n_2 \leqslant d_2$.

It can be easily verified by Lemma 2.1 that the FPD generated is orthogonal for the above $\boldsymbol{\beta}$ and has $N = N_0 \times 2 \times 3^0 = 2N_0$.

Example 1. Consider a $2^6 \times 3$ factorial experiment. Let $F_1, F_2, \ldots, F_6$ denote the six 2-level factors and F_7 denote the 3-level factor. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, F_3, F_4, F_5, F_6; F_7; F_1F_2, F_1F_3, F_1F_4; F_1F_7\}.$$

We can use the following $L_{12}(2^4 \times 3)$ to construct a PFD orthogonal for this $\boldsymbol{\beta}$.

$$L_{12}(2^4 \times 3) = \begin{bmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \\ \boldsymbol{l}_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}. \tag{3.1}$$

Step 1: Let $S_0 = \{F_2, F_3, F_4, F_5\}$ and $S_1 = \{F_6\}$. Clearly, $G_1 = \{F_2, F_3, F_4\}$ which is contained in S_0. $G_1 \cup S_1 = \{F_2, F_3, F_4, F_6\}$.

Step 2: $\boldsymbol{B}_1 = [1,0,0,0,0,1]'$ and $\boldsymbol{B}_2 = [0]$.

Step 3:

$$\boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{a}_{11} \\ \boldsymbol{a}_{12} \\ \boldsymbol{a}_{13} \\ \boldsymbol{a}_{14} \\ \boldsymbol{a}_{15} \\ \boldsymbol{a}_{16} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \\ \boldsymbol{l}_4 \end{bmatrix}, \qquad \boldsymbol{Z}_2 = [\boldsymbol{a}_{21}] = [\boldsymbol{l}_5].$$

The PFD determined by these pairs $(\boldsymbol{B}_1, \boldsymbol{Z}_1)$ and $(\boldsymbol{B}_2, \boldsymbol{Z}_2)$ is orthogonal for the $\boldsymbol{\beta}$ and has $N = 12 \times 2 \times 3^0 = 24$ runs for estimating $p = 14$ parameters.

Example 2. Consider a $2^2 \times 3^7$ factorial experiment. Let F_1, F_2 denote the two 2-level factors and $F_3, F_4, \ldots, F_9$ denote the seven 3-level factors. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \{\mu; F_1, F_2; F_3, F_4, \ldots, F_9; F_1F_2; F_1F_3, F_1F_4, F_1F_5, F_1F_6, F_1F_7, F_1F_8, F_1F_9\}.$$

We can use the following $L_{18}(2 \times 3^7)$, see Wu and Hamada (2000), to construct a PFD orthogonal for this $\boldsymbol{\beta}$.

$$L_{18}(2 \times 3^7) = \begin{bmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \\ \boldsymbol{m}_4 \\ \boldsymbol{m}_5 \\ \boldsymbol{m}_6 \\ \boldsymbol{m}_7 \\ \boldsymbol{m}_8 \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \\ 0&0&0&1&1&1&2&2&2&0&0&0&1&1&1&2&2&2 \\ 0&1&2&0&1&2&0&1&2&0&1&2&0&1&2&0&1&2 \\ 0&1&2&0&1&2&1&2&0&2&0&1&1&2&0&2&0&1 \\ 0&1&2&1&2&0&0&1&2&2&0&1&2&0&1&1&2&0 \\ 0&1&2&1&2&0&2&0&1&1&2&0&0&1&2&2&0&1 \\ 0&1&2&2&0&1&1&2&0&1&2&0&2&0&1&0&1&2 \\ 0&1&2&2&0&1&2&0&1&0&1&2&1&2&0&1&2&0 \end{bmatrix}. \tag{3.2}$$

Step 1: Let $S_0 = \{F_2\}$ and $S_1 = \phi$ (empty set).

Step 2: $\boldsymbol{B}_1 = [1,0]'$ and $\boldsymbol{B}_2 = [0,0,0,0,0,0,0]'$.

Step 3:

$$\boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{a}_{11} \\ \boldsymbol{a}_{12} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{m}_1 \end{bmatrix}, \qquad \boldsymbol{Z}_2 = \begin{bmatrix} \boldsymbol{a}_{21} \\ \boldsymbol{a}_{22} \\ \boldsymbol{a}_{23} \\ \boldsymbol{a}_{24} \\ \boldsymbol{a}_{25} \\ \boldsymbol{a}_{26} \\ \boldsymbol{a}_{27} \end{bmatrix} = \begin{bmatrix} \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \\ \boldsymbol{m}_4 \\ \boldsymbol{m}_5 \\ \boldsymbol{m}_6 \\ \boldsymbol{m}_7 \\ \boldsymbol{m}_8 \end{bmatrix}.$$

This PFD is orthogonal for the $\boldsymbol{\beta}$ and has $N = 36$ runs for estimating $p = 32$ parameters.

Switching the roles of 2-level factors and 3-level factors in the construction steps of Class 1, we have the following class of designs.

Class 2. For a given $L_{N_0}(2^{d_1} \times 3^{d_2})$, there is an orthogonal design with $N = 3N_0$ for the following $\boldsymbol{\beta}$ of $2^{n_1} \times 3^{n_2}$ factorial experiment.

$$\boldsymbol{\beta} = \{\mu\} \cup \{F_1, F_2, \ldots, F_{n_1}\} \cup \{F_{n_1+1}, F_{n_1+2}, \ldots, F_{n_1+n_2}\}$$
$$\cup \{F_j F_{n_1+1} \mid j = 1, 2, \ldots, n_1\} \cup \{F_{n_1+1}F_j, F_{n_1+1}F_j^2 \mid F_j \in G_1\},$$

where G_1 is a set consisting of some 3-level factors. Let $|G_1| = n_{21}$. Then $n_{21} \leqslant d_2$, $n_2 \leqslant 2d_2 - n_{21} + 1$ and $n_1 \leqslant d_1$.

Example 3. Consider a $2^4 \times 3^2$ factorial experiment. Let F_1, F_2, F_3, F_4 denote the four 2-level factors and F_5, F_6 denote the two 3-level factors. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, F_3, F_4; F_5, F_6; F_1F_5, F_2F_5, F_3F_5, F_4F_5; F_5F_6, F_5F_6^2\}.$$

The desired orthogonal design can be obtained from the following steps.

Step 1: Let $S_0 = \{F_6\}$ and $S_1 = \phi$. $G_1 = \{F_6\}$ which is contained in S_0. $G_1 \cup S_1 = \{F_6\}$.

Step 2: $\boldsymbol{B}_2 = [1, 0]'$ and $\boldsymbol{B}_1 = [0, 0, 0, 0]'$.

Step 3: $L_{12}(2^4 \times 3)$ of (3.1) is used to construct the $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_1$ matrices as follows.

$$\boldsymbol{Z}_2 = \begin{bmatrix} \boldsymbol{a}_{21} \\ \boldsymbol{a}_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{l}_5 \end{bmatrix}, \qquad \boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{a}_{11} \\ \boldsymbol{a}_{12} \\ \boldsymbol{a}_{13} \\ \boldsymbol{a}_{14} \end{bmatrix} = \begin{bmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \end{bmatrix}.$$

This orthogonal design has $N = 12 \times 2^0 \times 3 = 36$ runs for estimating $p = 21$ parameters.

Example 4. Consider a 2×3^{10} factorial experiment. Let F_1 denote the 2-level factor and $F_2, F_3, \ldots, F_{11}$ denote the ten 3-level factors. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \{\mu; F_1; F_2, F_3, \ldots, F_{11}; F_1F_2; F_2F_3, F_2F_3^2, F_2F_4, F_2F_4^2, F_2F_5, F_2F_5^2, F_2F_6, F_2F_6^2, F_2F_7, F_2F_7^2\}.$$

The desired orthogonal design can be obtained from the following steps.

Step 1: Let $S_0 = \{F_3, F_4, F_5, F_6, F_7, F_8, F_9\}$ and $S_1 = \{F_{10}, F_{11}\}$.

Step 2: $\boldsymbol{B}_2 = [1, 0, 0, 0, 0, 0, 0, 0, 1, 1]'$ and $\boldsymbol{B}_1 = [0]$.

Step 3: $L_{18}(2 \times 3^7)$ of (3.2) is used to construct the $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_1$ matrices as follows.

$$\boldsymbol{Z}_2 = \begin{bmatrix} \boldsymbol{a}_{21} \\ \boldsymbol{a}_{22} \\ \boldsymbol{a}_{23} \\ \boldsymbol{a}_{24} \\ \boldsymbol{a}_{25} \\ \boldsymbol{a}_{26} \\ \boldsymbol{a}_{27} \\ \boldsymbol{a}_{28} \\ \boldsymbol{a}_{29} \\ \boldsymbol{a}_{2,10} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{m}_2 \\ \boldsymbol{m}_3 \\ \boldsymbol{m}_4 \\ \boldsymbol{m}_5 \\ \boldsymbol{m}_6 \\ \boldsymbol{m}_7 \\ \boldsymbol{m}_8 \\ \boldsymbol{m}_7 \\ \boldsymbol{m}_8 \end{bmatrix}, \qquad \boldsymbol{Z}_1 = [\boldsymbol{a}_{11}] = [\boldsymbol{m}_1].$$

This orthogonal design has $N = 54$ runs for estimating $p = 43$ parameters.

Class 3. For a given $L_{N_0}(2^{d_1} \times 3^{d_2})$, there is an orthogonal design with $N = 4N_0$ for the following $\boldsymbol{\beta}$ of $2^{n_1} \times 3^{n_2}$ experiments:

$$\begin{aligned}\boldsymbol{\beta} = &\{\mu\} \cup \{F_1, F_2, \ldots, F_{n_1}\} \cup \{F_{n_1+1}, F_{n_1+2}, \ldots, F_{n_1+n_2}\} \cup \{F_1F_2\} \cup \{F_1F_j \mid F_j \in G_1\}\\ &\cup \{F_2F_j \mid F_j \in G_2\} \cup \{F_1F_{n_1+j}, F_2F_{n_1+j} \mid j = 1, 2, \ldots, n_2\},\end{aligned}$$

where G_1 and G_2 are two sets consisting of some 2-level factors. Let $|G_1| = n_{11}$, $|G_2| = n_{12}$, $G_{12} = G_1 \cup G_2$ and $|G_{12}| = n_0$. Then $n_0 \leqslant d_1$, $n_1 \leqslant 4d_1 - n_{11} - n_{12} + 2$ and $n_2 \leqslant d_2$.

The desired design can be obtained from the following approach.

Step 1: For this $\boldsymbol{\beta}$, all 2-factor interactions involve F_1 or F_2. Partition the remaining $(n_1 - 2)$ 2-level factors $F_3, F_4, \ldots, F_{n_1}$ into four distinct sets S_{00}, S_{01}, S_{10} and S_{11} such that S_{00} contains G_{12}; and none of the sets S_{00}, $G_1 \cup S_{01}$, $G_2 \cup S_{10}$ and S_{11} contains more than d_1 factors.

Step 2: For $\boldsymbol{B}_1$ matrix, let $\boldsymbol{b}_{11} = [0, 1]$, $\boldsymbol{b}_{12} = [1, 0]$ and $\boldsymbol{b}_{1i} = [\alpha_1, \alpha_2]$ for $F_i \in S_{\alpha_1\alpha_2}$, $\alpha_1, \alpha_2 = 0, 1$. For $\boldsymbol{B}_2$ matrix, let $\boldsymbol{b}_{2i} = 0$ for all i.

Step 3: For $\boldsymbol{Z}_1$ matrix, let both $\boldsymbol{a}_{11}$ and $\boldsymbol{a}_{12}$ be the zero row vector over GF[2]. Then assign each of the remaining $\boldsymbol{a}_{1i}$'s one of the first d_1 rows of $L_{N_0}(2^{d_1} \times 3^{d_2})$, such that the $\boldsymbol{a}_{1i}$'s corresponding to factors in each set of S_{00}, $G_1 \cup S_{01}$, $G_2 \cup S_{10}$ and S_{11} are all distinct. This leads to that $n_1 \leqslant 4d_1 - n_{11} - n_{12} + 2$. For $\boldsymbol{Z}_2$ matrix, let $\boldsymbol{a}_{2i} = \boldsymbol{l}_{d_1+i}$, for $i = 1, 2, \ldots, n_2$, where $n_2 \leqslant d_2$.

It can be verified by Lemma 2.1 that the obtained design is orthogonal for the $\boldsymbol{\beta}$ and has $N = N_0 \times 2^2 \times 3^0 = 4N_0$ runs.

Example 5. Consider a $2^{11} \times 3$ factorial experiment. Let $F_1, F_2, \ldots, F_{11}$ denote the eleven 2-level factors and F_{12} denote the 3-level factor. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, \ldots, F_{11}; F_{12}; F_1F_2; F_1F_3, F_1F_4, F_1F_5, F_1F_6; F_2F_3, F_2F_4, F_2F_5; F_1F_{12}, F_2F_{12}\}.$$

A PFD orthogonal for the above $\boldsymbol{\beta}$ can be obtained from the following steps.

Step 1: For this $\boldsymbol{\beta}$, $G_1 = \{F_3, F_4, F_5, F_6\}$ and $G_2 = \{F_3, F_4, F_5\}$. Let $S_{00} = \{F_3, F_4, F_5, F_6\}$ which contains $G_{12} = G_1 \cup G_2 = \{F_3, F_4, F_5, F_6\}$, $S_{01} = \phi$, $S_{10} = \{F_7\}$ and $S_{11} = \{F_8, F_9, F_{10}, F_{11}\}$. So $G_1 \cup S_{01} = \{F_3, F_4, F_5, F_6\}$ and $G_2 \cup S_{10} = \{F_3, F_4, F_5, F_7\}$.

Step 2:

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{b}_{11}\\ \boldsymbol{b}_{12}\\ \boldsymbol{b}_{13}\\ \boldsymbol{b}_{14}\\ \boldsymbol{b}_{15}\\ \boldsymbol{b}_{16}\\ \boldsymbol{b}_{17}\\ \boldsymbol{b}_{18}\\ \boldsymbol{b}_{19}\\ \boldsymbol{b}_{1,10}\\ \boldsymbol{b}_{1,11} \end{bmatrix} = \begin{bmatrix} 0 & 1\\ 1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 0\\ 1 & 1\\ 1 & 1\\ 1 & 1\\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{B}_2 = [\boldsymbol{b}_{21}] = [0].$$

Step 3: $L_{12}(2^4 \times 3)$ of (3.1) is used to construct the $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ matrices as follows:

$$\boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{a}_{11} \\ \boldsymbol{a}_{12} \\ \boldsymbol{a}_{13} \\ \boldsymbol{a}_{14} \\ \boldsymbol{a}_{15} \\ \boldsymbol{a}_{16} \\ \boldsymbol{a}_{17} \\ \boldsymbol{a}_{18} \\ \boldsymbol{a}_{19} \\ \boldsymbol{a}_{1,10} \\ \boldsymbol{a}_{1,11} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \\ \boldsymbol{l}_4 \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \end{bmatrix}, \qquad \boldsymbol{Z}_2 = [\boldsymbol{a}_{21}] = [\boldsymbol{l}_5].$$

The orthogonal design obtained has $N = 12 \times 2^2 \times 3^0 = 48$ runs for estimating $p = 27$ parameters.

Example 6. Consider a $2^{14} \times 3$ factorial experiment. Let $F_1, F_2, \ldots, F_{14}$ denote the fourteen 2-level factors and F_{15} denote the 3-level factor. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, \ldots, F_{14}; F_{15}; F_1F_2; F_1F_3, F_1F_4, F_2F_5, F_2F_6; F_1F_{15}, F_2F_{15}\}.$$

A PFD orthogonal for the above $\boldsymbol{\beta}$ can be obtained from the following steps.

Step 1: Let $S_{00} = \{F_3, F_4, F_5, F_6\}$, $S_{01} = \{F_7, F_8\}$, $S_{10} = \{F_9, F_{10}\}$ and $S_{11} = \{F_{11}, F_{12}, F_{13}, F_{14}\}$.

Step 2:

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{b}_{11} \\ \boldsymbol{b}_{12} \\ \boldsymbol{b}_{13} \\ \boldsymbol{b}_{14} \\ \boldsymbol{b}_{15} \\ \boldsymbol{b}_{16} \\ \boldsymbol{b}_{17} \\ \boldsymbol{b}_{18} \\ \boldsymbol{b}_{19} \\ \boldsymbol{b}_{1,10} \\ \boldsymbol{b}_{1,11} \\ \boldsymbol{b}_{1,12} \\ \boldsymbol{b}_{1,13} \\ \boldsymbol{b}_{1,14} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{B}_2 = [\boldsymbol{b}_{21}] = [0].$$

Step 3: $L_{12}(2^4 \times 3)$ of (3.1) is used to construct the $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ matrices as follows:

$$\boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{a}_{11} \\ \boldsymbol{a}_{12} \\ \boldsymbol{a}_{13} \\ \boldsymbol{a}_{14} \\ \boldsymbol{a}_{15} \\ \boldsymbol{a}_{16} \\ \boldsymbol{a}_{17} \\ \boldsymbol{a}_{18} \\ \boldsymbol{a}_{19} \\ \boldsymbol{a}_{1,10} \\ \boldsymbol{a}_{1,11} \\ \boldsymbol{a}_{1,12} \\ \boldsymbol{a}_{1,13} \\ \boldsymbol{a}_{1,14} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \end{bmatrix}, \qquad \boldsymbol{Z}_2 = [\boldsymbol{a}_{21}] = [\boldsymbol{l}_5].$$

The orthogonal design obtained has $N = 48$ runs for estimating $p = 26$ parameters.

Class 4. For a given $L_{N_0}(2^{d_1} \times 3^{d_2})$, there is an orthogonal design with $N = 4N_0$ for the following $\boldsymbol{\beta}$ of $2^{n_1} \times 3^{n_2}$ factorial experiments.

$$\boldsymbol{\beta} = \{\mu\} \cup \{F_1, F_2, F_3, \ldots, F_{n_1}\} \cup \{F_{n_1+1}, F_{n_1+2}, \ldots, F_{n_1+n_2}\} \cup \{F_1F_2, F_1F_3, F_2F_3, F_1F_2F_3\}$$
$$\cup \{F_1F_j \mid F_j \in G_1\} \cup \{F_2F_j \mid F_j \in G_2\} \cup \{F_3F_j \mid F_j \in G_3\} \cup \{F_1F_{n_1+j}, F_2F_{n_1+j}, F_3F_{n_1+j} \mid j = 1, 2, \ldots, n_2\},$$

where G_1, G_2 and G_3 are three sets consisting of some 2-level factors. Let $|G_1| = n_{11}$, $|G_2| = n_{12}$ and $|G_3| = n_{13}$. Also, let $G_{123} = G_1 \cup G_2 \cup G_3$ and $|G_{123}| = n_0$. Then $n_0 \leqslant d_1 - 1$, $n_1 \leqslant 4(d_1 - 1) - n_{11} - n_{12} - n_{13} + 3$ and $n_2 \leqslant d_2$.

The desired PFD can be constructed from the following approach.

Step 1: For this $\boldsymbol{\beta}$, the interactions involve at least one of F_1, F_2 and F_3. Partition the remaining $(n_1 - 3)$ 2-level factors $F_4, F_5, \ldots, F_{n_1}$ into four distinct sets S_{00}, S_{01}, S_{10} and S_{11} such that S_{00} contains G_{123}; and none of the sets S_{00}, $G_1 \cup S_{01}$, $G_2 \cup S_{10}$ and $G_3 \cup S_{11}$ contains more than $d_1 - 1$ factors.

Step 2: For $\boldsymbol{B}_1$ matrix, let $\boldsymbol{b}_{11} = [0, 1]$, $\boldsymbol{b}_{12} = [1, 0]$, $\boldsymbol{b}_{13} = [1, 1]$ and $\boldsymbol{b}_{1i} = [\alpha_1, \alpha_2]$ for $F_i \in S_{\alpha_1\alpha_2}$, $\alpha_1, \alpha_2 = 0, 1$. For $\boldsymbol{B}_2$ matrix, let $\boldsymbol{b}_{2i} = 0$ for all i.

Step 3: For $\boldsymbol{Z}_1$ matrix, let both $\boldsymbol{a}_{11}$ and $\boldsymbol{a}_{12}$ be the zero row vector over GF[2] and $\boldsymbol{a}_{13} = \boldsymbol{l}_1$. Then assign each of the remaining $\boldsymbol{a}_{1i}$'s one of $\boldsymbol{l}_2, \boldsymbol{l}_3, \ldots, \boldsymbol{l}_{d_1}$ such that the $\boldsymbol{a}_{1i}$'s corresponding to factors in each set of S_{00}, $G_1 \cup S_{01}$, $G_2 \cup S_{10}$ and $G_3 \cup S_{11}$ are all distinct. This leads to that $n_1 \leqslant 4(d_1 - 1) - n_{11} - n_{12} - n_{13} + 3$. For $\boldsymbol{Z}_2$ matrix, let $\boldsymbol{a}_{2i} = \boldsymbol{l}_{d_1+i}$, for $i = 1, 2, \ldots, n_2$, where $n_2 \leqslant d_2$. - This PFD has $N = N_0 \times 2^2 \times 3^0 = 4N_0$.

Example 7. Consider a $2^9 \times 3$ factorial experiment. Let $F_1, F_2, \ldots, F_9$ denote the nine 2-level factors and F_{10} denote the 3-level factor. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta} = \{\mu; F_1, F_2, F_3, \ldots, F_9; F_{10}; F_1F_2, F_1F_3, F_2F_3, F_1F_2F_3;$$
$$F_1F_4, F_1F_5, F_1F_6; F_2F_4, F_2F_5; F_3F_4; F_1F_{10}, F_2F_{10}, F_3F_{10}\}.$$

A PFD orthogonal for the above $\boldsymbol{\beta}$ can be constructed from the following approach.

Step 1: For this $\boldsymbol{\beta}$, $G_1=\{F_4,F_5,F_6\}$, $G_2=\{F_4,F_5\}$ and $G_3=\{F_4\}$. $G_{123}=G_1\cup G_2\cup G_3=\{F_4,F_5,F_6\}$. Let $S_{00}=\{F_4,F_5,F_6\}$ which contains G_{123}, $S_{01}=\phi$, $S_{10}=\{F_7\}$ and $S_{11}=\{F_8,F_9\}$. Thus, $G_1\cup S_{01}=\{F_4,F_5,F_6\}$, $G_2\cup S_{10}=\{F_4,F_5,F_7\}$ and $G_3\cup S_{11}=\{F_4,F_8,F_9\}$.

Step 2:

$$\boldsymbol{B}_1=\begin{bmatrix}\boldsymbol{b}_{11}\\ \boldsymbol{b}_{12}\\ \boldsymbol{b}_{13}\\ \boldsymbol{b}_{14}\\ \boldsymbol{b}_{15}\\ \boldsymbol{b}_{16}\\ \boldsymbol{b}_{17}\\ \boldsymbol{b}_{18}\\ \boldsymbol{b}_{19}\end{bmatrix}=\begin{bmatrix}0&1\\1&0\\1&1\\0&0\\0&0\\0&0\\1&0\\1&1\\1&1\end{bmatrix},\qquad \boldsymbol{B}_2=[\boldsymbol{b}_{21}]=[0].$$

Step 3: $L_{12}(2^4\times 3)$ of (3.1) is used to construct the $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ matrices as follows:

$$\boldsymbol{Z}_1=\begin{bmatrix}\boldsymbol{a}_{11}\\ \boldsymbol{a}_{12}\\ \boldsymbol{a}_{13}\\ \boldsymbol{a}_{14}\\ \boldsymbol{a}_{15}\\ \boldsymbol{a}_{16}\\ \boldsymbol{a}_{17}\\ \boldsymbol{a}_{18}\\ \boldsymbol{a}_{19}\end{bmatrix}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{0}\\ \boldsymbol{l}_1\\ \boldsymbol{l}_2\\ \boldsymbol{l}_3\\ \boldsymbol{l}_4\\ \boldsymbol{l}_4\\ \boldsymbol{l}_3\\ \boldsymbol{l}_4\end{bmatrix},\qquad \boldsymbol{Z}_2=[\boldsymbol{a}_{21}]=[\boldsymbol{l}_5].$$

This orthogonal design has $N=12\times 2^2\times 3^0=48$ runs for estimating $p=28$ parameters.

Example 8. Consider a $2^6\times 3$ factorial experiment. Let $F_1,F_2,\ldots,F_6$ denote the six 2-level factors and F_7 denote the 3-level factor. Suppose the nonnegligible factorial effects set $\boldsymbol{\beta}$ is given by

$$\boldsymbol{\beta}=\{\mu;F_1,F_2,F_3,\ldots,F_6;F_7;F_1F_2,F_1F_3,F_2F_3,F_1F_2F_3;F_1F_4,F_1F_5,F_1F_6;$$
$$F_2F_4,F_2F_5,F_2F_6;F_3F_4,F_3F_5,F_3F_6;F_1F_7,F_2F_7,F_3F_7\}.$$

A PFD orthogonal for the above $\boldsymbol{\beta}$ can be constructed from the following approach.

Step 1: Let $S_{00}=\{F_4,F_5,F_6\}$ and $S_{01}=S_{10}=S_{11}=\phi$.

Step 2:

$$\boldsymbol{B}_1=\begin{bmatrix}\boldsymbol{b}_{11}\\ \boldsymbol{b}_{12}\\ \boldsymbol{b}_{13}\\ \boldsymbol{b}_{14}\\ \boldsymbol{b}_{15}\\ \boldsymbol{b}_{16}\end{bmatrix}=\begin{bmatrix}0&1\\1&0\\1&1\\0&0\\0&0\\0&0\end{bmatrix},\qquad \boldsymbol{B}_2=[\boldsymbol{b}_{21}]=[0].$$

Step 3: $L_{12}(2^4 \times 3)$ of (3.1) is used to construct the $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ matrices as follows:

$$\boldsymbol{Z}_1 = \begin{bmatrix} \boldsymbol{a}_{11} \\ \boldsymbol{a}_{12} \\ \boldsymbol{a}_{13} \\ \boldsymbol{a}_{14} \\ \boldsymbol{a}_{15} \\ \boldsymbol{a}_{16} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \boldsymbol{l}_3 \\ \boldsymbol{l}_4 \end{bmatrix}, \qquad \boldsymbol{Z}_2 = [\boldsymbol{a}_{21}] = [\boldsymbol{l}_5].$$

This orthogonal design has $N = 48$ runs for estimating $p = 28$ parameters.

4. Concluding remarks

Similarly, the results of Class 1 can be applied to the case that the 2-factor interactions involve one specified 2-level or 3-level factor. And Class 3 and Class 4 can also be easily extended to the cases that the nonnegligible interactions involve at least one of two and three specified 3-level factors with run size $N = 9N_0$.

The classes of designs presented in this paper exactly belong to those of inter-effect orthogonality for the hierarchical models discussed in Dey and Mukerjee (1999). As proved in their paper, these designs are universally optimal for estimating the $\boldsymbol{\beta}$ within the class of designs with the fixed N.

Acknowledgements

The author thanks the referee for helpful suggestions on the manuscript. This research was supported by National Science Council of Taiwan, Grant NSC-88-2118-M-002-005.

References

Connor, W.S., Young, S., 1961. Fractional factorial designs for experiments with factor at two and three levels. Applied Mathematics Series. National Bureau of Standards, US Government Printing Office, Washington, DC.
Dey, A., Mukerjee, R., 1999. Inter-effect orthogonality and optimality in hierarchical models. Sankhya B 61, 460–468.
Liao, C.T., 1999. Orthogonal three-level parallel flats designs for user-specified resolution. Commun. Statist.: Theory Meth. 28 (8), 1945–1960.
Liao, C.T., 2000. Orthogonal parallel-flats designs for hierarchical models. Statist. Probab. Lett. 46, 329–335.
Liao, C.T., Iyer, H.K., 1998. Orthogonal parallel-flats for $s_1^{n_1} \times s_2^{n_2}$ mixed factorial experiments. J. Statist. Plann. Inference 69, 175–191.
Rao, C.R., 1947. Factorial experiments derivable from combinational arrangements of arrays. J. Roy. Statist. Soc. 9, 128–139.
Rao, C.R., 1973. Some combinatorial problems of arrays and applications to design of experiments. In: Srivastava, J.N. (Ed.), A survey of Combinatorial Theory. North-Holland, Amsterdam, pp. 349–359.
Srivastava, J.N., 1987. Advances in the general theory of factorial designs based on partial pencils in Euclidean n-space. Utilitas Math. 32, 75–94.
Srivastava, J.N., Li, F., 1996. Orthogonal designs of parallel flats type. J. Statist. Plann. Inference 53, 261–283.
Srivastava, J.N., Anderson, D.A., Mardekian, J., 1984. Theory of factorial designs of parallel flat type. I. The coefficient matrix. J. Statist. Plann. Inference 9, 229–252.
Sun, D.X., Wu, C.F.J., 1994. Interaction graphs for three-level fractional factorial designs. J. Quality Technol. 26, 297–307.
Taguchi, G., 1987. System of experimental Designs. UNIPUB, New York.
Wang, J.C., Wu, C.F.J., 1991. An approach to the construction of asymmetrical orthogonal arrays. JASA 86, 450–455.
Wu, C.F.J., Chen, Y., 1992. A graph-aided method for planning two-level experiments when certain interactions are important. Technometrics 34, 162–174.
Wu, C.F.J., Hamada, M., 2000. Experiment: Planning, Analysis and Parameter Design Optimization. John Wiley, New York.