
THE BINOMIAL BLACK–SCHOLES MODEL AND THE GREEKS

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This article returns to the choice of method for calculating option hedge ratios discussed by Pelsser and Vorst (1994). Where they demonstrated that numerical differentiation of a binomial model compared poorly to their design of an extended tree, this study shows that the Binomial Black–Scholes method advocated by Broadie and Detemple (1996) does not suffer from the same problem; therefore, it is very effective in the calculation of the Greeks. © 2002 John Wiley & Sons, Inc. *Jrl Fut Mark* 22:143–153, 2002

INTRODUCTION

Once an option has been priced and sold, calculation of the so-called Greeks (hedge ratios) or quantities of offsetting assets is key to the successful risk management of a naked option position. As *pricing* techniques advance in speed and accuracy, it is natural to consider how

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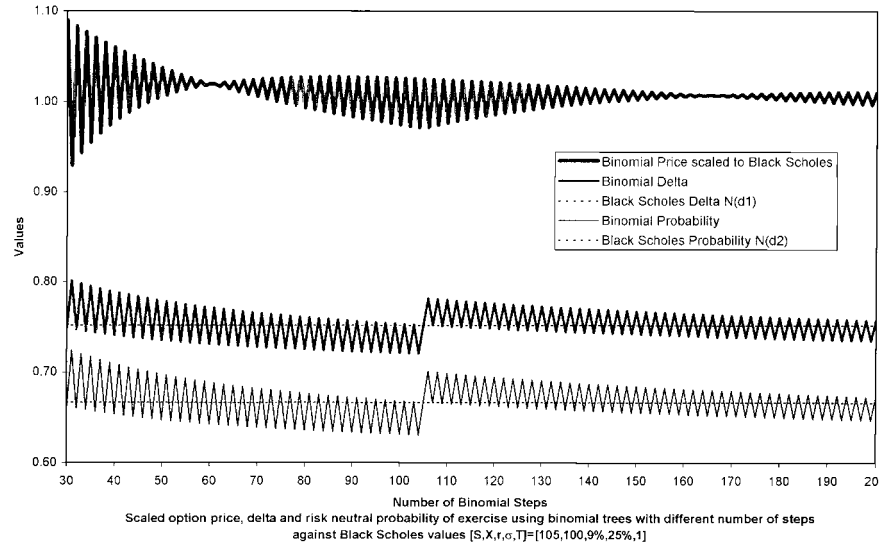


FIGURE 1
European (scaled) option pricing and Greeks.

these same improved techniques fare for *hedging* purposes in terms of speed and accuracy.

Binomial methods—as first advocated by Cox, Ross, and Rubinstein (1979)—are very quick and flexible for pricing options, even when free boundaries are present, as is the case for American (and other exotic) options. However, as Pelsser and Vorst (1994) showed, the use of such tree methods for calculation of partial derivatives may be flawed by the nature of the binomial discretization they encode.

Moreover, it has long been known (see Broadie and Detemple, 1996, for European and Ritchken, 1995, for barrier options) that convergence to the true price is not monotonic, but oscillatory, in the step size, and the same is true for option Greeks as well. Figure 1 shows a (rescaled) European option price against its Black–Scholes limit, along with the Δ and risk-neutral probability of exercise¹ compared to the continuous limits $N(d_1)$, $N(d_2)$. As can be seen, not only do the Δ and probability of exercise oscillate along with the price estimate, but it is also the case that there is a dependency between their errors; when the price estimate oscillates most, the Δ and probability have maximum bias. Estimates of the Greeks that did not oscillate in the number of steps used will be desirable because interaction between these errors may be problematic.

This study reviews the argument of Pelsser and Vorst (1994), who examined these discreteness issues, and then reapplies their comparison

¹We detail the calculation methods for Binomial pricing in the next section.

of differentiation techniques to another extension of the binomial tree method. Broadie and Detemple (1996) augmented the binomial tree method through the addition of Black–Scholes prices (or payoffs if greater) at the penultimate pricing node, arguing that these closed-form expressions would be useful in increasing the accuracy of American options because one period before maturity the option will revert to either its European value or payoff. For other options types as well, such a technique yields increased accuracy for the same tree size.

This article argues that while this so-called Binomial Black–Scholes method of Broadie and Detemple (hereafter referred to as BBS) yields more accurate prices, its main advantage is that it reinstates the use of the numerical differentiation method that Pelsser and Vorst showed worked so poorly for the straight binomial method. This is because the BBS method is based on smooth, not discrete, functions of stock price and time near maturity and thus, it produces smooth Greeks.

While examining only the numerical Greeks for European options against their Black–Scholes limiting values, the results may have more general relevance. American (and other) option Greeks were not analyzed here due to the unavailability of closed-form (continuous) benchmarks. However, our proofs and discussions in the following two sections seem to indicate that the results presented here are a consequence of the method used for evaluation, not of the option type itself.²

TWO METHODS FOR CALCULATING HEDGE RATIOS

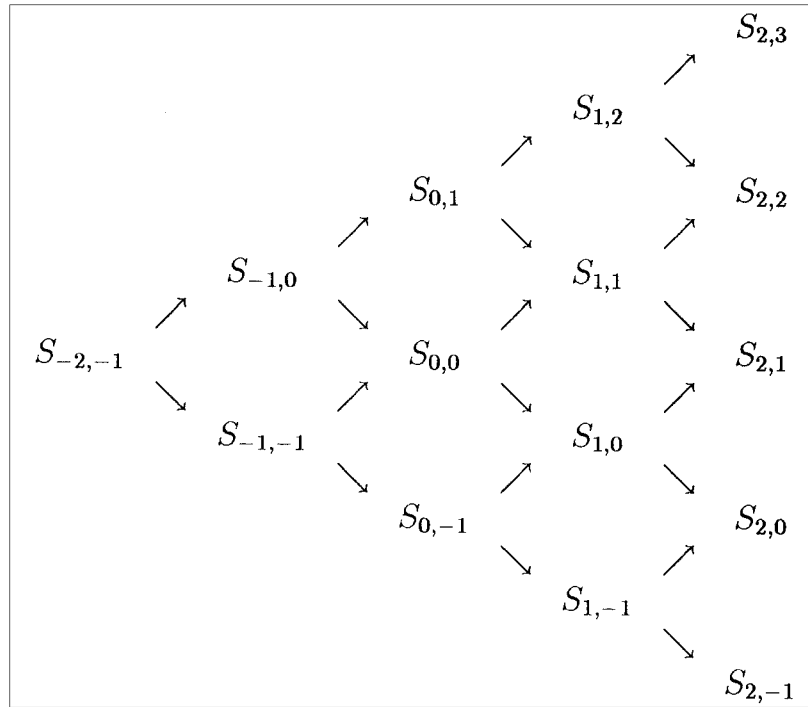
Table I shows a binomial tree of the type that frequently is used in the calculation of option prices; the time between intervals is Δt , time to final maturity is labeled N , and the volatility encoded by the tree is σ . The stock price $n \leq N$ periods ahead (at time $n\Delta t \leq N\Delta t$), and in-state j is labelled $S_{n,j}$ (with $S_{0,0}$ denoting the stock price at time zero). Furthermore, it is usual to fix the magnitude of up-and-down movements through constant multiples $u, d = e^{\pm\sigma\sqrt{\Delta t}}$ so that $S_{n,j} = S_{0,0}u^jd^{n-j}$.³

Cox et al. (1979) showed that current (call) option prices $C_{n,j}$ can be calculated as risk-neutral expectations (up probability of p) of

²As pointed out by the referee, since numerical prices of complex options (e.g., barrier options) are usually much less stable than European options, it may be not always the case that the BBS method provides better numerical Greeks for complex options.

³The tree drawn in Table I has been extended to include two prior time intervals $n = -2$ so that it represents the extended tree of Pelsser and Vorst (1994). In the extended tree, we still have $S_{n,j} = S_{0,0}u^jd^{n-j}$, but n and j may be negative for dates prior to time 0. This feature will be used after demonstrating the problem with numerical differentiation.

TABLE I
 $N = 2$ -Period Binomial Tree Extended to Predate Time Zero and
 Thus Encompass the Current State $S_{0,0}$



The first subscripted index is time and the second is the number of up steps.

future prices $pC_{n+1,j+1} + (1 - p)C_{n+1,j}$ if a continuous risk-free rate r is used for discounting, repeating the recursion back from fixed payoffs at maturity.

Having arrived at a current option price, this method then can be repeated to obtain the price of options with slightly different characteristics and the numerical differential inferred from the change in price. The partial derivative with respect to the stock price (delta Δ) and time (theta θ) require the following (note that these differentials are defined symmetrically about the current value $S_{0,0}$)

$$\Delta = \frac{C(S_{0,0} + h, T) - C(S_{0,0} - h, T)}{2h}$$

$$\theta = \frac{C(S_{0,0}, T - \tau) - C(S_{0,0}, T + \tau)}{2\tau}$$

However, Pelsser and Vorst (1994) showed graphically that implementation of such numerical differentiation will fail for Δ because the tree takes discrete payoffs, and, as such, for a small perturbation h , the option value is not convex in $S_{0,0}$. Taking the numerical derivative of the probability weighted call payoffs (the maximum of the terminal stock price less the strike price and zero) shows that the probability of exercise does not change for small perturbation h , effectively the Δ is locally constant because the option price itself is a locally linear function of the stock price.

Defining the binomial probability of arriving at a final node (N, k) as $\text{prob}(N, k) = \frac{N!p^k(1-p)^{N-k}}{k!(N-k)!}$, call prices are derived from discounted risk-neutral expected values of the option payoff. This requires the definition of i , the minimum number of up steps required for the option to finish in the money, that is, $i = \min_k S_{N,k} > K$ (the strike price)

$$C(S_{0,0}, T) = e^{-rT} \sum_{k=i}^N \text{prob}(N, k) (S_{0,0} d^{(N-k)} u^k - K) \quad (1)$$

When the perturbation h is arbitrarily small, the minimum number (i) of up steps required for the $(S_{0,0} + h)$ and $(S_{0,0} - h)$ trees is the same as that for the $S_{0,0}$ tree. Therefore, Δ is locally constant⁴ and Γ (the second numerical differential) is zero,

$$\begin{aligned} C(S_{0,0} \pm h, T) &= e^{-rT} \sum_{k=i}^N \text{prob}(N, k) ((S_{0,0} \pm h) d^{(N-k)} u^k - K) \\ \Delta &= e^{-rT} \sum_{k=i}^N \text{prob}(N, k) d^{(N-k)} u^k \\ \Gamma &= 0 \end{aligned}$$

This problem also is present when allowing for early exercise and pricing American options.⁵ In the following derivation, $i^*(n)$ —a state reference—represents the minimum state in which the option is exercised early at time $n\Delta t$ and j^* —a time reference—represents the

⁴The same is also true of the Binomial risk-neutral probability of exercise corresponding to $N(d_2)$, calculated as $\sum_{k=i}^N \text{prob}(N, k)$.

⁵The derivation shows that this is less likely to happen with American than European options.

first time that the binomial tree price can exceed the early exercise threshold⁶

$$C^A(S_{0,0}, T) = \sum_{n=j^*}^{N-1} \text{prob}(n, i^*(n) | \text{no ex. before } n\Delta T) (S_{0,0} d^{(n-i^*)} u^{i^*} - K) e^{-rn\Delta T} \\ + e^{-rT} \sum_{k=i}^{i^*(N)} \text{prob}(N, k | \text{no ex. before } T) (S_{0,0} d^{(N-k)} u^k - K),$$

where $\text{prob}(\cdot|\cdot)$ is the conditional probability of no exercise before a certain time. When h is arbitrarily small, the i^* and j^* for the $(S_{0,0} + h)$ and $(S_{0,0} - h)$ trees are again the same as that for the $S_{0,0}$ tree. Therefore, Δ is locally constant, that is,

$$\Delta = \frac{C^A(S_{0,0} + h, T) - C^A(S_{0,0} - h, T)}{2h} \\ = \sum_{n=j^*}^{N-1} \text{prob}(n, i^*(n) | \text{no ex. before } n\Delta T) d^{(n-i^*)} u^{i^*} e^{-rn\Delta T} \\ + e^{-rT} \sum_{k=j}^{i^*(N)} \text{prob}(N, k | \text{no ex. before } T) d^{(N-k)} u^k$$

One alternative to a straight binomial method is that of Figlewski and Gao (1999), who, in their so-called Adaptive Mesh Method (AMM), sought to reduce this discretization problem through the addition of more grid points near the strike K , thus improving pricing and restoring the sensitivity of Δ to $S_{0,0}$. From the above proof, it is straightforward to show that although the AMM method can increase the accuracy of option prices, it still suffers from perturbation discreteness for any finite expanded mesh. This discreteness becomes smaller the larger the number of extra nodes; however, the computation time also increases.

Pelsser and Vorst (1994) argued that contrary to initial intuition when working with numerical derivatives (when small h is preferred), the error displayed in their figure did not result because h was too big, but because h was too small already and effectively suggested increasing h until it was aligned with the step sizes u and d themselves. This is done by recalculating prices over the full extended tree shown in Table I and

⁶Note that j^* could be greater than or equal to N if early exercise is never optimal.

using $C(S_{0,1})$, $C(S_{0,-1})$ and $C(S_{-2,-1})$, $C(S_{2,1})$ rather than by recalculating the initial tree.

$$\Delta = \frac{C(S_{0,1}) - C(S_{0,-1})}{S_{0,1} - S_{0,-1}}$$

$$\theta = \frac{C(S_{-2,-1}) - C(S_{2,1})}{4\tau}$$

Not only does this have the benefit of being a quicker calculation (a slightly larger tree is quicker than calculating a new tree over again), but also the resulting Greeks do not suffer from the discreteness problem associated with numerical differentiation.

Although this extended-tree method is much more reliable than the numerical-differentiation method of an existing binomial tree, it still has a remaining error that is dependent on the magnitude of the tree intervals chosen in time and stock price. If the discreteness is too large, the measure obtained for the Greek will be biased because of its nonlinear nature and, therefore, fine trees still are preferred to sparse trees. This discretization error is reduced by averaging the Δ from above and below and the θ from left and the right, but it cannot be eliminated for any finite step size.

For a given pricing mechanism, a method of differentiation that could be made arbitrarily accurate would still be desirable because it would allow evaluation using any h and arbitrarily small h in particular. An adapted Binomial tree that admitted numerical differentiation would be desirable.

A FURTHER TREE METHOD AND ITS GREEKS

Broadie and Detemple (1996) showed how inclusion of Black–Scholes prices in the penultimate node expedites a more accurate tree; this is known as the Binomial Black–Scholes or BBS method. This article shows that not only does it yield more accurate prices, but it also allows accurate calculation of the Greeks because, unlike the piecewise linear payoff presumed in Equation (1), nonlinear and smoothly differentiable Black–Scholes functions are inserted into the final pricing node

$$C(S_{0,0}, T) = e^{-r(T-\Delta t)} \sum_{i=0}^{N-1} \text{prob}(N-1, i) \text{BS}(S_{0,0}d^{(N-1-i)}u^i, K, r, \sigma, \Delta t) \quad (2)$$

$$C(S_{0,0} \pm h, T)$$

$$= e^{-r(T-\Delta t)} \sum_{i=0}^{N-1} \text{prob}(N-1, i) \text{BS}((S_{0,0} \pm h)d^{(N-1-i)}u^i, K, r, \sigma, \Delta t)$$

Because there is now a chance that the option may end in the money from even the smallest of penultimate stock prices, summations must be taken over all penultimate nodes.

NUMERICAL RESULTS

To illustrate the point, European option deltas were calculated in a manner similar to those in Pelsser and Vorst (1994) using $(K, r, \sigma, T) = (100, 9\%, 25\%, 1)$ over a small range of $S_{0,0}$ using $N = 50$. Their figure (Exhibit 3) showed that numerical differentiation of the binomial model failed and option deltas were highly discrete in $S_{0,0}$; however, Figure 2 here, where differences between calculated Δ s and Black–Scholes Δ s against $S_{0,0}$, shows that numerical differentiation of the BBS method does not yield discrete values of Δ . The numerical differentials are very close to the Black–Scholes analytic values and indeed closer than either the Pelsser and Vorst (1994) extended tree values or the same extended tree where the BBS modification was made. Thus, the addition of the smooth function in the BBS tree and numerical differentiation with $h = 0.01$ outperforms either a straight extended tree or a BBS extended tree, with the caveat that the calculation time is, of course, longer (recall that numerical differentiation requires the tree to be recalculated while the extended tree does not).

Although not shown here, the resulting Γ , or second difference, estimates using numerical differentiation ($h = 0.01$) also outperformed extended binomial and extended BBS tree Γ s.

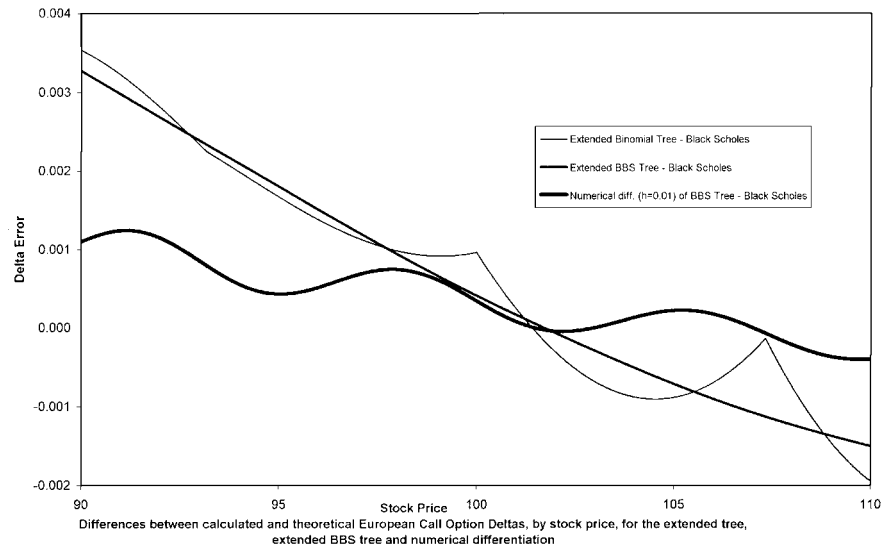


FIGURE 2
 Δ estimation error for a European Call Option.

Option θ s also were calculated. Although Pelsser and Vorst (1994) did not recommend a particular way to calculate θ , the Figures show the right- and left-hand derivatives and their average that can be calculated from their extended tree.

$$\begin{aligned} \theta_- &= \frac{C(S_{-2,-1}) - C(S_{0,0})}{2\tau} \\ \theta_+ &= \frac{C(S_{0,0}) - C(S_{2,1})}{2\tau} \\ \theta &= \frac{C(S_{-2,-1}) - C(S_{2,1})}{4\tau} \\ &= \frac{\theta_+ + \theta_-}{2} \end{aligned}$$

Figure 3 shows the extended binomial tree θ s from backward (θ_-), forward (θ_+), and average (θ) methods compared to the theoretical Black-Scholes value. The average of forward and backward differences performs best compared to the continuous Black-Scholes limit although some form of oscillation in the number of steps is present.

Figure 4 shows the extended BBS tree θ s. This method is superior to the extended binomial tree used in Figure 3 because no oscillation is present in the number of tree steps. To reiterate, this is because of the insertion of a smooth function before payoff. The average of forward

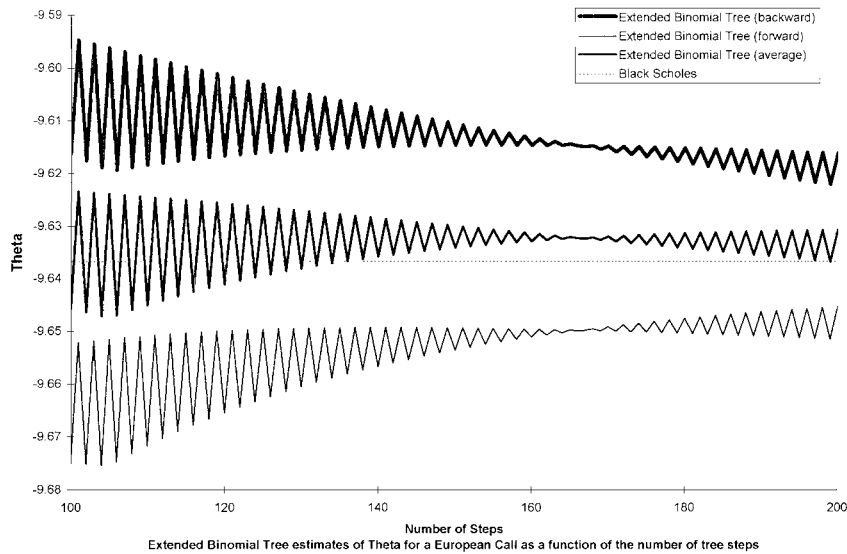


FIGURE 3
European Call Option θ estimates.

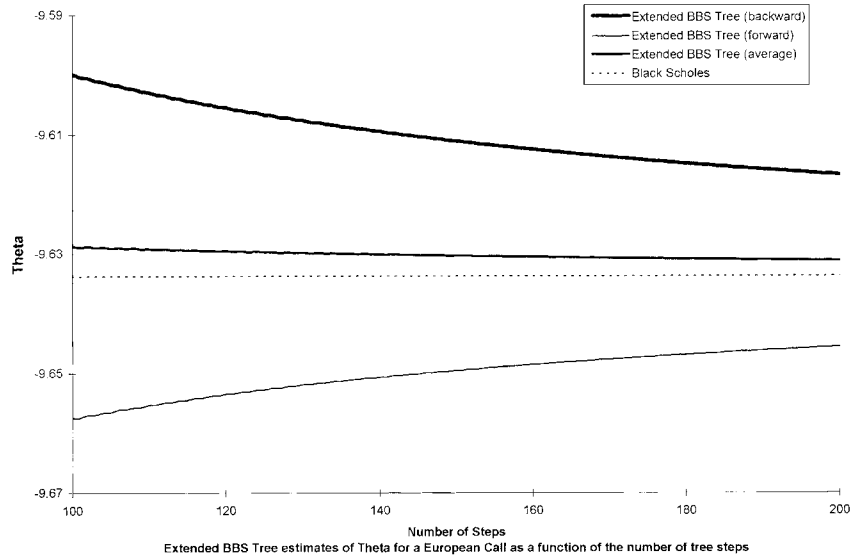


FIGURE 4
European Call Option θ estimates.

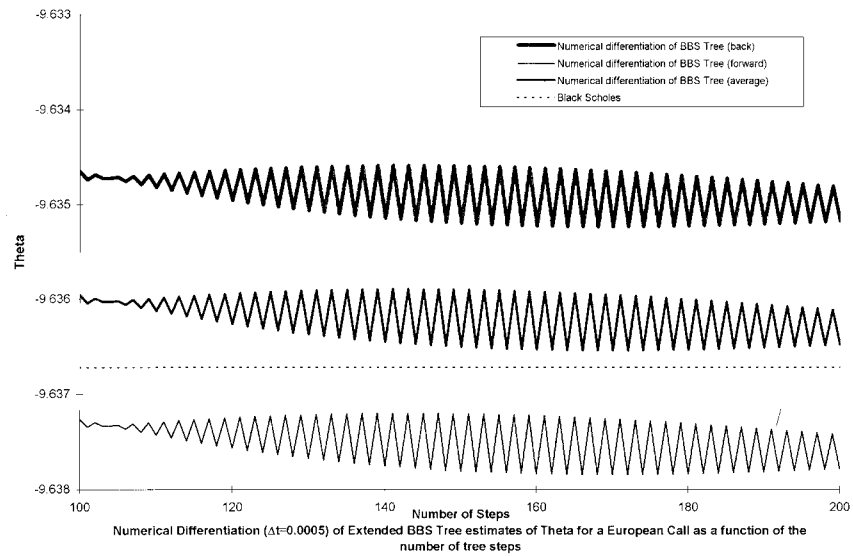


FIGURE 5
European Call Option θ estimates.

and backward methods also is recommended, as it is closer to the Black–Scholes continuous value.

Now that the use of numerical differentiation can be reinstated, it is possible, however, to improve θ estimates further. Figure 5 shows θ estimates calculated from numerical differentiation of the BBS extended

tree (with $\Delta t = 0.0005$). Although requiring the calculation of a second tree⁷, the estimates are significantly more accurate for limited numbers of steps.

All θ estimates are shown as a function of the number of steps in the tree, in itself a measure of accuracy of the actual price, but a method that gives more accurate Greeks using a smaller number of steps is more desirable.

CONCLUSION AND REMARKS

This short article shows that the binomial Black–Scholes (BBS) method of Broadie and Detemple (1996) is not only very useful for *pricing*, but also for calculating the Greeks via *numerical differentiation*. Unlike the binomial method, it does not suffer from the problem of discreteness. This allows the same tool to be used for hedging (via numerical differentiation), as well as for pricing.

For plain vanilla options, this article demonstrates another advantage of pinning a tree method on closed-form approximations before maturity that are smooth rather than on precise payoffs at expiry that are not smooth. In future research, it would be interesting to examine the usefulness of the BBS method when applied to other tree approaches that incorporate uncertain volatility features or complex payoffs. For example, Ritchken and Trevor (1999) used tree techniques to price options under GARCH volatility, and the inclusion of Black–Scholes prices one period before maturity (from when volatility would have little time to change) may both increase pricing accuracy and facilitate the calculation of the Greeks via numerical differentiation.

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⁷Three trees are needed in total in order to calculate a backward, a forward, and an average θ . Note that the scale of Figure 5 is about 10× larger than that of Figure 4.