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# RECONSTRUCTING $J_2$ FLOW MODEL FOR ELASTOPLASTIC MATERIALS

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# 重建彈塑性材料之 $J_2$ 流動模式

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## 摘　要

本文旨在重建 $J_2$ 彈塑性組成模式，由選定之公設出發，導出彈塑損傷破壞模式，以兼具等向與走動的方式一體適用非線性軟硬化與完全塑性。文中證得最大耗散率原理，找出其 Kuhn-Tucker 條件。不需假手人爲規定，本文發現由公設可推得嶄新的加應變條件、可逆不可逆判準、弱穩定條件、强穩定條件、絕對破壞條件、加載破壞條件、及彼此間之密切關連。文中亦探究材料性質函數及其在材料試驗的應力應變曲線的意義。

**關鍵詞**：加應變條件，穩定條件，破壞條件，可逆不可逆判準，最大耗散率。

## Abstract

We reconstruct the constitutive relations for $J_2$ elastoplasticity. From a few axioms we deduce a consistent model of elastic-plastic-damage-failure, which can transparently handle nonlinear softening and perfect plasticity as well as nonlinear hardening in the combined-isotropic-kinematic sense. We assert maximum dissipation rate in the set of admissible stress states and find out its Kuhn-Tucker conditions. In a natural way, we discover the straining condition, irreversibility-reversibility criterion, weak stability condition, strong stability condition, absolute failure condition, loading failure condition, and their close relationships. We also investigate material properties and explore their physical meanings in stress-strain curves of material tests.

**Keywords** : Straining condition, stability condition, failure condition, irreversibility-reversibility criterion, maximum dissipation rate.

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# 1. Introduction

In the development of the concept and constitutive theory of plasticity, the limit concept emerged first and has been evolving into today's various yield conditions. A second but decisive stride taken in the development of the theory was the proposal of incremental stress-strain relations, which has been developing into today's plastic-flow rules. The third important step may be the joint usage of the loading (and unloading and neutral loading) conditions and the yield conditions, thus establishing loading-unloading criteria [1] [2] [3]. Other significant contributions include the replacement of rigidity by incorporating the elastic part, the enhancement of perfect plasticity by introducing the rules of hardening, and the consideration of Bauschinger's effect by dividing stress into active and back stresses. Nowadays, despite there exist other variant models and theories [3] [4], a conventional flow model of plasticity generally includes such ingredients as the elastic part, yield condition, loading-unloading criterion, plastic flow and hardening rule, and kinematic rule.

As parts of a single and integral model, the ingredients are necessarily interwoven but, since they are separately specified, their interrelations are not so clear. To establish a correct theory, the specification of the ingredients should be consistent with each other and in harmony with natural laws, and, if necessary, some closely related topics such as material stability, damage, failure and so on should be treated. In the present paper, we, examining constitutive ingredients of $J_2$ elastoplasticity, choose parts of them as axioms and derive the rest. In doing so, we correct some inconsistencies in conventional models and arrive at enlightening results regarding the nature of plasticity, irreversibility, stability and failure.

We first in section 2 present the axiomatic part of the model including the first and second laws of thermodynamics. Then, in the rest of the paper we deduce from the axioms other important ingredients of the model and explore the properties of the modeled materials. In order to complete the description of the model, we collect in section 3 the nonaxiomatic ingredients which will later on be deduced from the axiomatic part. For convenience we also furnish a compendium of important propositions to be proved in the paper in section 4, at the end of which previews of the sections that follow can be found.

# 2. $J_2$ elastic-plastic-damage-failure model: axiomatic part

The law of energy balance asserts that in a mechanical process,

$$\sigma_{ij}\,\dot{\epsilon}_{ij} = \dot{W} + \dot{\Lambda}, \tag{1}$$

in which $\sigma_{ij}$ is stress, $\epsilon_{ij}$ is strain, $\sigma_{ij}\,\dot{\epsilon}_{ij}$ is the mechanical power, $W$ is the strain energy (per unit volume or density), $\Lambda$ is the dissipated energy (density) or briefly the dissipation (density), and $\dot{\Lambda}$ is the dissipation (density) rate. For a function of one argument, a superscript prime denotes a differentiation of the function with respect to the argument, for example, $k'(\bar{\epsilon}^p) = dk/d\bar{\epsilon}^p$, while a superimposed dot stands especially for a material time differentiation. As usual, the stress deviator is denoted by $s_{ij}$ and the strain deviator is denoted by $e_{ij}$. The dissipation rate $\dot{\Lambda}$ is postulated to be nonnegative:

$$\dot{\Lambda} \geq 0, \tag{2}$$

as the second law of thermodynamics implies. According to the Legendre transformation, the complementary energy (density) $W^c$ is connected to the strain energy $W$ by the following relation:

$$\sigma_{ij}\,\epsilon_{ij} = W + W^c. \tag{3}$$

In the stress-space formulation, stress plays the role of independent variable and is divided into two parts, active and back (or kinematic):

$$\sigma_{ij} = \sigma^a_{ij} + \sigma^b_{ij}, \quad s_{ij} = s^a_{ij} + s^b_{ij}, \tag{4}$$

the plastic-flow potential

$$g \equiv \frac{3}{2} J_2/h^2(\bar{\epsilon}^p) \equiv \frac{3}{4} s^a_{ij}\, s^a_{ij}/h^2(\bar{\epsilon}^p), \tag{5}$$

its induced yield function [5]

$$f \equiv \sigma^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}} = \frac{3}{2} s^a_{ij}\, s^a_{ij}/h^2(\bar{\epsilon}^p), \tag{6}$$

and the yield condition

$$f = 1 \quad \text{or} \quad \frac{3}{2} s^a_{ij}\, s^a_{ij} = h^2(\bar{\epsilon}^p), \tag{7}$$

are all defined on stress space, and stress is permissible only in the unit level set [5]:

$$f \leq 1 \quad \text{or} \quad \frac{3}{2} s^a_{ij}\, s^a_{ij} \leq h^2(\bar{\epsilon}^p), \tag{8}$$

which defines the set of admissible stress states in stress space.

While, corresponding to Eq. (1), the strain rate, the conjugate to stress, is postulated to be composed of two parts:

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^p_{ij}, \quad \dot{e}_{ij} = \dot{e}^{\,e}_{ij} + \dot{e}^{\,p}_{ij}, \tag{9}$$

where the superscripts $e$ and $p$ signify, respectively, the elastic and plastic parts. For simplicity, small deformation is assumed throughout. The elastic part is governed by

$$2G(\bar{\epsilon}^p)\dot{e}^{\,e}_{ij} = \dot{s}_{ij}, \quad 3K(\bar{\epsilon}^p)\dot{\epsilon}^e_{kk} = \dot{\sigma}_{ll}, \tag{10}$$

where
$$G(\bar{e}^p) \geq 0, \quad K(\bar{e}^p) \geq 0 \tag{11}$$

are two experimentally determined material functions. The plastic part is governed by the evolution rules for the back stress $\sigma_{ij}^b$ and the plastic strain $\epsilon_{ij}^p$, i.e., the kinematic rule:
$$\dot{s}_{ij}^b = \frac{2}{3} k'(\bar{e}^p) \dot{e}_{ij}^p, \quad \sigma_{kk}^b = 0, \tag{12}$$

and the plastic-flow rule:
$$\dot{e}_{ij}^p = \dot{\Lambda} \frac{\partial g}{\partial \sigma_{ij}^a} = \dot{\Lambda} \frac{3}{2h^2(\bar{e}^p)} s_{ij}^a, \quad \epsilon_{kk}^p = 0, \tag{13}$$

where
$$\dot{\Lambda} = 0 \quad \text{if} \quad f < 1, \tag{14}$$

and
$$k'(\bar{e}^p) \geq 0, \quad h(\bar{e}^p) > 0 \tag{15}$$

are two experimentally determined material functions.

The equivalent plastic strain $\bar{e}^p$,
$$\bar{e}^p = \int_{t_0}^{t} \dot{\bar{e}}^p(\xi) d\xi, \quad \dot{\bar{e}}^p \equiv \left( \frac{2}{3} \dot{e}_{ij}^p \dot{e}_{ij}^p \right)^{\frac{1}{2}} \geq 0, \tag{16}$$

is a time-like parameter because its time derivative $\dot{\bar{e}}^p$ is nonnegative and hence is an indicator for irreversible change of material properties. The symbols $t_0$, $\tau$ and $t$ denote the zero-value time, the initial time and (the current) time, respectively, and $t_0 \leq \tau \leq t$. It is postulated that *conceptually* there is a time instant called the zero-value time $t_0$ before and at which all relevant values are zeroes [6]. As a consequence, the model is thus general enough to deal with the problems of nonzero initial values if the initial conditions of stress $\sigma_{ij}(\tau)$ (or strain $\epsilon_{ij}(\tau)$), the active stress $s_{ij}^a(\tau)$, and the equivalent plastic strain $\bar{e}^p(\tau)$ at the initial time $\tau$ are prescribed. In particular, initial isotropy of plastic response [4] is not assumed. From the first-order differential equations (1) and (2) and Eq. (3), it is obvious that the three energy quantities, i.e., the strain energy $W$, the complementary energy $W^c$, and the dissipated energy $\Lambda$, are uniquely determined up to an additive constant each, which leaves room to the postulate of the zero-value state: $W(t_0) = W^c(t_0) = \Lambda(t_0) = 0$, without loss of generality [5]. In fact more than often the initial values of $W(\tau)$, $W^c(\tau)$, $\Lambda(\tau)$, and $\bar{e}^p(\tau)$ are not known and what really matter are the relative values of $W(t) - W(\tau)$, $W^c(t) - W^c(\tau)$, $\Lambda(t) - \Lambda(\tau)$, and $\bar{e}^p(t) - \bar{e}^p(\tau)$.

## 3. $J_2$ elastic-plastic-damage-failure model: nonaxiomatic part

A mechanical model is displayed in Fig. 1 which may help understand, hopefully without misleading, the meanings of Eqs. (4) and (9)~(14). The model proposed in the preceding section is hardly different from conventional models except that the conventional ones prescribe loading-unloading criteria whereas the proposed model merely prescribes $\dot{\Lambda} = 0$ if $f < 1$ (see section 6 for more discussions) and identifies the dissipation rate $\dot{\Lambda}$ in the plastic-flow rule (13) [5]. It is amazing and significant to note that it can be *proved* as will be elaborated in sections 9 and 10 that, given only those in the axiomatic part, there come out *naturally* a series of *deduced rules*: If the material is weakly stable, that is, if the weak stability conditions

$$\text{(shear weak stability)} \qquad G(\bar{e}^p) > 0, \tag{17}$$

$$\text{(bulk weak stability)} \qquad K(\bar{e}^p) > 0 \tag{18}$$

are satisfied, the mechanical behavior of the material is governed by the proposed model in which the dissipation rate $\dot{\Lambda}$ abides by the following naturally deduced criterion:

$$\dot{\Lambda} \begin{cases} > 0 & \text{iff} \quad f = 1 \quad \text{and} \quad G s^a_{ij}\, \dot{e}_{ij} > 0 \\ = 0 & \text{otherwise} \end{cases}, \tag{19}$$

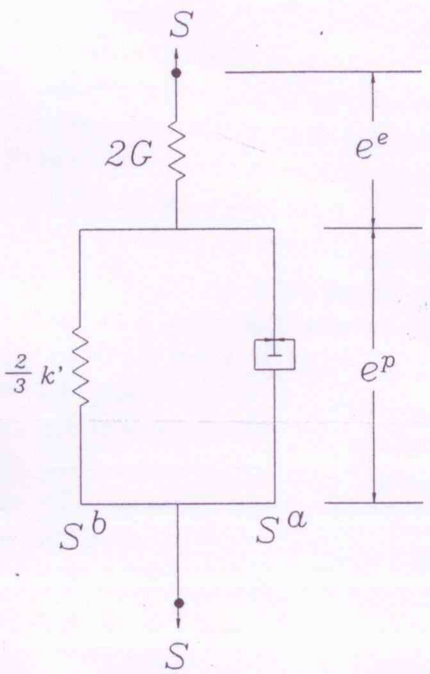


**Fig. 1 Mechanical model**

which may be termed the irreversibility-reversibility criterion. As such, the conventional loading-unloading criteria are not only artificial but also ambiguous or incorrect. It is observed that the demarcation in the criterion (19) is simple and, the most important, the criterion is versatilely applicable to perfect plasticity and softening as well as to hardening without any additional effort; no modified versions have to be made as the conventional ones, no discrimination between hardening and softening or perfect plasticity and no difficulty in transition from one to the other. These advantages may be attributed to the straining condition $Gs^a_{ij}\,\dot{e}_{ij} > 0$ which can distinguish irreversibility from reversibility even in the cases of softening and perfect plasticity, while the loading condition $s^a_{ij}\,\dot{s}_{ij} > 0$ used in the conventional loading-unloading criteria fails to tell. In summary, with this natural criterion (19) and the two material functions $k(\bar{e}^p)$ and $h(\bar{e}^p)$ in Eqs. (12) and (13), the model is capable of describing the behaviors of nonlinear hardening and softening and perfect plasticity in the combined-isotropic-kinematic sense.

It is further observed that, being functions of the equivalent plastic strain which account for the dependence of the shear and bulk moduli on the plastic deformation, the material functions $G(\bar{e}^p)$ and $K(\bar{e}^p)$ reflect the phenomenon of progressive *damage* through elastic-plastic coupling. Moreover, it will be shown in section 9 that the material is alive in *no* case if the *absolute* failure conditions

$$\text{(shear absolute failure)} \quad G(\bar{e}^p) = 0, \tag{20}$$

$$\text{(bulk absolute failure)} \quad K(\bar{e}^p) = 0 \tag{21}$$

are satisfied. In addition to the preceding failure modes, perfectly plastic or softening materials as analyzed in section 14 are more likely to fail in the so-called loading failure mode:

$$\text{(shear loading failure)} - 3G < h' + k' \leq 0 \quad \text{and} \quad f = 1 \quad \text{and} \quad s^a_{ij}\,\dot{s}_{ij} > 0. \tag{22}$$

It is worth noting that the above failure conditions (20)~(22) are an integral part of the proposed model, which is in harmony with the rest of the model. This cautions any attempt to arbitrarily join a failure model to the elastoplastic model.

## 4. Material properties

The proposed model employs totally four experimentally determined functions for material properties, i.e., the bulk modulus function $K(\bar{e}^p)$, the shear modulus function $G(\bar{e}^p)$, the yield-stress function $h(\bar{e}^p)$, and the back-stress function $k(\bar{e}^p)$. Given only those axioms stated in section 2, the four material functions and the three energy quantities can be proved to hold the following important properties:

$$\dot{\Lambda} = s^a_{ij}\,\dot{e}^p_{ij} = h\dot{\bar{e}}^p \geq 0, \tag{23}$$

$$\left\{f = \sigma^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}}\right\} \quad \text{and} \quad \{\dot{\Lambda} = 0 \quad \text{if} \quad f < 1\} \Rightarrow \{\dot{\Lambda} = s^a_{ij}\,\dot{e}^{\,p}_{ij}\}, \tag{24}$$

$$\left\{f = \sigma^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}}\right\} \quad \text{and} \quad \{\dot{\Lambda} = s^a_{ij}\,\dot{e}^{\,p}_{ij}\} \Rightarrow \{\dot{\Lambda} = 0 \quad \text{if} \quad f < 1\}, \tag{25}$$

$$\{\dot{\Lambda} = 0 \quad \text{if} \quad f < 1\} \quad \text{and} \quad \{\dot{\Lambda} = s^a_{ij}\,\dot{e}^{\,p}_{ij}\} \Rightarrow \left\{f = \sigma^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}} \quad \text{if} \quad \dot{\Lambda} > 0\right\}, \tag{26}$$

$$2\dot{\Lambda} = \max_{s^a_{ij}}\{2s^a_{ij}\,\dot{e}^{\,p}_{ij} \mid f \leq 1\}, \tag{27}$$

$$(h' + k' + 3G)\dot{\Lambda} = 3G\,s^a_{ij}\,\dot{e}_{ij} \quad \text{if} \quad f = 1, \tag{28}$$

$$h' + k' + 3G \geq 0, \tag{29}$$

$$G > 0 \Leftrightarrow h' + k' + 3G > 0, \tag{30}$$

$$G = 0 \Leftrightarrow h' + k' + 3G = 0 \Rightarrow h' + k' = 0, \tag{31}$$

$$s^a_{ij}\,\dot{s}_{ij} > 0 \Rightarrow G\,s^a_{ij}\,\dot{e}_{ij} > 0, \tag{32}$$

$$f = 1 \quad \text{and} \quad s^a_{ij}\,\dot{s}_{ij} > 0 \Rightarrow \dot{\Lambda} > 0, \tag{33}$$

$$h' + k' > 0 \Rightarrow h' + k' + 3G > 0, \tag{34}$$

$$\bar{e}^{\,p}_{lf} \leq \bar{e}^{\,p}_{af}, \tag{35}$$

$$\frac{2}{3}(h' + k')\dot{\Lambda} = s^a_{ij}\,\dot{s}_{ij} \quad \text{if} \quad f = 1, \tag{36}$$

$$k' + 3G \geq 0, \tag{37}$$

$$\oint_{s^a_{ij}} 2G\,s^a_{ij}\,\dot{e}_{ij}\,dt \geq 0, \tag{38}$$

$$W \geq 0, \tag{39}$$

$$W^c \geq 0, \tag{40}$$

$$\Lambda \geq 0. \tag{41}$$

Most of these are to be proved below by taking advantage of the rule of *reductio ad absurdum*. Eq. (23) is proved in section 5. An important comment on axioms is then given in section 6, where Eqs. (24)~(26) are proved. Section 7 discusses the Kuhn-Tucker conditions for the assertion of maximum dissipation rate, Eq. (27), in the set of admissible stress states. Eqs. (28)~(31) are proved in section 8, after which the absolute failure condition and the weak stability condition come out naturally in section 9. Then we find out the sufficient and necessary conditions for irreversibility and the irreversibility-reversibility criterion in section 10. In section 11 we make comparison between the straining condition and the loading condition, as a consequence Eqs. (32) and (33) verified, with which we are able to categorize hardening, softening and perfect plasticity in section 12 and also to discuss the strong and weak stability conditions in section 13, verifying the propositions (34) and (35). The important assertion (36) and the loading failure condition are investigated in section 14. In section 15 the expansion and shrinkage of the yield surface are briefly discussed. In section 16 Eqs. (37) and (38) are proved. Finally Eqs. (39)~(41) are proved in section 17.

# 5. Constitutive law for the dissipation rate

From Eqs. $(16)_2$, $(13)_1$ and (6), we have

$$(h\dot{\bar{e}}^p)^2 = \dot{\Lambda}^2 f. \tag{42}$$

If $f = 1$, since $h > 0$, $\dot{\bar{e}}^p \geq 0$, and $\dot{\Lambda} \geq 0$, we have $\dot{\Lambda} = h\dot{\bar{e}}^p$. And if $f < 1$, in view of Eqs. (42) and (14) and $h > 0$, we have $\dot{\Lambda} = h\dot{\bar{e}}^p = \dot{\bar{e}}^p = 0$. Thus, no matter which case,

$$\dot{\Lambda} = h\dot{\bar{e}}^p. \tag{43}$$

If $f = 1$, it follows from Eqs. (6), $(13)_1$, (43), and (2) that

$$\dot{\Lambda} = f\dot{\Lambda} = \sigma^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}}\dot{\Lambda} = s^a_{ij}\,\dot{e}^p_{ij} = h\dot{\bar{e}}^p \geq 0. \tag{44}$$

It is interesting to know that even if $f < 1$, the above equation still holds, because the axiom (14) stipulates that $\dot{\Lambda} = 0$ if $f < 1$ and because of Eqs. $(13)_1$ and (42). The equation

$$\dot{\Lambda} = s^a_{ij}\,\dot{e}^p_{ij} \tag{45}$$

is indeed the constitutive law for the dissipation rate.

# 6. A comment on axioms

A few words of comment on axioms follows. Among the three equations (6), (14), and (45), the present paper has picked up Eqs. (6) and (14) as axioms and then with the aid of some other axioms derived Eq. (45). In fact, any two of the three equations can be chosen as axioms and then with the aid of some other axioms the third equation can be derived, as just shown in the preceding section and as will be shown in the rest of the present section.

If Eqs. (6) and (45) are chosen as axioms, substituting the plastic-flow rule $(13)_1$ for $\dot{e}^{p}_{ij}$ in Eq. (45) and then using Eq. (6) give

$$\dot{\Lambda} = s^a_{ij}\,\dot{e}^{p}_{ij} = s^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}}\dot{\Lambda} = f\,\dot{\Lambda}, \tag{46}$$

or

$$(f-1)\dot{\Lambda} = 0, \tag{47}$$

which renders the statement (14) upon considering the axiom $f \leq 1$.

If Eqs. (14) and (45) are chosen as axioms, we know from Eq. (14) and the axioms $f \leq 1$ and $\dot{\Lambda} \geq 0$ that

$$\dot{\Lambda} > 0 \Rightarrow f = 1, \tag{48}$$

and, on the other hand, substitute the plastic-flow rule $(13)_1$ for $\dot{e}^{p}_{ij}$ in Eq. (45), giving

$$\dot{\Lambda} = s^a_{ij}\,\dot{e}^{p}_{ij} = s^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}}\dot{\Lambda}, \tag{49}$$

or

$$\left(1 - s^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}}\right)\dot{\Lambda} = 0, \tag{50}$$

which gives

$$\dot{\Lambda} > 0 \Rightarrow s^a_{ij}\frac{\partial g}{\partial \sigma^a_{ij}} = 1. \tag{51}$$

Because of Eqs. (48) and (51), thus we have proved Eq. (26). It is interesting to note that the relation between $f$ and $g$ exists only in the irreversibile stage. In the reversible stage there is no need of $g$ and, moreover, it is impossible to know $\frac{\partial g}{\partial \sigma^a_{ij}}$ theoretically and experimentally.

## 7. Kuhn-Tucker conditions for maximum dissipation rate

It is observed that Eqs. $(13)_1$, (47), (2) and (8) constitute the Kuhn-Tucker conditions [7] for the following constrained optimization problem: minimize $-2s^a_{ij}\,\dot{e}^{p}_{ij}$ subject to Eq. (8). In other words, the plastic-flow rule, the equality $(f-1)\dot{\Lambda} = 0$, and the inequalities $\dot{\Lambda} \geq 0$ and $f \leq 1$ are sufficient and necessary for the assertion of maximum dissipation rate in the set of admissible (active) stress states. Because these conditions are either axioms or deduced statements of the model, we have proved the assertion for the *associated* model investigated. The *associativity* refers to that $\frac{\partial f}{\partial \sigma^a_{ij}} = c\frac{\partial g}{\partial \sigma^a_{ij}}$ where $c$ is a positive real. In the present model $c = 2$. As an extension, for a general associated model, the convex function to be minimized is $-c\sigma^a_{ij}\,\dot{\epsilon}^{p}_{ij}$, and it is asserted that

$$c\dot{\Lambda} = \max_{\sigma^a_{ij}}\{c\sigma^a_{ij}\,\dot{\epsilon}^{p}_{ij} \mid f \leq 1\}. \tag{52}$$

# 8. Nonnegativity of $h' + k' + 3G$

By virtue of Eqs. $(10)_1$ and (9),

$$\dot{s}_{ij} = 2G\,\dot{e}^{\,e}_{ij} = 2G\,\dot{e}_{ij} - 2G\,\dot{e}^{\,p}_{ij}. \tag{53}$$

Substituting Eqs. (4), (12) and (53) into the identity

$$(\dot{s}_{ij} - \dot{s}^b_{ij}) + \dot{s}^b_{ij} = \dot{s}_{ij}, \tag{54}$$

yields

$$\dot{s}^a_{ij} + \frac{2}{3}(k' + 3G)\dot{e}^{\,p}_{ij} = 2G\,\dot{e}_{ij}, \tag{55}$$

or

$$s^a_{ij}\,\dot{s}^a_{ij} + \frac{2}{3}(k' + 3G)s^a_{ij}\,\dot{e}^{\,p}_{ij} = 2G\,s^a_{ij}\,\dot{e}_{ij}. \tag{56}$$

From Eqs. (7) and (43), the consistency condition is $\dot{f} = 0$ if $f = 1$, i.e.

$$s^a_{ij}\,\dot{s}^a_{ij} = \frac{2}{3}h'\dot{\Lambda} \quad \text{if} \quad f = 1. \tag{57}$$

From Eqs. (56), (57) and (45),

$$(h' + k' + 3G)\dot{\Lambda} = 3G\,s^a_{ij}\,\dot{e}_{ij} \quad \text{if} \quad f = 1. \tag{58}$$

Now we are ready to prove Eq. (29). In the case of $f = 1$ and $G\,s^a_{ij}\,\dot{e}_{ij} > 0$ it follows from Eq. (58) that $(h' + k' + 3G)\dot{\Lambda} > 0$. This would imply $\dot{\Lambda} < 0$, which would contradict Eq. (2), $\dot{\Lambda} \geq 0$, if we should assume $h' + k' + 3G < 0$. Thus we have proved that

$$h' + k' + 3G \geq 0. \tag{59}$$

Rewritten as

$$\frac{d(\frac{\sqrt{3}}{3}h + \frac{\sqrt{3}}{3}k)}{d(\frac{\sqrt{3}}{2}\bar{e}^{\,p})} \geq -2G, \tag{60}$$

the assertion (59) means profoundly that in a torsional-shear test of a thin-walled circular tube under monotonic rotation the slope of the curve of shear stress versus plastic shear strain is not smaller than $-2G$; in other words, the shear stress versus shear strain curve goes down at most vertically and cannot snap back in the direction of shear strain [6], as illustrated in Fig. 2.

We are now going to prove $h' + k' + 3G > 0 \Rightarrow G > 0$. The premise is $h' + k' + 3G > 0$. In the case of $f = 1$ and $\dot{\Lambda} > 0$, since $h' + k' + 3G > 0$, Eq. (58) says $(h' + k' + 3G)\dot{\Lambda} = 3G\,s^a_{ij}\,\dot{e}_{ij} > 0$. Should $G = 0$, it would follow that $3G\,s^a_{ij}\,\dot{e}_{ij} = 0$. Contradiction. Since $G \geq 0$, the conclusion is $G > 0$. Thus we have proved that $h' + k' + 3G > 0 \Rightarrow G > 0$.

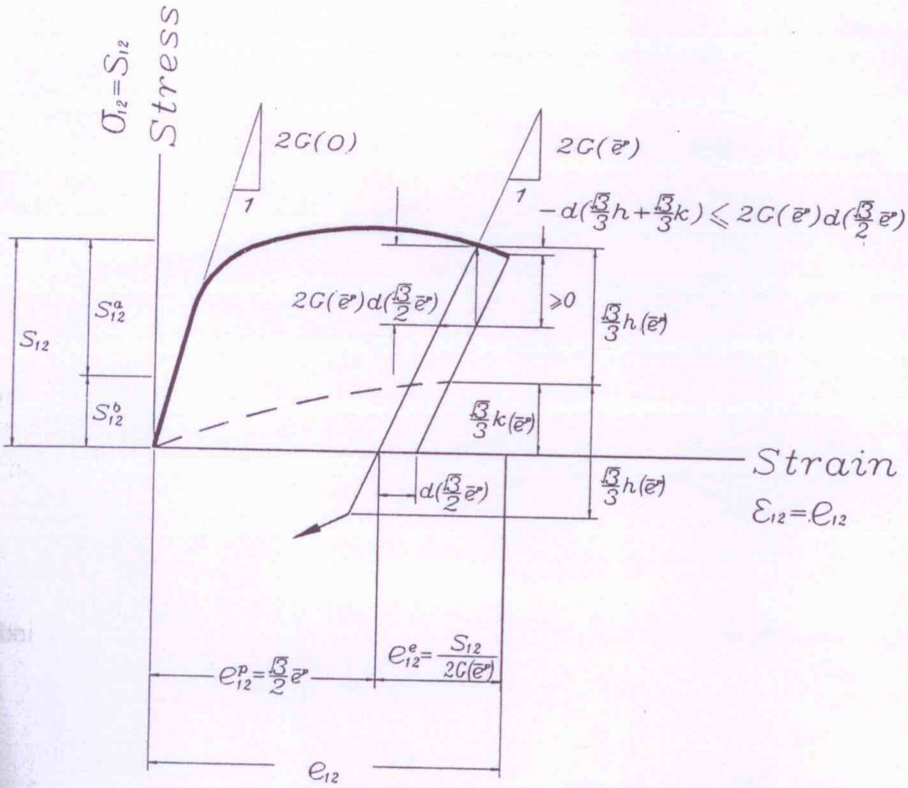


**Fig. 2 Relations of material functions and constraints**

Next, we are to prove $G>0 \Rightarrow h'+k'+3G>0$. The premise is $G>0$. In the case of $f=1$ and $s_{ij}^a\,\dot{e}_{ij}>0$, since $G>0$, Eq. (58) says $(h'+k'+3G)\dot{\Lambda}=3G\,s_{ij}^a\,\dot{e}_{ij}>0$. Should $h'+k'+3G=0$, it would follow that $(h'+k'+3G)\dot{\Lambda}=3G\,s_{ij}^a\,\dot{e}_{ij}=0$. Contradiction. Since $h'+k'+3G\geq 0$, the conclusion is $h'+k'+3G>0$. Thus we have proved that $G>0 \Rightarrow h'+k'+3G>0$.

Hence,

$$G>0 \Leftrightarrow h'+k'+3G>0. \tag{61}$$

Since $h'+k'+3G\geq 0$ and $G\geq 0$, Eq. (61) is logically equivalent to

$$G=0 \Leftrightarrow h'+k'+3G=0 \Rightarrow h'+k'=0. \tag{62}$$

## 9. Absolute failure and weak stability

Once $h'+k'+3G=0$ and $G=0$, $\dot{s}_{ij}=0$ due to Eq. $(10)_1$ and Eq. (58) is trivial. Moreover, the straining condition $G\,s_{ij}^a\,\dot{e}_{ij}>0$ can no longer be valid so that we cannot strain the material further without failure in any case, and the shear stress-shear strain curve drops down vertically, since the shear stress-shear plastic strain curve declines at a slope of $-2G$ [6], i.e.

$$\frac{d(\frac{\sqrt{3}}{3}h+\frac{\sqrt{3}}{3}k)}{d(\frac{\sqrt{3}}{2}\bar{e}^p)}=-2G. \tag{63}$$

Therefore, we conclude that at the time the model breaks down completely and the material fails in all cases. Hence

$$G=0 \Leftrightarrow h'+k'+3G=0 \Leftrightarrow \text{shear absolute failure} \tag{64}$$

at a particular value of $\bar{e}^p$ denoted by $\bar{e}^p_{af}$ or at a particular value of $\Lambda$ denoted by $\Lambda_{af}$ indicates shear absolute failure, as illustrated in Figs. 3 and 4. On the other hand, once $K=0$, $\dot{\sigma}_{ll}=0$ due to Eq. $(10)_2$ and the spherical stress-volumetric strain curve declines to be trivially horizontal. Therefore we conclude that

$$K=0 \Leftrightarrow \text{bulk absolute failure} \tag{65}$$

at a particular value of $\bar{e}^p$ denoted by $\bar{e}^p_{*bulk}$ or at a particular value of $\Lambda$ indicates bulk absolute failure. On the contrary, if $0\le\bar{e}^p<\bar{e}^p_{af}$, then

$$G>0 \Leftrightarrow h'+k'+3G>0 \Leftrightarrow \text{shear weak stability,} \tag{66}$$

and if $0\le\bar{e}^p<\bar{e}^p_{*bulk}$, then

$$K>0 \Leftrightarrow \text{bulk weak stability} \tag{67}$$

such that the governing relations of the proposed model do have chances to work, and we say that the material is weakly stable and Eqs. (66) and (67) are the weak stability conditions.

## 10. Sufficient and necessary conditions for irreversibility

If weak stability is ensured, $h'+k'+3G>0$ and $G>0$, and it is easily seen from Eq. (58) that

$$\text{if} \quad f=1, \quad \text{then} \quad \dot{\Lambda}>0 \Leftrightarrow G\,s^a_{ij}\,\dot{e}_{ij}>0. \tag{68}$$

Upon considering $f\le1$ and $\dot{\Lambda}\ge0$, Eq. (68) and the axiom (14) together give

$$\dot{\Lambda}>0 \Rightarrow f=1 \quad \text{and} \quad G\,s^a_{ij}\,\dot{e}_{ij}>0, \tag{69}$$

that is, the sufficient and necessary conditions for irreversibility are the yield condition $f=1$ and the straining condition

$$G\,s^a_{ij}\,\dot{e}_{ij}>0. \tag{70}$$

In view of Eq. (2), the proved statement (69) is logically equivalent to the following irreversibility-reversibility criterion:

$$\dot{\Lambda}\begin{cases} >0 & \text{iff} \quad f=1 \quad \text{and} \quad G\, s^a_{ij}\, \dot{e}_{ij} > 0 \\ =0 & \text{otherwise} \end{cases} \tag{71}$$

Referring to Eq. (71), the process in which $\dot{\Lambda} > 0$ is referred to as in the irreversible stage and the one in which $\dot{\Lambda} = 0$ as in the reversible stage.

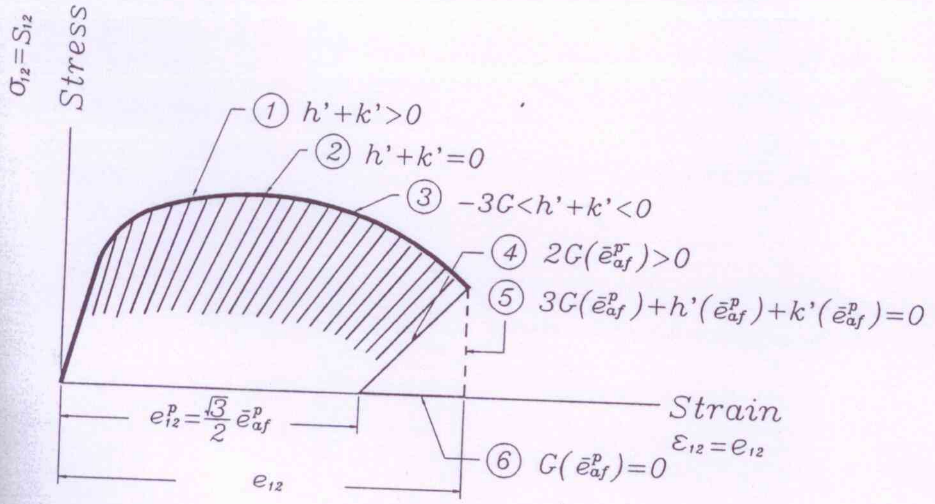


**Fig. 3 Stress-strain diagram exhibiting hardening (1), perfect plasticity (2), softening (3), yielding (1,2,3), probable failure on loading (3), absolute failure (5), and degradation of moduli**

## 11. Loading condition versus straining condition

From Eqs. (9), $(10)_1$ and (45), it is easy to see that

$$2G\, s^a_{ij}\, \dot{e}_{ij} = s^a_{ij}\, \dot{s}_{ij} + 2G\, s^a_{ij} \dot{e}^{\,p}_{ij} = s^a_{ij}\, \dot{s}_{ij} + 2G\, \dot{\Lambda}. \tag{72}$$

Hence due to Eqs. (72), (2) and $(11)_1$,

$$2G\, s^a_{ij}\, \dot{e}_{ij} \geq s^a_{ij}\, \dot{s}_{ij}. \tag{73}$$

Thus it follows immediately:

$$s^a_{ij}\, \dot{s}_{ij} > 0 \Rightarrow G\, s^a_{ij}\, \dot{e}_{ij} > 0, \tag{74}$$

where $s^a_{ij}\, \dot{s}_{ij}$ is known as the loading condition. On the other hand, if $G\, s^a_{ij}\, \dot{e}_{ij} > 0$, we cannot tell the sign of $s^a_{ij}\, \dot{s}_{ij}$ in view of Eq. (73). Therefore, loading implies straining, but straining doesn't correspond to loading, as shown schematically in Fig. 5. Combining Eqs. (74) and (71) gives

$$\dot{\Lambda} > 0 \quad \text{if} \quad f = 1 \quad \text{and} \quad s^a_{ij}\,\dot{s}_{ij} > 0. \tag{75}$$

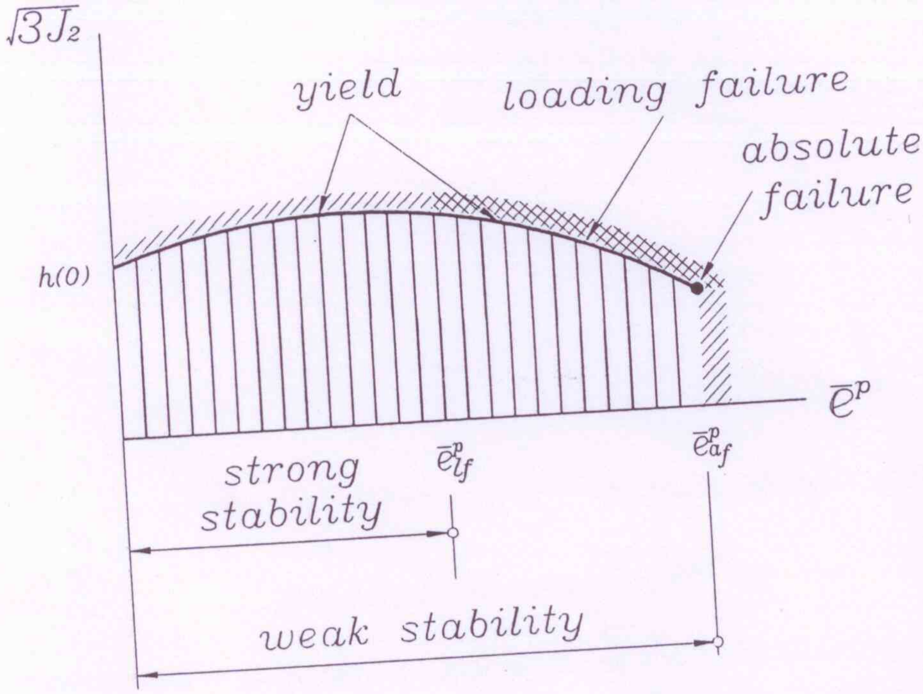


**Fig. 4 Stability and failure**

Therefore, we conclude that the loading condition $s^a_{ij}\,\dot{s}_{ij} > 0$ and the yield condition $f = 1$ are sufficient conditions for irreversibility, $\dot{\Lambda} = h\dot{\bar{e}}^p > 0$, but not necessary conditions, whereas the straining condition $G\,s^a_{ij}\,\dot{e}_{ij} > 0$ and the yield condition $f = 1$ are sufficient and necessary conditions for irreversibility, $\dot{\Lambda} = h\dot{\bar{e}}^p > 0$. In this regard, some conventional criteria are cumbersome because they employ the correct but partial statement (75), etc. and, even worse, some others, ignorant of the role of the loading or straining condition, use the wrong statement that $\dot{\Lambda} > 0$ if $f = 1$, etc. Figure 5 illustrates why in a material softening case the loading condition does not work as is supposed to do, while the straining condition works, since it is capable of distinguishing irreversibility from reversibility no matter softening or hardening. Due to strain determinacy and stress indeterminacy for perfect plasticity and softening depicted in Fig.5 the straining condition replaces without doubt the conventional role of the loading condition.

The straining condition $G\,s^a_{ij}\,\dot{e}_{ij} > 0$ is easier to handle, since $G(\bar{e}^p(t))$ and $s^a_{ij}(t)$ have been just obtained and $\dot{e}_{ij}(t)$ is prescribed in a strain-controlled process. It is perfectly suitable for stiffness formulations, on which most matrix structural analyses and finite element methods are based [2] [5].

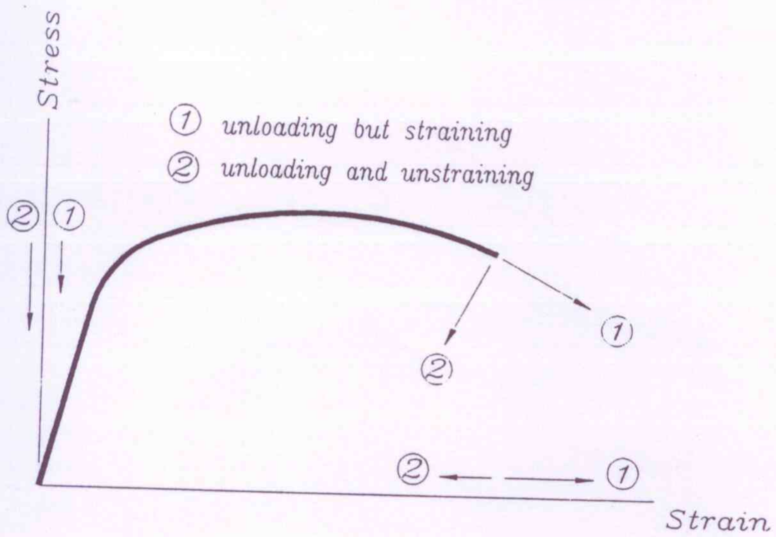


**Fig. 5 Strain determinacy and stress indeterminacy for a softening case**

## 12. Hardening and softening

If the yield condition $f = 1$ and the loading condition $s^a_{ij}\,\dot{s}_{ij} > 0$ are satisfied, Eq. (74) indicates that the straining condition is also satisfied. Thus, strain increases with stress; the material is said hardening. At the time, $s^a_{ij}\,\dot{s}_{ij} = s^a_{ij}\,\dot{s}^a_{ij} + s^a_{ij}\,\dot{s}^b_{ij} = \frac{2}{3}(h' + k')\dot{\Lambda} > 0$, and $\dot{\Lambda} > 0$; therefore, $h' + k' > 0$. Hence we have the following criterion for categorizing material behaviors:

$$\text{hardening} \quad \Leftrightarrow \quad h' + k' > 0, \tag{76}$$

$$\text{perfect} \quad \Leftrightarrow \quad h' + k' = 0, \tag{77}$$

$$\text{softening} \quad \Leftrightarrow \quad -3G < h' + k' < 0, \tag{78}$$

$$\text{absolute failure} \quad \Leftrightarrow \quad -3G = h' + k' = 0, \tag{79}$$

as illustrated in Fig. 3.

## 13. Strong stability versus weak stability

A material in the states of Eq. (76) is stable in any case, no matter it is stress-controlled or strain-controlled, loading or unloading, or straining or unstraining. Therefore, we call Eq. (76) the strong stability condition, i.e.

$$h' + k' > 0 \Leftrightarrow \text{shear strong stability}. \tag{80}$$

It qualifies unconditionally stable states and is a stable condition in Drucker's sense [8] [9]. It cannot be overemphasized that the material which is perfectly plastic or softening is not necessarily unstable and stability is not always hardening. For examples, for a strain-controlled process, or a straining process, an unstraining process, or an unloading process, a material in the states of Eqs. (77) or (78) is stable, but it is unstable for a loading or reloading process, failing as soon as yielding. This explains why shear failure may occur at any value of $\bar{e}^p \in [\bar{e}^p_{lf}, \bar{e}^p_{af}]$ with $\bar{e}^p_{lf}$ satisfying both $h' + k' = 0$ and $f = 1$ and $\bar{e}^p_{af}$ satisfying both $h' + k' + 3G = 0$ and $f = 1$, as shown in Fig. 4. Nevertheless, the material may survive up to right before $\bar{e}^p_{af}$ in some cases and survives in no case at $\bar{e}^p_{af}$. In other words, the material in the states of Eqs. (77) or (78) is conditionally stable. Therefore, Eq. (17) is called the (shear) weak stability condition.

Since $G \geq 0$, it is easy to show that

$$h' + k' > 0 \Rightarrow h' + k' + 3G > 0, \tag{81}$$

which asserts that strong stability implies weak stability and that

$$\bar{e}^p_{lf} \leq \bar{e}^p_{af}, \tag{82}$$

as depicted in Fig. 4.

## 14. Loading failure

In the foregoing section, we have mentioned probability of failure before absolute failure if the material is perfectly plastic and/or softening. We have known in section 12 that

$$\frac{2}{3}(h' + k')\dot{\Lambda} = s^a_{ij}\,\dot{s}_{ij} \quad \text{if} \quad f = 1. \tag{83}$$

It is clear that the above equality breaks down in the case of simultaneous softening, yielding, and loading. Hence

$$-3G < h' + k' \leq 0 \quad \text{and} \quad f = 1 \quad \text{and} \quad s^a_{ij}\,\dot{s}_{ij} > 0 \Leftrightarrow \text{shear loading failure}, \tag{84}$$

which may be called the (shear) loading failure condition. Figure 4 displays the product space of stress and $\bar{e}^p$ in which three curves, the elastic boundary $f = 1$, the strong stability boundary $h' + k' = 0$, and the weak stability boundary $h' + k' + 3G = 0$, are plotted. The two curves $f = 1$ and $h' + k' + 3G = 0$ separate the forbidden from the set of weakly stable states, which is indeed a fiber bundle, fibered into elastic regions for various irreversible stages and $f=1$ being the base manifold. The loading failure curve (or hypersurface)

$$\{(\sqrt{3J_2}, \bar{e}^p) \mid f=1 \quad \text{and} \quad -3G < h' + k' \le 0\} \tag{85}$$

represents *probable* loading failure, the states on which do not lead to failure unless loading is the case, whereas the absolute failure point (or hypercurve)

$$\{(\sqrt{3J_2}, \bar{e}^p) \mid f=1 \quad \text{and} \quad h' + k' + 3G = 0\}, \tag{86}$$

represents absolute failure as depicted in Fig. 4.

## 15. Expansion and shrinkage of yield surface

Since the yield condition, Eq. (7), defines the yield (hyper)surface in stress space and correspondingly the unit level set, Eq. (8), defines the set of admissible stress states, we can observe the relationship between the sign of $h'$ and their size in stress space as follows:

$$\text{expanding} \quad \Leftrightarrow \quad h' > 0, \tag{87}$$

$$\text{rigid} \quad \Leftrightarrow \quad h' = 0, \tag{88}$$

$$\text{shrinking} \quad \Leftrightarrow \quad h' < 0. \tag{89}$$

## 16. Nonnegativity of modulus-weighted active work in the large

Since $G \ge 0$ and $k' \ge 0$, we immediately have

$$k' + 3G \ge 0. \tag{90}$$

Rewritten as

$$\frac{d(\frac{\sqrt{3}}{3}k)}{d(\frac{\sqrt{3}}{2}\bar{e}^p)} \ge -2G, \tag{91}$$

the assertion (90) means profoundly that, in a torsional-shear test of a thin-walled circular tube under monotonic rotation, the back stress versus shear strain curve goes down at most vertically and cannot snap back in the direction of shear strain.

From Eqs. (56) and (45),

$$2G\, s^a_{ij}\, \dot{e}_{ij} = s^a_{ij}\, \dot{s}^a_{ij} + \frac{2}{3}(k' + 3G)\dot{\Lambda}. \tag{92}$$

Consider a closed cycle (or path) of active stress in the time interval $[t_1, t_2]$ in which $s^a_{ij}(t_1) = s^a_{ij}(t_2)$. Then the first integral in the right-hand side of the following equation:

$$\oint_{t_1}^{t_2} 2G\, s^a_{ij}\, \dot{e}_{ij}\, dt = \oint_{t_1}^{t_2} s^a_{ij}\, \dot{s}^a_{ij}\, dt + \oint_{t_1}^{t_2} \frac{2}{3}(k' + 3G)\dot{\Lambda}\, dt \tag{93}$$

vanishes and the second integral is nonnegative because both $k' + 3G$ and $\dot{\Lambda}$ are nonnegative. Accordingly, we have proved that

$$\oint_{t_1}^{t_2} 2G\, s^a_{ij}\, \dot{e}_{ij}\, dt \geq 0, \tag{94}$$

which asserts the nonnegativity of the modulus-weighted active work in a closed cycle of active stress (or in the large). It is somehow similar to but different from Ilyushin's postulate [10]. For comparison purposes, Eq. (45) is recalled that

$$d\Lambda = s^a_{ij}\, de^p_{ij} \geq 0, \qquad \oint d\Lambda = \oint s^a_{ij}\, de^p_{ij} \geq 0, \tag{95}$$

which assert the nonnegativities of active plastic works both in the small and in the large.

## 17. Nonnegativities of strain and complementary energies and dissipation

To prove Eqs. (39) and (40), consider a case near the zero-value state under pure volumetric deformation $\epsilon_{kk}(t) \neq 0$, in which, since given $K(\bar{e}^p) \geq 0$, $W(t) = W^c(t) = K(0)\epsilon^2_{kk}(t) \geq 0$. This would contradict the assumption. Thus we have proved that $W \geq 0$ and $W^c \geq 0$.

Since given $\dot{\Lambda} \geq 0$ and $\Lambda(t_0) = 0$, it is clear that

$$\Lambda(t) = \int_{t_0}^{t} \dot{\Lambda}(\xi) d\xi \geq 0. \tag{96}$$

Thus we have proved Eq. (41).

## 18. Conclusions

In the present paper, we have reconstructed the $J_2$ flow model for elastoplastic materials by distilling axiomatic ingredients and derive from the axioms a consistent model of elastic-plastic-damage-failure and investigating the properties of the model. The model accepts as its founding stones and thereby satisfies the first and second laws of thermodynamics. It has been found that any two of (1) the relation between the yield function and the plastic-flow potential function, (2) the sufficient condition for reversibility, and (3) the constitutive law for the dissipation rate can be chosen as axioms and with aid of some other axioms the third one can then be deduced. The assertion of maximum dissipation rate in the set of admissible stress states has been proved and its Kuhn-Tucker conditions formulated.

Eqs. (58) and (83) are two very enlightening assertions which reveal much of the nature of the modeled materials concerning irreversibility, straining, loading, stability, and failure. It has been found that if weak stability is ensured, the yield condition and the straining condition are sufficient and necessary for irreversibility. It is worth noting that instead of arbitrary assignment the stability conditions and the failure conditions emerge naturally in the deduction of the model. We have identified two shear failure modes: loading failure and absolute failure. The loading failure occurs on simultaneous softening, yielding, and loading. The absolute failure means material loss of weak stability. For a hardening material, there is no loading failure mode. The nonnegativities of $h' + k' + 3G$ and $k' + 3G$ have been proved and found profound meanings in stress-strain curves. It is interesting to note that the modulus-weighted active work in the large is nonnegative but that in the small may be nonnegative or negative, while the active plastic work either in the small or in the large is known to be nonnegative.

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